PHY 712 Electrodynamics
9-9:50 AM MWF Olin 103

Plan for Lecture 34:

Special Topics in Electrodynamics:

Electromagnetic aspects of superconductivity

- London equations
- Tunneling between two superconductors

20	Mon: 03/16/2015	Chap. 8	Review Exam; Wave guides	#19	03/18/2015
21	Wed: 03/18/2015	Chap. 8	Wave guides	#20	03/20/2015
22	Fri: 03/20/2015	Chap. 9	Radiation sources	#21	03/23/2015
23	Mon: 03/23/2015	Chap. 9 & 10	Radiation and scattering	#22	03/25/2015
24	Wed: 03/25/2015	Chap. 9 & 10	Radiation and scattering		
25	Fri: 03/27/2015	Chap. 11	Special relativity	#23	03/30/2015
26	Mon: 03/30/2015	Chap. 11	Special relativity	#24	04/01/2015
27	Wed: 04/01/2015	Chap. 11	Special relativity	#25	04/06/2015
	Fri: 04/03/2015	Good Friday	No class		
28	Mon: 04/06/2015	Chap. 14	Radiation from moving charges	#26	04/08/2015
29	Wed: 04/08/2015	Chap. 14	Radiation from moving charges	#27	04/10/2015
30	Fri: 04/10/2015	Chap. 14	Radiation from moving charges	#28	04/13/2015
31	Mon: 04/13/2015	Chap. 15	Radiation due to scattering	#29	04/15/2015
32	Wed: 04/16/2015	Chap. 13	Cherenkov radiation	#30	04/17/2015
33	Fri: 04/17/2015		Special topics -- superconductivity		
34	Mon: 04/20/2015		Special topics -- superconductivity		
35	Wed: 04/22/2015		Review		
36	Fri: 04/24/2015		Review		
	Mon: 04/27/2015		Presentations I		
	Wed: 04/29/2015		Presentations II		
	Fri: 05/01/2015		Presentations III & Take home exam		

London model of superconducting state

Recall Drude model of conductivity in "normal" materials

$$m\frac{d\mathbf{v}}{dt} = -e\mathbf{E} - m\frac{\mathbf{v}}{\tau}$$

$$\mathbf{v}(t) = \mathbf{v}_0 e^{-t/\tau} - \frac{e\mathbf{E}\tau}{m}$$

$$\mathbf{J} = -ne\mathbf{v} \qquad \text{for } t >> \tau \qquad \mathbf{J} = \frac{ne^2\tau}{m}\mathbf{E} \equiv \sigma\mathbf{E}$$

Suppose $\tau \to 0$; London model of conductivity in superconducting materials

$$m\frac{d\mathbf{v}}{dt} = -e\mathbf{E}$$

$$\frac{d\mathbf{v}}{dt} = -\frac{e\mathbf{E}}{m} \qquad \frac{d\mathbf{J}}{dt} = -ne\frac{d\mathbf{v}}{dt} = \frac{ne^2\mathbf{E}}{m}$$

From Maxwell's equations:

$$\nabla\times\mathbf{B} = \frac{4\pi}{c}\mathbf{J} + \frac{1}{c}\frac{\partial\mathbf{E}}{\partial t} \qquad \nabla\times\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}$$

London model, continued

London model of conductivity in superconducting materials

$$\frac{d\mathbf{J}}{dt} = -ne\frac{d\mathbf{v}}{dt} = \frac{ne^2\mathbf{E}}{m}$$

