

**PHY 712 Electrodynamics**  
**9-9:50 AM MWF Olin 103**

**Plan for Lecture 27:**

**Continue reading Chap. 11 –**  
**Theory of Special Relativity**

**A. Lorentz transformation relations**  
**B. Electromagnetic field transformations**  
**C. Connection to Liénard-Wiechert potentials**  
**for constant velocity sources**

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	Fri: 03/17/2017		APS Meeting - no class		
23	Mon: 03/20/2017	Chap. 9	Sources of Radiation	#17	03/24/2017
24	Wed: 03/22/2017	Chap. 9 & 10	Radiation and Scattering		
25	Fri: 03/24/2017	Chap. 9 & 10	Radiation and Scattering	#18	03/27/2017
26	Mon: 03/27/2017	Chap. 11	Special relativity	#19	03/31/2017
27	Wed: 03/29/2017	Chap. 11	Special relativity		
28	Fri: 03/31/2017				
29	Mon: 04/03/2017				
30	Wed: 04/05/2017				
31	Fri: 04/07/2017				
32	Mon: 04/10/2017				
33	Wed: 04/12/2017				
	Fri: 04/14/2017		Good Friday Holiday -- no class		
34	Mon: 04/17/2017				
35	Wed: 04/19/2017				
36	Fri: 04/21/2017				
	Mon: 04/24/2017		Presentations I		
	Wed: 04/26/2017		Presentations II		

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**OREST**  
CITY

Department of Physics

**News**

**Events**

**Visiting Assistant Professor**  
Opening in Physics

**Part-time Instructor** Opening in Physics

**Angela Harper awarded NSF**  
Graduate Research Fellowship

**Wed. Mar. 29, 2017**  
Neutrinos  
**Physics Colloquium**  
**Prof. Walker, Duke U.**  
Olin 101 4:00 PM  
**Refreshments:**  
3:30 PM Olin Lobby

**Thur. Mar. 30, 2017**  
**Small Chiral Spintronic Devices**  
**Wenxiao Huang**  
Ph. D. Defense  
(Mentor: D. Carroll)  
**Public Talk:**  
Olin 101 3:00 PM

**Wed. Apr. 5, 2017**  
**Hydrogen Storage**  
**Evan Welchman**  
Ph. D. Defense

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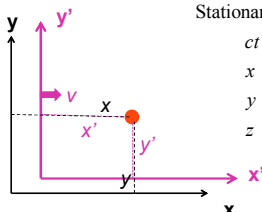
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**Lorentz transformations**      Convenient notation :

$$\beta_v \equiv \frac{v}{c}$$

$$\gamma_v \equiv \frac{1}{\sqrt{1-\beta_v^2}}$$


Stationary frame      Moving frame

$$ct = \gamma(ct' + \beta x')$$

$$x = \gamma(x' + \beta ct')$$

$$y = y'$$

$$z = z'$$

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**Lorentz transformations -- continued**

For the moving frame with  $v = v\hat{x}$  :

$$\mathcal{L}_v = \begin{pmatrix} \gamma_v & \gamma_v \beta_v & 0 & 0 \\ \gamma_v \beta_v & \gamma_v & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \mathcal{L}_v^{-1} = \begin{pmatrix} \gamma_v & -\gamma_v \beta_v & 0 & 0 \\ -\gamma_v \beta_v & \gamma_v & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \mathcal{L}_v \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} \quad \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \mathcal{L}_v^{-1} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

Notice :

$$c^2 t'^2 - x'^2 - y'^2 - z'^2 = c^2 t^2 - x^2 - y^2 - z^2$$

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**Special theory of relativity and Maxwell's equations**

Continuity equation:  $\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0$

Lorentz gauge condition:  $\frac{1}{c} \frac{\partial \Phi}{\partial t} + \nabla \cdot \mathbf{A} = 0$

Potential equations:

$$\frac{1}{c^2} \frac{\partial^2 \Phi}{\partial t^2} - \nabla^2 \Phi = 4\pi \rho$$

$$\frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla^2 \mathbf{A} = \frac{4\pi}{c} \mathbf{J}$$

Field relations:

$$\mathbf{E} = -\nabla \Phi - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t}$$

$$\mathbf{B} = \nabla \times \mathbf{A}$$

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More 4-vectors:

Time and position :  $\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \Rightarrow x^\alpha$

Charge and current :  $\begin{pmatrix} c\rho \\ J_x \\ J_y \\ J_z \end{pmatrix} \Rightarrow J^\alpha$

