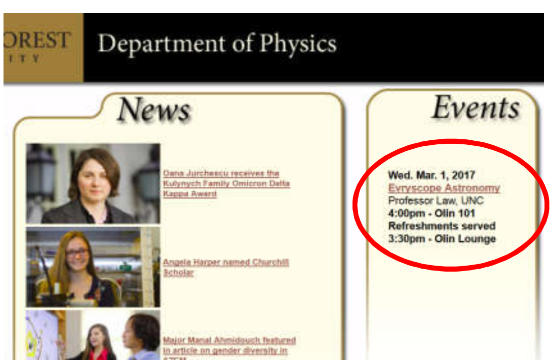
PHY 712 Electrodynamics
9-9:50 AM Olin 103

Plan for Lecture 21:

Chap. 8 in Jackson – Wave Guides

1. Electromagnetic waves near an ideal conductor
2. Electromagnetic waves within an ideal rectangular wave guide

12	Wed: 02/08/2017	Chap. 5	Magnetostatics and the Hyperfine Interaction	#12	02/10/2017
13	Fri: 02/10/2017	Chap. 5	Magnetic dipoles and dipolar fields	#13	02/13/2017
14	Mon: 02/13/2017	Chap. 6	Maxwell's Equations	#14	02/15/2017
15	Wed: 02/15/2017	Chap. 6	Electromagnetic energy and forces	#15	02/17/2017
16	Fri: 02/17/2017	Chap. 7	Electromagnetic plane waves	#16	02/20/2017
17	Mon: 02/20/2017	Chap. 7	Dielectric media		
18	Wed: 02/22/2017	Chap. 7	Complex dielectrics		
19	Fri: 02/24/2017	Chap. 1-7	Review -- Take home exam distributed		
20	Mon: 02/27/2017	Chap. 8	Wave guides		Exam
21	Wed: 03/01/2017	Chap. 8	Wave guides		Exam
22	Fri: 03/03/2017	Chap. 8	Wave guides		Exam Due
	Mon: 03/06/2017		*Spring break - no class*		
	Wed: 03/08/2017		*Spring break - no class*		
	Fri: 03/10/2017		*Spring break - no class*		
	Mon: 03/13/2017		*APS Meeting - no class*		
	Wed: 03/15/2017		*APS Meeting - no class*		
	Fri: 03/17/2017		*APS Meeting - no class*		
23	Mon: 03/20/2017				
24	Wed: 03/22/2017				



Fields near the surface on an ideal conductor

Suppose for an isotropic medium : $\mathbf{D} = \varepsilon_b \mathbf{E}$ $\quad \mathbf{J} = \sigma \mathbf{E}$

Maxwell's equations in terms of $\mathbf{H}$ and $\mathbf{E}$:

$$\nabla \cdot \mathbf{E} = 0 \qquad \nabla \cdot \mathbf{H} = 0$$

$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t} \qquad \nabla \times \mathbf{H} = \sigma \mathbf{E} + \varepsilon_b \frac{\partial \mathbf{E}}{\partial t}$$

$$\left(\nabla^2 - \mu\sigma \frac{\partial}{\partial t} - \mu\varepsilon_b \frac{\partial^2}{\partial t^2} \right) \mathbf{F} = 0 \qquad \mathbf{F} = \mathbf{E}, \mathbf{H}$$

Plane wave form for $\mathbf{E}$:

$$\mathbf{E}(\mathbf{r},t) = \Re\left(\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t}\right) \qquad \text{where } \mathbf{k} = (n_R + i n_I)\frac{\omega}{c}\hat{\mathbf{k}}$$

$$\Rightarrow \mathbf{E}(\mathbf{r},t) = e^{-\hat{\mathbf{k}}\cdot\mathbf{r}/\delta}\,\Re\left(\mathbf{E}_0 e^{i n_R (\omega/c)\hat{\mathbf{k}}\cdot\mathbf{r} - i\omega t}\right)$$

