PHY 712 Electrodynamics
9-9:50 AM MWF Olin 103

Plan for Lecture 11:

Start reading Chapter 5

- **A. Magnetostatics**
- **B. Vector potential**
- **C. Example: current loop**

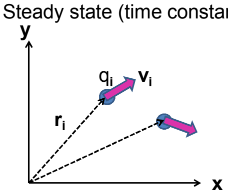
Course schedule for Spring 2017

(Preliminary schedule -- subject to frequent adjustment.)

	Lecture date	JDJ Reading	Topic	HW	Due date
1	Wed: 01/11/2017	Chap. 1	Introduction, units and Poisson equation	#1	01/18/2017
2	Fri: 01/13/2017	Chap. 1	Electrostatic energy calculations	#2	01/18/2017
	Mon: 01/16/2017		*MLK Holiday - no class*		
3	Wed: 01/18/2017	Chap. 1	Poisson equation and Green's theorem	#3	01/20/2017
4	Fri: 01/20/2017	Chap. 1 and 2	Poisson equation in 2 and 3 dimensions	#4	01/23/2017
5	Mon: 01/23/2017	Chap. 1 and 2	Brief introduction to grid solution methods	#5	01/25/2017
6	Wed: 01/25/2017	Chap. 2	Method of images	#6	01/27/2017
7	Fri: 01/27/2017	Chap. 3	Cylindrical and spherical geometries	#7	01/30/2017
8	Mon: 01/30/2017	Chap. 3 & 4	Multipole analysis	#8	02/01/2017
9	Wed: 02/01/2017	Chap. 4	Dipoles and dielectrics	#9	02/03/2017
10	Fri: 02/03/2017	Chap. 4	Dipoles and dielectrics	#10	02/06/2017
11	Mon: 02/06/2017	Chap. 5	Magnetostatics	#11	02/08/2017
12	Wed: 02/08/2017				
13	Fri: 02/10/2017				
14	Mon: 02/13/2017				

Magnetostatics

Magnetic flux density or magnetic induction field **B**
Steady state (time constant) current density **J**



$$\mathbf{J}(\mathbf{r}) = \sum_i q_i \mathbf{v}_i \delta^3(\mathbf{r} - \mathbf{r}_i)$$

Note that "statics" implies that $\nabla \cdot \mathbf{J} = 0$.

This follows from the continuity equation :

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0$$

Comparison of electrostatics and magnetostatics

Electrostatic field due to charge density $\rho(\mathbf{r})$:

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int d^3r'\rho(\mathbf{r}')\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}$$

Magnetostatic field due to current density $\mathbf{J}(\mathbf{r})$:

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int d^3r'\,\mathbf{J}(\mathbf{r}')\times\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}$$

Alternative forms magnetostatic equations

Magnetostatic field due to current density $\mathbf{J}(\mathbf{r})$:

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int d^3r'\,\mathbf{J}(\mathbf{r}')\times\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} = \frac{\mu_0}{4\pi}\int d^3r'\left(\nabla\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)\times\mathbf{J}(\mathbf{r}')$$

Note that: $\nabla\frac{1}{|\mathbf{r}-\mathbf{r}'|} = -\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}$

Also note that: $\nabla\times\left(s(\mathbf{r})\mathbf{V}(\mathbf{r})\right) = \nabla s(\mathbf{r})\times\mathbf{V}(\mathbf{r}) + s(\mathbf{r})\nabla\times\mathbf{V}(\mathbf{r})$

$$\Rightarrow \mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\nabla\times\int d^3r'\,\frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$$

In this case $\mathbf{V}(\mathbf{r}) \equiv \mathbf{J}(\mathbf{r}')$ so that $\nabla\times\mathbf{V}(\mathbf{r}) = 0$

Alternative forms magnetostatic equations -- continued

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\nabla\times\int d^3r'\,\frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$$

$\Rightarrow \nabla\cdot\mathbf{B}(\mathbf{r}) = 0$ No magnetic monopoles

$\Rightarrow \nabla\times\mathbf{B}(\mathbf{r}) = \mu_0\mathbf{J}(\mathbf{r})$ Ampere's law

"Proof" of Ampere's law for magnetostatic system:

$$\nabla\times\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\nabla\times\nabla\times\int d^3r'\,\frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$$

Note that: $\nabla\times\nabla\times\mathbf{V} = \nabla(\nabla\cdot\mathbf{V}) - \nabla^2\mathbf{V}$