From Maxwell's equations:

$$\nabla\times\mathbf{B} = \frac{4\pi}{c}\mathbf{J} + \frac{1}{c}\frac{\partial\mathbf{E}}{\partial t} \qquad \nabla\times\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla\times(\nabla\times\mathbf{B}) = -\nabla^2\mathbf{B} = \frac{4\pi}{c}\nabla\times\mathbf{J} - \frac{1}{c^2}\frac{\partial^2\mathbf{B}}{\partial t^2}$$

$$-\nabla^2\frac{\partial\mathbf{B}}{\partial t} = \frac{4\pi}{c}\nabla\times\frac{\partial\mathbf{J}}{\partial t} - \frac{1}{c^2}\frac{\partial^3\mathbf{B}}{\partial t^3}$$

$$-\nabla^2\frac{\partial\mathbf{B}}{\partial t} = \frac{4\pi ne^2}{mc}\nabla\times\mathbf{E} - \frac{1}{c^2}\frac{\partial^3\mathbf{B}}{\partial t^3}$$

$$-\nabla^2\frac{\partial\mathbf{B}}{\partial t} = -\frac{4\pi ne^2}{mc^2}\frac{\partial\mathbf{B}}{\partial t} - \frac{1}{c^2}\frac{\partial^3\mathbf{B}}{\partial t^3}$$

$$\frac{\partial}{\partial t}\left(\nabla^2 - \frac{1}{\lambda_L^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{B} = 0 \qquad \text{with } \lambda_L^2 \equiv \frac{mc^2}{4\pi ne^2}$$

London model – continued

London model of conductivity in superconducting materials

$$\frac{d\mathbf{J}}{dt} = -ne\frac{d\mathbf{v}}{dt} = \frac{ne^2\mathbf{E}}{m}$$

$$\frac{\partial}{\partial t}\left(\nabla^2 - \frac{1}{\lambda_L^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{B} = 0 \qquad \text{with } \lambda_L^2 \equiv \frac{mc^2}{4\pi ne^2}$$

For slowly varying solution:

$$\frac{\partial}{\partial t}\left(\nabla^2 - \frac{1}{\lambda_L^2}\right)\mathbf{B} = 0 \qquad \text{for } \frac{\partial\mathbf{B}}{\partial t} = \hat{\mathbf{z}}\frac{\partial B_z(x,t)}{\partial t}:$$

$$\Rightarrow \frac{\partial B_z(x,t)}{\partial t} = \frac{\partial B_z(0,t)}{\partial t}e^{-x/\lambda_L}$$

London leap: $B_z(x,t) = B_z(0,t)e^{-x/\lambda_L}$

Consistent results for current density:

$$\frac{4\pi}{c}\nabla\times\mathbf{J} = -\nabla^2\mathbf{B} = -\frac{1}{\lambda_L^2}\mathbf{B} \qquad \mathbf{J} = \hat{\mathbf{y}}J_y(x) \quad \Rightarrow \quad J_y(x) = \lambda_L\frac{ne^2}{mc}\mathrm{B}_z(0)\mathrm{e}^{-x/\lambda_L}$$

Behavior of superconducting material – exclusion of magnetic field according to the London model

Penetration length for superconductor: $\lambda_L^2 \equiv \dfrac{mc^2}{4\pi ne^2}$

$B_z(x,t) = B_z(0,t)e^{-x/\lambda_L}$

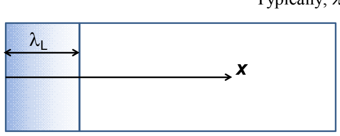
Vector potential for $\nabla\cdot\mathbf{A} = 0$:

$\mathbf{A} = \hat{\mathbf{y}}A_y(x) \qquad A_y(x) = -\lambda_L B_z(0)e^{-x/\lambda_L}$

Current density: $J_y(x) = \lambda_L\dfrac{ne^2}{mc}\mathrm{B}_z(0)\mathrm{e}^{-x/\lambda_L}$

$$\Rightarrow \mathbf{J} + \frac{ne^2}{mc}\mathbf{A} = 0 \quad \text{or} \quad \frac{ne}{m}\left(m\mathbf{v} + \frac{e}{c}\mathbf{A}\right) = 0$$

Typically, $\lambda_L \approx 10^{-7}m$



Magnetization field

Treating London current in terms of corresponding magnetization field **M**:

$\mathbf{B} = \mathbf{H} + 4\pi\mathbf{M}$

$\Rightarrow$ For $x >> \lambda_L$, $\mathbf{H} = -4\pi\mathbf{M}$

Gibbs free energy associated with magnetization for superconductor:

$$G_S(H_a) = G_S(H=0) - \int_0^{H_a} dH M(H) = G_S(0) + \frac{1}{8\pi} H_a^2$$

Gibbs free energy associated with magnetization for normal conductor:

$$G_N(H_a) \approx G_N(H=0)$$

Condition at phase boundary between normal and superconducting states:

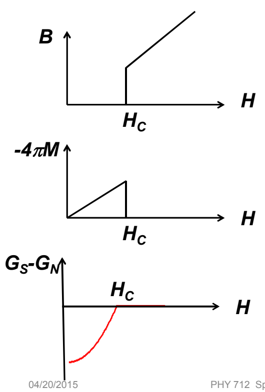
$$G_N(H_C) \approx G_N(0) = G_S(H_C) = G_S(0) + \frac{1}{8\pi} H_C^2$$

(At *T=0K*)

$$\Rightarrow G_S(0) - G_N(0) = -\frac{1}{8\pi} H_C^2$$

$$G_S(H_a) - G_N(H_a) = \begin{cases} -\frac{1}{8\pi}\left(H_C^2 - H_a^2\right) & \text{for } H_a < H_C \\ 0 & \text{for } H_a > H_C \end{cases}$$

Magnetization field (for "type I" superconductor at T=0K)



Inside superconductor

$\mathbf{B} = 0 = \mathbf{H} + 4\pi\mathbf{M}$ for $H < H_C$

Temperature dependence of critical field

$$H_c(T) \approx H_c(0)\left(1 - \left(\frac{T}{T_c}\right)^2\right)$$

From PR **108**, 1175 (1957) Bardeen, Cooper, and Schrieffer, "Theory of Superconductivity"

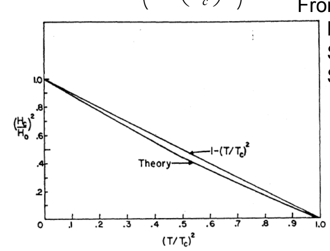


FIG. 2. Ratio of the critical field to its value at $T=0°$K *vs* $(T/T_c)^2$. The upper curve is the $1-(T/T_c)^2$ law of the Gorter-Casimir theory and the lower curve is the law predicted by the theory in the weak-coupling limit. Experimental values generally lie between the two curves.

$$T_c \approx \frac{\hbar\omega}{k} e^{-2/(N(E_F)V)}$$

characteristic phonon energy

density of electron states at E_F

attraction potential between electron pairs

Type I elemental superconductors

http://wuphys.wustl.edu/~jss/NewPeriodicTable.pdf

Periodic Table of Superconductivity

(dedicated to the memory of Bernd Matthias; compiled by James S. Schilling)

30 elements superconduct at ambient pressure, 23 more superconduct at high pressure.

ambient pressure superconductor: T_c(K), T_c^{max}(K), P(GPa)

high pressure superconductor: T_c^{max}(K), P(GPa)