Plane wave form for $\mathbf{E}$:

$$\mathbf{E}(\mathbf{r},t) = \Re\left(\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t}\right) \qquad \text{where } \mathbf{k} = (n_R + i n_I)\frac{\omega}{c}\hat{\mathbf{k}}$$

$$\left(\nabla^2 - \mu\sigma \frac{\partial}{\partial t} - \mu\varepsilon_b \frac{\partial^2}{\partial t^2} \right) \mathbf{E} = 0$$

$$-(n_R + i n_I)^2 + i\frac{\mu\sigma c^2}{\omega} + \mu\varepsilon_b c^2 = 0$$

Fields near the surface on an ideal conductor -- continued

For our system :

$$\frac{\omega}{c} n_R = \omega \sqrt{\frac{\mu\varepsilon_b}{2}} \left(\sqrt{1 + \left(\frac{\sigma}{\omega\varepsilon_b}\right)^2} + 1 \right)^{1/2}$$

$$\frac{\omega}{c} n_I = \omega \sqrt{\frac{\mu\varepsilon_b}{2}} \left(\sqrt{1 + \left(\frac{\sigma}{\omega\varepsilon_b}\right)^2} - 1 \right)^{1/2}$$

$$\text{For } \frac{\sigma}{\omega} >> 1 \qquad \frac{\omega}{c} n_R \approx \frac{\omega}{c} n_I \approx \sqrt{\frac{\mu\sigma\omega}{2}} \equiv \frac{1}{\delta}$$

$$\Rightarrow \mathbf{E}(\mathbf{r},t) = e^{-\hat{\mathbf{k}}\cdot\mathbf{r}/\delta}\,\Re\left(\mathbf{E}_0 e^{i\hat{\mathbf{k}}\cdot\mathbf{r}/\delta - i\omega t}\right)$$

$$\Rightarrow \mathbf{H}(\mathbf{r},t) = \frac{n}{c\mu}\hat{\mathbf{k}} \times \mathbf{E}(\mathbf{r},t) = \frac{1+i}{\delta\mu\omega}\hat{\mathbf{k}} \times \mathbf{E}(\mathbf{r},t)$$

Some representative values of skin depth
Ref: Lorrain[2] and Corson

	σ (10^7 S/m)	μ/μ_0	δ (0.001m) at 60 Hz	δ (0.001m) at 1 MHz
Al	3.54	1	10.9	84.6
Cu	5.80	1	8.5	66.1
Fe	1.00	100	1.0	10.0
Mumetal	0.16	2000	0.4	3.0
Zn	1.86	1	15.1	117

Relative energies associated with field

Electric energy density: $\varepsilon_b|\mathbf{E}|^2$

Magnetic energy density: $\mu|\mathbf{H}|^2$

Ratio inside conducting media: $$\frac{\varepsilon_b|\mathbf{E}|^2}{\mu|\mathbf{H}|^2} = \frac{\varepsilon_b}{\mu\left|\frac{1+i}{\delta\mu\omega}\right|^2} = \frac{\varepsilon_b\mu\omega^2\delta^2}{2}$$

$$= 2\pi^2 \frac{\varepsilon_b}{\varepsilon_0}\frac{\mu}{\mu_0}\frac{\delta^2}{\lambda^2}$$

For $\frac{\varepsilon_b|\mathbf{E}|^2}{\mu|\mathbf{H}|^2} \ll 1 \Rightarrow$ magnetic energy dominates

Note that in free space, $\frac{\varepsilon_0|\mathbf{E}|^2}{\mu_0|\mathbf{H}|^2} = 1$

Fields near the surface on an ideal conductor -- continued

For $\frac{\sigma}{\omega} >> 1 \quad \frac{\omega}{c}n_R \approx \frac{\omega}{c}n_I \approx \sqrt{\frac{\mu\sigma\omega}{2}} \equiv \frac{1}{\delta}$