Recall that: $\nabla^2\frac{1}{|\mathbf{r}-\mathbf{r}'|} = -4\pi\delta^3(\mathbf{r}-\mathbf{r}')$ and $\nabla\cdot\mathbf{J}(\mathbf{r}) = 0$

Differential forms of magnetostatic equations:

$$\Rightarrow \nabla \cdot \mathbf{B}(\mathbf{r}) = 0 \qquad \text{No magnetic monopoles}$$
$$\Rightarrow \nabla \times \mathbf{B}(\mathbf{r}) = \mu_0 \mathbf{J}(\mathbf{r}) \qquad \text{Ampere's law}$$

Magnetostatic vector potential

$$\mathbf{B}(\mathbf{r}) = \nabla \times \mathbf{A}(\mathbf{r})$$
$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \nabla \times \int d^3r' \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$$
$$\Rightarrow \mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int d^3r' \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} + \nabla s(\mathbf{r})$$

Non uniqueness of the magnetostatic vector potential

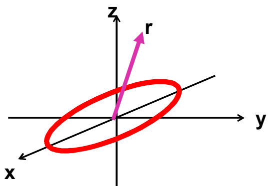
$$\text{Note that: } \quad \mathbf{B}(\mathbf{r}) = \nabla \times \mathbf{A}(\mathbf{r}) = \nabla \times \mathbf{A}'(\mathbf{r})$$
$$\text{if } \quad \mathbf{A}'(\mathbf{r}) = \mathbf{A}(\mathbf{r}) + \nabla s(\mathbf{r})$$

$$\text{Example: } \quad \text{for } \mathbf{B}(\mathbf{r}) = B_0 \hat{\mathbf{z}}$$
$$\mathbf{A}(\mathbf{r}) = \tfrac{1}{2} B_0 (x\hat{\mathbf{y}} - y\hat{\mathbf{x}})$$
$$\text{or} \quad \mathbf{A}(\mathbf{r}) = B_0 x \hat{\mathbf{y}}$$
$$\text{or} \quad \mathbf{A}(\mathbf{r}) = -B_0 y \hat{\mathbf{x}}$$

Differential form of Ampere's law in terms of vector potential:

$$\nabla \times \mathbf{B}(\mathbf{r}) = \nabla \times \nabla \times \mathbf{A}(\mathbf{r}) = \mu_0 \mathbf{J}(\mathbf{r})$$
$$\Rightarrow \nabla(\nabla \cdot \mathbf{A}(\mathbf{r})) - \nabla^2 \mathbf{A}(\mathbf{r}) = \mu_0 \mathbf{J}(\mathbf{r})$$
$$\text{If} \quad \nabla \cdot \mathbf{A}(\mathbf{r}) = 0 \quad \text{(Coulomb gauge)} \quad \Rightarrow \nabla^2 \mathbf{A}(\mathbf{r}) = -\mu_0 \mathbf{J}(\mathbf{r})$$
$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int d^3r' \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$$

Magnetostatics example: current loop



$$\mathbf{J}(\mathbf{r}') = \frac{I}{a}\sin\theta'\,\delta(\cos\theta')\,\delta(r'-a)\left(-\sin\phi'\,\hat{\mathbf{x}} + \cos\phi'\,\hat{\mathbf{y}}\right)$$

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int d^3r'\,\frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$$

Magnetostatics example: current loop -- continued

$$\mathbf{J}(\mathbf{r}') = \frac{I}{a}\sin\theta'\,\delta(\cos\theta')\,\delta(r'-a)\left(-\sin\varphi'\,\hat{\mathbf{x}} + \cos\varphi'\,\hat{\mathbf{y}}\right)$$

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int d^3r'\,\frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$$

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I}{4\pi a}\int r'^2\,dr'\,d\cos\theta'\,d\varphi'\,\frac{\sin\theta'\,\delta(\cos\theta')\,\delta(r'-a)\left(-\sin\varphi'\,\hat{\mathbf{x}}+\cos\varphi'\,\hat{\mathbf{y}}\right)}{\left(r^2+r'^2-2rr'\left(\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos(\varphi-\varphi')\right)\right)^{1/2}}$$

Completing integration over r' and θ':

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I a^2}{4\pi a}\int_0^{2\pi} d\varphi'\,\frac{\left(-\sin\varphi'\,\hat{\mathbf{x}}+\cos\varphi'\,\hat{\mathbf{y}}\right)}{\left(r^2+a^2-2ra\left(\sin\theta\cos(\varphi-\varphi')\right)\right)^{1/2}}$$