H																	He
Li 0.0004 14 30	Be 0.026 3.7 30											B 11 250	C	N	O 0.6 100	F	Ne
Na	Mg											Al 1.14	Si 8.2 15.2	P 13 30	S 17.3 190	Cl	Ar
K	Ca 29 217	Sc 19.6 106	Ti 0.39 3.35 56.0	V 5.38 16.5 120	Cr	Mn	Fe 2.1 21	Co	Ni	Cu	Zn 0.875	Ga 1.091 7 1.4	Ge 5.35 11.5	As 2.4 32	Se 8 150	Br 1.4 100	Kr
Rb	Sr 7 50	Y 19.5 115	Zr 0.546 11 30	Nb 9.50 9.9 10	Mo 0.92	Tc 7.77	Ru 0.51	Rh .00033	Pd	Ag	Cd 0.56	In 3.404	Sn 3.722 5.3 11.3	Sb 3.9 25	Te 7.5 35	I 1.2 25	Xe
Cs 1.3 12	Ba 5 18	insert La-Lu	Hf 0.12 8.6 62	Ta 4.483 4.5 43	W 0.012	Re 1.4	Os 0.655	Ir 0.14	Pt	Au	Hg-α 4.153	Tl 2.39	Pb 7.193	Bi 8.5 9.1	Po	At	Rn
Fr	Ra	insert Ac-Lr	Rf	Ha													
		La-fcc 6.00 13 15	Ce 1.7 5	Pr	Nd	Pm	Sm	Eu 2.75 142	Gd	Tb	Dy	Ho	Er	Tm	Yb	Lu 12.4 174	
		Ac	Th 1.368	Pa 1.4	U 0.8(β) 2.4(α) 1.2	Np	Pu	Am 0.79 2.2 6	Cm	Bk	Cf	Es	Fm	Md	No	Lr	

M. Debessai, T. Matsuoka, J.J. Hamlin, W. Bi, Y. Meng, K. Shimizu, and J.S. Schilling, J. Phys.: Conf. Series **215**, 012034 (2010).
High pressure data for Ca and Be: K. Shimizu email from 9 Dec 2013.

Quantization of current flux associated with the superconducting state (Ref: Ashcroft and Mermin, ***Solid State Physics***)

From the London equations for the interior of the superconductor:

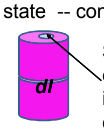
$$\left(m\mathbf{v}+\frac{e}{c}\mathbf{A}\right)=0$$

Now suppose that the current carrier is a pair of electrons characterized by a wavefunction of the form $\psi=|\psi|e^{i\phi}$

The quantum mechanical current associated with the electron pair is

$$\mathbf{j}=-\frac{e\hbar}{2mi}\left(\psi^*\nabla\psi-\psi\nabla\psi^*\right)-\frac{2e^2}{mc}\mathbf{A}|\psi|^2$$
$$=-\left(\frac{e\hbar}{m}\nabla\phi+\frac{2e^2}{mc}\mathbf{A}\right)|\psi|^2$$

Quantization of current flux associated with the superconducting state -- continued



Suppose a superconducting material has a cylindrical void. Evaluate the integral of the current in a closed path within the superconductor containing the void.

$$\oint \mathbf{j}\cdot d\mathbf{l}=0=-\oint\left(\frac{e\hbar}{m}\nabla\phi+\frac{2e^2}{mc}\mathbf{A}\right)|\psi|^2\cdot d\mathbf{l}$$

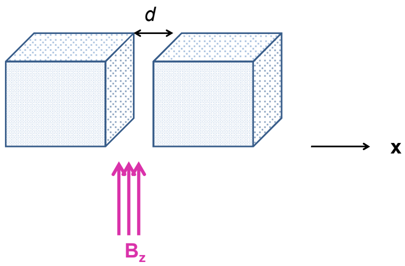
$$\oint \mathbf{A}\cdot d\mathbf{l}=\int(\nabla\times\mathbf{A})\cdot d\mathbf{a}=\int\mathbf{B}\cdot d\mathbf{a}=\Phi \quad \text{magnetic flux}$$

$$\oint \nabla\phi\cdot d\mathbf{l}=2\pi n \quad \text{for some integer } n$$

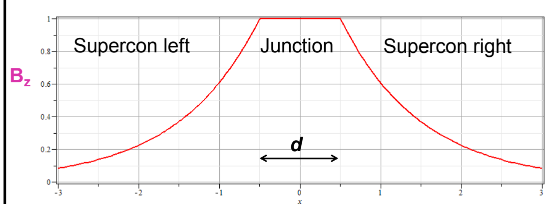
$$\Rightarrow \text{Quantization of flux in the void: } |\Phi|=n\frac{hc}{2e}\equiv n\Phi_0$$

Such “vortex” fields can exist within type II superconductors.