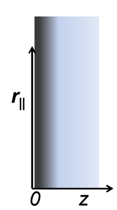
In this limit, $\sqrt{\frac{\mu\varepsilon}{\mu_0\varepsilon_0}} = c\sqrt{\mu\varepsilon} = n_R + in_I = \frac{c}{\omega}\frac{1}{\delta}(1+i)$

$$\mathbf{E}(\mathbf{r},t) = e^{-\hat{\mathbf{k}}\cdot\mathbf{r}/\delta}\Re\left(\mathbf{E}_0 e^{i\hat{\mathbf{k}}\cdot\mathbf{r}/\delta - i\omega t}\right)$$

$$\mathbf{D}(\mathbf{r},t) = \varepsilon\mathbf{E}(\mathbf{r},t) = \frac{i\sigma}{\omega}\mathbf{E}(\mathbf{r},t)$$

$$\mathbf{H}(\mathbf{r},t) = \frac{n}{c\mu}\hat{\mathbf{k}}\times\mathbf{E}(\mathbf{r},t) = \frac{1+i}{\delta\mu\omega}\hat{\mathbf{k}}\times\mathbf{E}(\mathbf{r},t)$$

$$\mathbf{B}(\mathbf{r},t) = \mu\mathbf{H}(\mathbf{r},t) = \frac{n}{c}\hat{\mathbf{k}}\times\mathbf{E}(\mathbf{r},t) = \frac{1+i}{\delta\omega}\hat{\mathbf{k}}\times\mathbf{E}(\mathbf{r},t)$$



Fields near the surface on an ideal conductor -- continued

$$\mathbf{E}(\mathbf{r},t) = e^{-\hat{\mathbf{k}}\cdot\mathbf{r}/\delta}\,\Re\left(\mathbf{E}_0 e^{i\hat{\mathbf{k}}\cdot\mathbf{r}/\delta - i\omega t}\right)$$

$$\mathbf{D}(\mathbf{r},t) = \varepsilon\mathbf{E}(\mathbf{r},t) = \frac{i\sigma}{\omega}\mathbf{E}(\mathbf{r},t)$$

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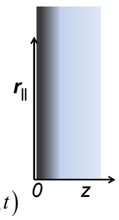


Note that the **H** field is larger than **E** field so we can write:

$$\mathbf{H}(\mathbf{r},t) = e^{-\hat{\mathbf{k}}\cdot\mathbf{r}/\delta}\,\Re\left(\mathbf{H}_0 e^{i\hat{\mathbf{k}}\cdot\mathbf{r}/\delta - i\omega t}\right)$$

$$\mathbf{E}(\mathbf{r},t) = \delta\mu\omega\frac{1-i}{2}\hat{\mathbf{k}}\times\mathbf{H}(\mathbf{r},t)$$

Boundary values for ideal conductor

Inside the conductor:

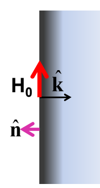
$$\mathbf{H}(\mathbf{r},t) = e^{-\hat{\mathbf{k}}\cdot\mathbf{r}/\delta}\,\Re\left(\mathbf{H}_0 e^{i\hat{\mathbf{k}}\cdot\mathbf{r}/\delta - i\omega t}\right)$$

$$\mathbf{E}(\mathbf{r},t) = \delta\mu\omega\frac{1-i}{2}\hat{\mathbf{k}}\times\mathbf{H}(\mathbf{r},t)$$

At the boundary of an ideal conductor, the **E** and **H** fields decay in the direction normal to the interface.