Let $\varphi-\varphi' \equiv \phi$

$\sin\varphi' = \sin(\varphi-\phi) = \sin\varphi\cos\phi - \cos\varphi\sin\phi$

$\cos\varphi' = \cos(\varphi-\phi) = \cos\varphi\cos\phi + \sin\varphi\sin\phi$

Remaining non-trivial terms

$$\mathbf{A}(\mathbf{r}) = -\frac{\mu_0 I a}{4\pi}\left(\sin\varphi\hat{\mathbf{x}} - \cos\varphi\hat{\mathbf{y}}\right)\int_0^{2\pi} d\phi\,\frac{\cos\phi}{\left(r^2+a^2-2ra\left(\sin\theta\cos\phi\right)\right)^{1/2}}$$

Magnetostatics example: current loop -- continued

$$\mathbf{A}(\mathbf{r}) = -\frac{\mu_0 I a}{4\pi}\left(\sin\varphi\hat{\mathbf{x}} - \cos\varphi\hat{\mathbf{y}}\right)\int_0^{2\pi} d\phi\,\frac{\cos\phi}{\left(r^2+a^2-2ra\left(\sin\theta\cos\phi\right)\right)^{1/2}}$$

Elliptic integrals:

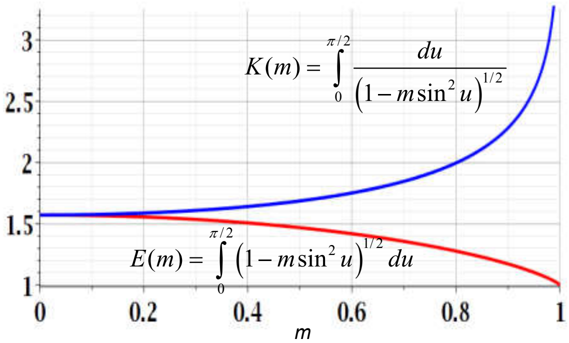
$$K(m) = \int_0^{\pi/2}\frac{du}{\left(1-m\sin^2 u\right)^{1/2}}$$

$$E(m) = \int_0^{\pi/2}\left(1-m\sin^2 u\right)^{1/2} du$$

$$\mathbf{A}(\mathbf{r}) = -\frac{\mu_0}{4\pi}4Ia\,\frac{\left(\sin\varphi\hat{\mathbf{x}}-\cos\varphi\hat{\mathbf{y}}\right)}{\left(r^2+a^2+2ra\sin\theta\right)^{1/2}}\left[\frac{\left(2-k^2\right)K(k)-2E(k)}{k^2}\right]$$

where: $k^2 \equiv \dfrac{4ar\sin\theta}{r^2+a^2+2ra\sin\theta}$

$$\mathbf{B}(\mathbf{r}) = \nabla\times\mathbf{A}(\mathbf{r})$$



Magnetostatics example: current loop -- continued

$$\mathbf{A}(\mathbf{r})=-\frac{\mu_0}{4\pi}4Ia\frac{(\sin\varphi\hat{\mathbf{x}}-\cos\varphi\hat{\mathbf{y}})}{\left(r^2+a^2+2ra\sin\theta\right)^{1/2}}\left[\frac{\left(2-k^2\right)K(k)-2E(k)}{k^2}\right]$$

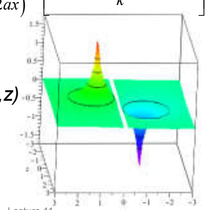
where: $k^2\equiv\frac{4ar\sin\theta}{r^2+a^2+2ra\sin\theta}$

For $\varphi=0$: $x=r\sin\theta,\ y=0$

$$\mathbf{A}(\mathbf{r})=A_y(x,z)\hat{\mathbf{y}}=\frac{\mu_0}{4\pi}4Ia\hat{\mathbf{y}}\frac{1}{\left(x^2+z^2+a^2+2ax\right)^{1/2}}\left[\frac{\left(2-k^2\right)K(k)-2E(k)}{k^2}\right]$$

where: $k^2\equiv\frac{4ax}{x^2+z^2+a^2+2ax}$

$A_y(x,z)$



Magnetostatics example: current loop -- continued

$$\mathbf{A}(\mathbf{r})=-\frac{\mu_0}{4\pi}4Ia\frac{(\sin\varphi\hat{\mathbf{x}}-\cos\varphi\hat{\mathbf{y}})}{\left(r^2+a^2+2ra\sin\theta\right)^{1/2}}\left[\frac{\left(2-k^2\right)K(k)-2E(k)}{k^2}\right]$$

$$=\hat{\boldsymbol{\varphi}}\frac{\mu_0}{4\pi}\frac{4Ia}{\left(r^2+a^2+2ra\sin\theta\right)^{1/2}}\left[\frac{\left(2-k^2\right)K(k)-2E(k)}{k^2}\right]$$

where: $k^2\equiv\frac{4ar\sin\theta}{r^2+a^2+2ra\sin\theta}$

$$\mathbf{B}=\nabla\times\mathbf{A}=\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(r\sin\theta A_\varphi(r,\theta)\right)\hat{\mathbf{r}}-\frac{1}{r}\frac{\partial\left(rA_\varphi(r,\theta)\right)}{\partial r}\hat{\boldsymbol{\theta}}$$