Josephson junction -- tunneling current between two superconductors (Ref. Teplitz, ***Electromagnetism*** (1982))

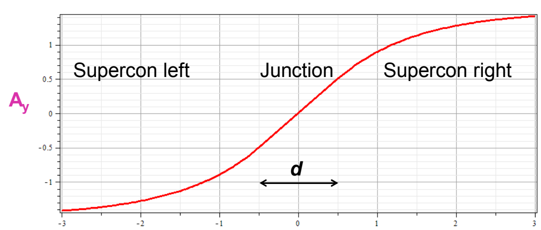


Josephson junction -- continued



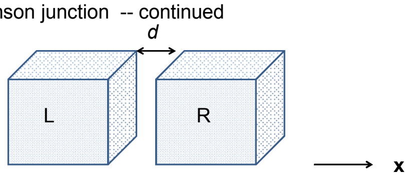
$$B_z(x) = \begin{cases} B_0 e^{(x+d/2)/\lambda_L} & x < -d/2 \\ B_0 & -d/2 < x < d/2 \\ B_0 e^{(-x+d/2)/\lambda_L} & x > d/2 \end{cases}$$

Josephson junction -- continued



$$A_y(x) = \begin{cases} B_0\left(\lambda_L e^{(x+d/2)/\lambda_L} - \left(\lambda_L + d/2\right)\right) & x < -d/2 \\ B_0 x & -d/2 < x < d/2 \\ B_0\left(-\lambda_L e^{(-x+d/2)/\lambda_L} + \left(\lambda_L + d/2\right)\right) & x > d/2 \end{cases}$$

Josephson junction -- continued



Quantum mechanical model of tunnelling current

Let $\Psi_L = \Psi_L{}^0 e^{i\phi_L}$ denote a wavefunction for a Cooper pair on left

Let $\Psi_R = \Psi_R{}^0 e^{i\phi_R}$ denote a wavefunction for a Cooper pair on right

$$-i\hbar\frac{\partial \Psi_L}{\partial t} = E_L\Psi_L + \varepsilon\Psi_R$$

$$-i\hbar\frac{\partial \Psi_R}{\partial t} = E_R\Psi_R + \varepsilon\Psi_L$$

Coupling parameter (ε)

Josephson junction -- continued

Solving for wavefunctions

$$\frac{1}{2}\frac{\partial\left|\Psi_L{}^0\right|^2}{\partial t} + i\left|\Psi_L{}^0\right|^2\frac{\partial \phi_L}{\partial t} = -\frac{i}{\hbar}\left(E_L\left|\Psi_L{}^0\right|^2 + \varepsilon\Psi_L{}^0\Psi_R{}^0 e^{i(\phi_R-\phi_L)}\right)$$

$$\frac{1}{2}\frac{\partial\left|\Psi_R{}^0\right|^2}{\partial t} + i\left|\Psi_R{}^0\right|^2\frac{\partial \phi_R}{\partial t} = -\frac{i}{\hbar}\left(E_R\left|\Psi_R{}^0\right|^2 + \varepsilon\Psi_L{}^0\Psi_R{}^0 e^{-i(\phi_R-\phi_L)}\right)$$

$$\left|\Psi_L{}^0\right|^2 \equiv n_L \qquad \left|\Psi_R{}^0\right|^2 \equiv n_R \qquad \phi_{LR} \equiv \phi_L - \phi_R$$

$$\frac{\partial n_L}{\partial t} = -\frac{\partial n_R}{\partial t} = -\frac{2\varepsilon}{\hbar}\sqrt{(n_L n_R)}\sin\phi_{LR}$$

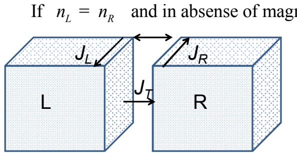
$$\frac{\partial \phi_L}{\partial t} = -\frac{E_L}{\hbar} - \varepsilon\sqrt{\frac{n_R}{n_L}}\cos\phi_{LR}$$

$$\frac{\partial \phi_R}{\partial t} = -\frac{E_R}{\hbar} - \varepsilon\sqrt{\frac{n_L}{n_R}}\cos\phi_{LR}$$