Ideal conductor boundary conditions:

$$\hat{\mathbf{n}}\times\mathbf{E}\Big|_S = 0 \qquad \hat{\mathbf{n}}\cdot\mathbf{H}\Big|_S = 0$$

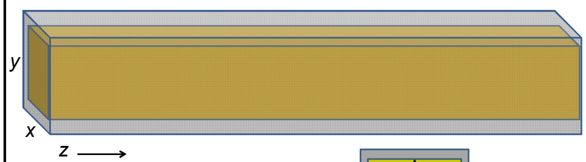


Waveguide terminology

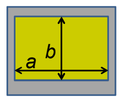
- TEM: transverse electric and magnetic (both E and H fields are perpendicular to wave propagation direction)
- TM: transverse magnetic (H field is perpendicular to wave propagation direction)
- TE: transverse electric (E field is perpendicular to wave propagation direction)

Analysis of rectangular waveguide

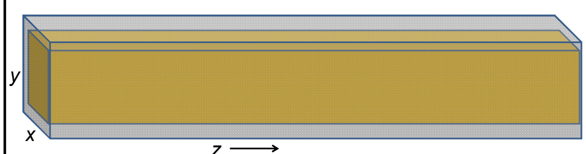
Boundary conditions at surface of waveguide:
$\mathbf{E}_{tangential}=0$, $\mathbf{B}_{normal}=0$



Cross section view



Analysis of rectangular waveguide



$$\mathbf{B} = \Re\left\{\left(B_x(x,y)\hat{\mathbf{x}} + B_y(x,y)\hat{\mathbf{y}} + B_z(x,y)\hat{\mathbf{z}}\right)e^{ikz-i\omega t}\right\}$$
$$\mathbf{E} = \Re\left\{\left(E_x(x,y)\hat{\mathbf{x}} + E_y(x,y)\hat{\mathbf{y}} + E_z(x,y)\hat{\mathbf{z}}\right)e^{ikz-i\omega t}\right\}$$

Inside the dielectric medium:

$$\nabla\cdot\mathbf{E}=0 \qquad \nabla\cdot\mathbf{B}=0$$
$$\nabla\times\mathbf{E}+\frac{\partial\mathbf{B}}{\partial t}=0 \qquad \nabla\times\mathbf{B} - \varepsilon\frac{\partial\mathbf{E}}{\partial t}=0$$

Solution of Maxwell's equations within the pipe:

Combining Faraday's Law and Ampere's Law, we find that each field component must satisfy a two-dimensional Helmholz equation:

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}-k^2+\mu\varepsilon\omega^2\right)E_x(x,y) = 0.$$

For the rectangular wave guide discussed in Section 8.4 of your text a solution for a TE mode can have:

$$E_z(x,y)\equiv 0 \quad \text{and} \quad B_z(x,y)=B_0\cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right),$$

$$\text{with } k^2\equiv k_{mn}^2 = \mu\varepsilon\omega^2 - \left[\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right]$$

Maxwell's equations within the pipe in terms of all 6 components:

$$\frac{\partial B_x}{\partial x}+\frac{\partial B_y}{\partial y}+ikB_z=0.$$

$$\frac{\partial E_x}{\partial x}+\frac{\partial E_y}{\partial y}+ikE_z=0.$$

For TE mode with $E_z \equiv 0$

$$B_x=-\frac{k}{\omega}E_y$$

$$B_y=\frac{k}{\omega}E_x$$

$$\frac{\partial E_z}{\partial y}-ikE_y=i\omega B_x. \qquad \frac{\partial B_z}{\partial y}-ikB_y=-i\mu\varepsilon\omega E_x.$$

$$ikE_x-\frac{\partial E_z}{\partial x}=i\omega B_y. \qquad ikB_x-\frac{\partial B_z}{\partial x}=-i\mu\varepsilon\omega E_y.$$

$$\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}=i\omega B_z. \qquad \frac{\partial B_y}{\partial x}-\frac{\partial B_x}{\partial y}=-i\mu\varepsilon\omega E_z.$$