Evaluation for special cases

For $k^2\rightarrow 0$:

$$\frac{\left(2-k^2\right)K(k)-2E(k)}{k^2}\approx\frac{\pi}{16}k^2$$

Evaluation for special case $k^2 \to 0$

$$\mathbf{A}(\mathbf{r}) = \hat{\boldsymbol{\varphi}} \frac{\mu_0 I a^2}{4} \frac{r\sin\theta}{\left(r^2+a^2+2ra\sin\theta\right)^{3/2}}$$

$$\mathbf{B} = \nabla\times\mathbf{A} = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(r\sin\theta A_\phi(r,\theta)\right)\hat{\mathbf{r}} - \frac{1}{r}\frac{\partial\left(rA_\phi(r,\theta)\right)}{\partial r}\hat{\boldsymbol{\theta}}$$

$$= \frac{\mu_0 I a^2}{4} \frac{\left(\hat{\mathbf{r}}\cos\theta\left(2a^2+2r^2+ar\,\sin\theta\right) - \hat{\boldsymbol{\theta}}\sin\theta\left(2a^2-r^2+ar\,\sin\theta\right)\right)}{\left(r^2+a^2+2ra\sin\theta\right)^{5/2}}$$

where $k^2 \equiv \frac{4ar\sin\theta}{r^2+a^2+2ra\sin\theta} \to 0$

For $\theta = 0$: $r = z$

$$\mathbf{B} = \nabla\times\mathbf{A} = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(r\sin\theta A_\phi(r,\theta)\right)\hat{\mathbf{r}} - \frac{1}{r}\frac{\partial\left(rA_\phi(r,\theta)\right)}{\partial r}\hat{\boldsymbol{\theta}}$$

$$= \hat{\mathbf{z}}\frac{\mu_0 I a^2}{2}\frac{1}{\left(z^2+a^2\right)^{3/2}}$$

Evaluation for special case $k^2 \to 0$

$$\mathbf{A}(\mathbf{r}) = \hat{\boldsymbol{\varphi}} \frac{\mu_0 I a^2}{4} \frac{r\sin\theta}{\left(r^2+a^2+2ra\sin\theta\right)^{3/2}}$$

$$\mathbf{B} = \nabla\times\mathbf{A} = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(r\sin\theta A_\phi(r,\theta)\right)\hat{\mathbf{r}} - \frac{1}{r}\frac{\partial\left(rA_\phi(r,\theta)\right)}{\partial r}\hat{\boldsymbol{\theta}}$$

$$= \frac{\mu_0 I a^2}{4} \frac{\left(\hat{\mathbf{r}}\cos\theta\left(2a^2+2r^2+ar\,\sin\theta\right) - \hat{\boldsymbol{\theta}}\sin\theta\left(2a^2-r^2+ar\,\sin\theta\right)\right)}{\left(r^2+a^2+2ra\sin\theta\right)^{5/2}}$$

where $k^2 \equiv \frac{4ar\sin\theta}{r^2+a^2+2ra\sin\theta} \to 0$

For $r \to \infty$

$$\mathbf{B} = \nabla\times\mathbf{A} = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(r\sin\theta A_\phi(r,\theta)\right)\hat{\mathbf{r}} - \frac{1}{r}\frac{\partial\left(rA_\phi(r,\theta)\right)}{\partial r}\hat{\boldsymbol{\theta}}$$

$$= \frac{\mu_0 I a^2}{4r^3}\left(\hat{\mathbf{r}}\left(2\cos\theta\right) + \hat{\boldsymbol{\theta}}\sin\theta\right)$$

Magnetostatics example: current loop -- continued

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}4Ia\frac{\left(\sin\phi\hat{\mathbf{x}} - \cos\phi\hat{\mathbf{y}}\right)}{\left(r^2+a^2+2ra\sin\theta\right)^{1/2}}\left[\frac{\left(2-k^2\right)K(k) - 2E(k)}{k^2}\right]$$

where: $k \equiv \frac{4ar\sin\theta}{r^2+a^2+2ra\sin\theta}$

$\mathbf{B}(\mathbf{r}) = \nabla\times\mathbf{A}(\mathbf{r})$

Note that for spherical polar coordinates: $\hat{\boldsymbol{\varphi}} = \sin\phi\hat{\mathbf{x}} - \cos\phi\hat{\mathbf{y}}$