Josephson junction -- continued

Tunneling current: $J_T = 2e\frac{\partial n_L}{\partial t} = -\frac{4e\varepsilon}{\hbar}\sqrt{(n_L n_R)}\sin\phi_{LR}$

If $n_L = n_R$ and in absense of magnetic field, $\phi_{LR}(t) = \phi_{LR}(0) + \frac{E_R - E_L}{\hbar}t$



$$\Rightarrow J_L = \frac{2e}{2m}\left|\Psi_L{}^0\right|^2\left(\hbar\nabla\phi_L - \frac{2e}{c}\mathbf{A}\right)$$

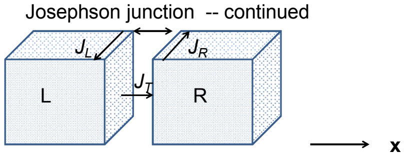
$$\Rightarrow J_R = \frac{2e}{2m}\left|\Psi_R{}^0\right|^2\left(\hbar\nabla\phi_R - \frac{2e}{c}\mathbf{A}\right)$$

x

Relationship between superconductor currents J_L and J_R and tunneling current. Within the superconductor, denote the generalize current operator acting on pair wavefunction $\Psi = \Psi^0 e^{i\phi}$

$$\hat{\mathbf{v}} \equiv \frac{1}{2m}\left(-i\hbar\nabla - \frac{2e}{c}\mathbf{A}\right) \quad \text{with current } J = \frac{2e}{2}\left(\Psi^*\left(\hat{\mathbf{v}}\Psi\right) + \Psi\left(\hat{\mathbf{v}}\Psi\right)^*\right)$$

Josephson junction -- continued



$$\Rightarrow J_L = \frac{2e}{2m}\left|\Psi_L^0\right|^2\left(\hbar\nabla\phi_L - \frac{2e}{c}\mathbf{A}\right) \equiv 2en_L\mathbf{v}_L$$

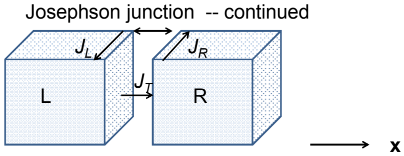
$$\Rightarrow J_R = \frac{2e}{2m}\left|\Psi_R^0\right|^2\left(\hbar\nabla\phi_R - \frac{2e}{c}\mathbf{A}\right) \equiv 2en_R\mathbf{v}_R$$

$$\nabla\phi_L = \frac{2m\mathbf{v}_L}{\hbar} + \frac{2e}{\hbar c}\mathbf{A} \qquad \nabla\phi_R = \frac{2m\mathbf{v}_R}{\hbar} + \frac{2e}{\hbar c}\mathbf{A}$$

Tunneling current: $J_T = 2e\frac{\partial n_L}{\partial t} = -\frac{4e\varepsilon}{\hbar}\sqrt{(n_L n_R)}\sin\phi_{LR}$

Need to evaluate ϕ_{LR} in presence of magnetic field

Josephson junction -- continued



$$\nabla\phi_L = \frac{2m\mathbf{v}_L}{\hbar} + \frac{2e}{\hbar c}\mathbf{A} \qquad \nabla\phi_R = \frac{2m\mathbf{v}_R}{\hbar} + \frac{2e}{\hbar c}\mathbf{A}$$

Recall that for $x \to -\infty$ $\mathbf{v}_L \to 0$ and $\mathbf{A} \to -\left(\lambda_L + d/2\right)B_0\hat{\mathbf{y}}$