TE modes for retangular wave guide continued:

$$E_z(x,y)\equiv 0 \quad \text{and} \quad B_z(x,y)=B_0\cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right),$$

$$E_x=\frac{\omega}{k}B_y=\frac{-i\omega}{k^2-\mu\varepsilon\omega^2}\frac{\partial B_z}{\partial y}=\frac{-i\omega}{\left[\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right]}\frac{n\pi}{b}B_0\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right),$$

$$E_y=-\frac{\omega}{k}B_x=\frac{i\omega}{k^2-\mu\varepsilon\omega^2}\frac{\partial B_z}{\partial x}=\frac{i\omega}{\left[\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right]}\frac{m\pi}{a}B_0\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right).$$

Check boundary conditions:

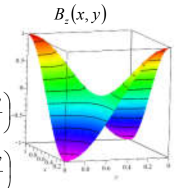
$\mathbf{E}_{\text{tangential}}=0$ because: $E_x(x,0)=E_x(x,b)=0$

and $E_y(0,y)=E_y(a,y)=0.$

$\mathbf{B}_{\text{normal}}=0$

Solution for m=n=1

$B_z(x,y)$

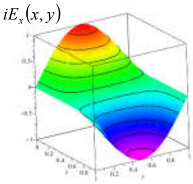


$$B_z(x,y)=B_0\cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)$$

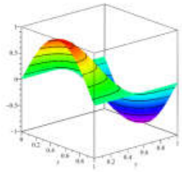
$$iE_x(x,y)=B_0\left(\frac{\omega n\pi/b}{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2}\right)\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$$

$$iE_y(x,y)=B_0\left(\frac{-\omega m\pi/a}{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2}\right)\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)$$

$iE_x(x,y)$

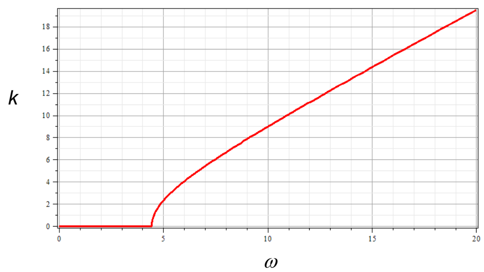


$iE_y(x,y)$

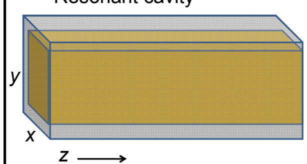


Solution for m=n=1

$$k^2 \equiv k_{mn}^2 = \mu\varepsilon\omega^2 - \left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]$$



Resonant cavity



$$0 \le x \le a$$
$$0 \le y \le b$$
$$0 \le z \le d$$

$$\mathbf{B} = \Re\left\{\left(B_x(x,y,z)\hat{\mathbf{x}} + B_y(x,y,z)\hat{\mathbf{y}} + B_z(x,y,z)\hat{\mathbf{z}}\right)e^{-i\omega t}\right\}$$

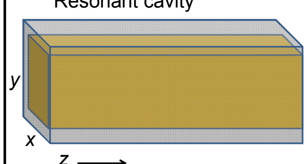
$$\mathbf{E} = \Re\left\{\left(E_x(x,y,z)\hat{\mathbf{x}} + E_y(x,y,z)\hat{\mathbf{y}} + E_z(x,y,z)\hat{\mathbf{z}}\right)e^{-i\omega t}\right\}$$

In general: $E_i(x,y,z) = E_i(x,y)\sin(kz)$ or $E_i(x,y)\cos(kz)$

$B_i(x,y,z) = B_i(x,y)\sin(kz)$ or $B_i(x,y)\cos(kz)$

$$\Rightarrow k = \frac{p\pi}{d}$$

Resonant cavity



$$0 \le x \le a$$
$$0 \le y \le b$$
$$0 \le z \le d$$

$$k^2 = \left(\frac{p\pi}{d}\right)^2 = \mu\varepsilon\omega^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2$$

$$\Rightarrow \omega^2 = \frac{1}{\mu\varepsilon}\left(\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{p\pi}{d}\right)^2\right)$$