$\mathbf{A}(\mathbf{r}) = A_\phi(\mathbf{r})\hat{\boldsymbol{\varphi}}$

where $A_\phi(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{4Ia}{\left(r^2+a^2+2ra\sin\theta\right)^{1/2}}\left[\frac{\left(2-k^2\right)K(k) - 2E(k)}{k^2}\right]$

$$\mathbf{B}(\mathbf{r}) = \frac{1}{r\sin\theta}\frac{\partial\left(\sin\theta A_\phi(\mathbf{r})\right)}{\partial\theta}\hat{\mathbf{r}} - \frac{1}{r}\frac{\partial\left(rA_\phi(\mathbf{r})\right)}{\partial r}\hat{\boldsymbol{\theta}}$$

For $r \to \infty$:

$$\mathbf{B}(\mathbf{r}) \approx \frac{\mu_0}{4\pi}\frac{I\pi a^2}{r^3}\left(2\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\boldsymbol{\theta}}\right)$$

Other examples of current density sources:

Quantum mechanical expression for current density

for a particle of mass M and charge e and of probability amplitude $\Psi(\mathbf{r})$:

$$\mathbf{J}(\mathbf{r}) = -\frac{e\hbar}{2Mi}\left(\Psi^*(\mathbf{r})\nabla\Psi(\mathbf{r}) - \Psi(\mathbf{r})\nabla\Psi^*(\mathbf{r})\right)$$

For an electron in a spherical potential (such as in an atom):

$$\Psi(\mathbf{r}) \equiv \Psi_{nlm_l}(\mathbf{r}) = R_{nl}(r)Y_{lm_l}(\hat{\mathbf{r}})$$

$$\mathbf{J}(\mathbf{r}) = \frac{e\hbar}{2Mi}|R_{nl}(r)|^2 \frac{1}{r\sin\theta}\left(Y_{lm_l}^*(\hat{\mathbf{r}})\frac{\partial Y_{lm_l}(\hat{\mathbf{r}})}{\partial\phi} - Y_{lm_l}(\hat{\mathbf{r}})\frac{\partial Y_{lm_l}^*(\hat{\mathbf{r}})}{\partial\phi}\right)\hat{\boldsymbol{\varphi}}$$

$$= \frac{e\hbar}{M}\frac{m_l}{r\sin\theta}\left|\Psi_{nlm_l}(\mathbf{r})^2\right|\hat{\boldsymbol{\varphi}}$$

Note that: $\hat{\boldsymbol{\varphi}} = -\sin\phi\hat{\mathbf{x}} + \cos\phi\hat{\mathbf{y}} = \frac{\hat{\mathbf{z}}\times\mathbf{r}}{r\sin\theta}$

$$\mathbf{J}(\mathbf{r}) = \frac{e\hbar}{M}\frac{m_l}{r^2\sin^2\theta}\left|\Psi_{nlm_l}(\mathbf{r})^2\right|(\hat{\mathbf{z}}\times\mathbf{r})$$

Magnetic vector potential for this case:

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int d^3r' \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$$

$$\mathbf{J}(\mathbf{r}') = \frac{e\hbar}{M}\frac{m_l}{r'^2\sin^2\theta'}\left|\Psi_{nlm_l}(\mathbf{r}')^2\right|(\hat{\mathbf{z}}\times\mathbf{r}')$$

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{e\hbar m_l}{M}\int d^3r' \frac{(\hat{\mathbf{z}}\times\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\frac{\left|\Psi_{nlm_l}(\mathbf{r}')^2\right|}{r'^2\sin^2\theta'}$$

For example: electron in the $nlm_l = 211$ state of H:

$$\left|\Psi_{211}(\mathbf{r}')^2\right| = \frac{1}{64\pi a^3}\left(\frac{r'}{a}\right)^2 e^{-r'/a}\sin^2\theta'$$

$$\mathbf{A}(\mathbf{r}) = -\frac{\mu_0}{8\pi}\frac{e\hbar}{M}\frac{(\hat{\mathbf{z}}\times\mathbf{r})}{r^3}\left[1 - e^{-r/a}\left(1 + \frac{r}{a} + \frac{r^2}{2a^2} + \frac{r^3}{8a^3}\right)\right]$$