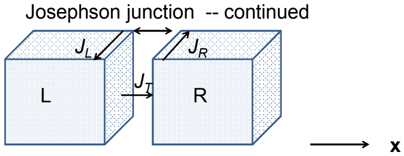
for $x \to \infty$ $\mathbf{v}_R \to 0$ and $\mathbf{A} \to \left(\lambda_L + d/2\right)B_0\hat{\mathbf{y}}$

Integrating the difference of the phase angles along y:

$$\phi_{LR} = \phi_{LR}^0 + B_0(2\lambda_L + d)y$$

Tunneling current: $J_T = -\frac{4e\varepsilon}{\hbar}\sqrt{(n_L n_R)}\sin\phi_{LR}$

Josephson junction -- continued



Integrating the difference of the phase angles along y:

$$\phi_{LR} = \phi_{LR}^0 + \frac{2e}{\hbar c}B_0(2\lambda_L + d)y$$

Tunneling current density: $J_T = \frac{4e\varepsilon}{\hbar}n_L\sin\phi_{LR}$

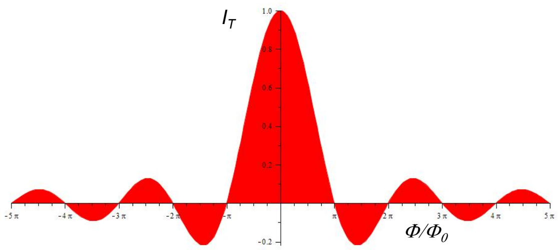
Integrating current density throughout width w of superconductors

$$I_T = h\int_{-w/2}^{w/2} J_T dy = hwJ_{T0}\sin(\phi_{LR}^0)\frac{\sin(\pi\Phi/\Phi^0)}{\pi\Phi/\Phi^0}$$

where $\Phi = B_0 w(2\lambda_L + d)$ and $\Phi^0 = \frac{2\pi\hbar c}{2e}$

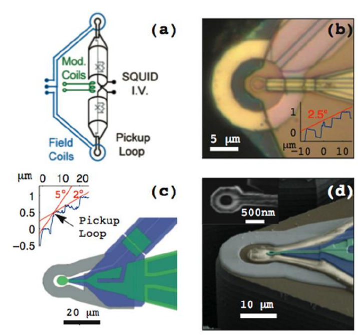
Josephson junction -- continued

SQUID =*superconducting quantum interference device*



Note: This very sensitive "SQUID" technology has been used in scanning probe techniques. See for example, J. R. Kirtley, Rep. Prog. Physics 73, 126501 (2010).

Scanning SQIUD microscopy
Ref. J. R. Kirtley, Rep. Prog. Phys. **73** 126501 (2010)



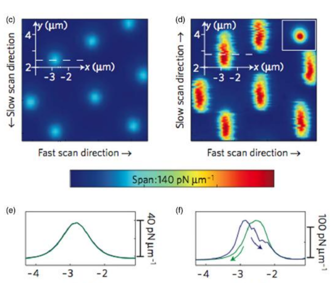


Figure 31. MFM imaging and manipulation of individual vortices in YBCO at $T = 22.3$ K. (*a*), (*b*) Schematic drawings of an MFM tip (triangle) that attracts a vortex (thick lines) in a sample with randomly distributed pinning sites (dots): at large heights (*a*) the force between tip and vortex is too weak to move the vortex; at small heights (*b*) the vortex moves right and then left as the tip rasters over it. (*c*) MFM image taken at $z = 420$ nm (maximum applied lateral force $F^{lat}_{max} \approx 6$ nN). (*d*) $z = 170$ nm ($F^{lat}_{max} \approx 12$ nm). Inset: scan taken at a comparable height at $T = 5.2$ K. (*e*) Line cut through the data in (*c*) along the dashed line, showing the signal from a stationary vortex (solid curve with peak to the left, blue online). Overlapping it is a line cut from the reverse scan (solid curve with peak to the right, green online). (*f*) Line cut through the data in (*d*) along the dashed line, showing a typical signal from a dragged vortex. Figure reprinted by permission from [312]. Copyright 2008 Macmillan Publishers Ltd.