Vector and scalar potentials :  $\begin{pmatrix} \Phi \\ A_x \\ A_y \\ A_z \end{pmatrix} \Rightarrow A^\alpha$

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Lorentz transformations  $\mathcal{L}_v = \begin{pmatrix} \gamma_v & \gamma_v \beta_v & 0 & 0 \\ \gamma_v \beta_v & \gamma_v & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Time and space :  $x^\alpha = \mathcal{L}_v x'^\alpha \equiv \mathcal{L}_v^{\alpha\beta} x'^\beta$

Charge and current :  $J^\alpha = \mathcal{L}_v J'^\alpha \equiv \mathcal{L}_v^{\alpha\beta} J'^\beta$

Vector and scalar potential :  $A^\alpha = \mathcal{L}_v A'^\alpha \equiv \mathcal{L}_v^{\alpha\beta} A'^\beta$

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4-vector relationships

$\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \Leftrightarrow \begin{pmatrix} A^0 \\ A^1 \\ A^2 \\ A^3 \end{pmatrix} \Leftrightarrow (A^0, \mathbf{A})$ : upper index 4 - vector  $A^\alpha$  for  $(\alpha = 0, 1, 2, 3)$

Keeping track of signs -- lower index 4 - vector  $A_\alpha = (A^0, -\mathbf{A})$

Derivative operators :

$\partial^\alpha = \left( \frac{\partial}{c\partial t}, -\nabla \right) \quad \partial_\alpha = \left( \frac{\partial}{c\partial t}, \nabla \right)$

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**Special theory of relativity and Maxwell's equations**

Continuity equation :  $\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0 \quad \partial_\alpha J^\alpha = 0$

Lorentz gauge condition :  $\frac{1}{c} \frac{\partial \Phi}{\partial t} + \nabla \cdot \mathbf{A} = 0 \quad \partial_\alpha A^\alpha = 0$

Potential equations :  $\frac{1}{c^2} \frac{\partial^2 \Phi}{\partial t^2} - \nabla^2 \Phi = 4\pi \rho$   
 $\frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla^2 \mathbf{A} = \frac{4\pi}{c} \mathbf{J}$   $\partial_\alpha \partial^\alpha A^\beta = \frac{4\pi}{c} J^\beta$

Field relations :  $\mathbf{E} = -\nabla \Phi - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t}$   
 $\mathbf{B} = \nabla \times \mathbf{A}$

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**Electric and Magnetic field relationships**

$\mathbf{E} = -\nabla \Phi - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t}$

$E_x = -\frac{\partial \Phi}{\partial x} - \frac{\partial A_x}{c \partial t} = -(\partial^0 A^1 - \partial^1 A^0)$

$E_y = -\frac{\partial \Phi}{\partial y} - \frac{\partial A_y}{c \partial t} = -(\partial^0 A^2 - \partial^2 A^0)$

$E_z = -\frac{\partial \Phi}{\partial z} - \frac{\partial A_z}{c \partial t} = -(\partial^0 A^3 - \partial^3 A^0)$

$\mathbf{B} = \nabla \times \mathbf{A}$

$B_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = -(\partial^2 A^3 - \partial^3 A^2)$

$B_y = \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} = -(\partial^3 A^1 - \partial^1 A^3)$

$B_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = -(\partial^1 A^2 - \partial^2 A^1)$

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**Field strength tensor**  $F^{\alpha\beta} \equiv (\partial^\alpha A^\beta - \partial^\beta A^\alpha)$

$F^{\alpha\beta} \equiv \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$

**Transformation of field strength tensor**

$F^{\alpha\beta} = \Lambda^\alpha{}_\nu \Lambda^\nu{}_\delta F^{\nu\delta} \Lambda^\delta{}_\beta$   $\Lambda_\nu = \begin{pmatrix} \gamma_\nu & \gamma_\nu \beta_\nu & 0 & 0 \\ \gamma_\nu \beta_\nu & \gamma_\nu & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

$F^{\alpha\beta} = \begin{pmatrix} 0 & -E'_x & -\gamma_\nu(E'_y + \beta_\nu B'_z) & -\gamma_\nu(E'_z - \beta_\nu B'_y) \\ E'_x & 0 & -\gamma_\nu(B'_z + \beta_\nu E'_y) & \gamma_\nu(B'_y - \beta_\nu E'_z) \\ \gamma_\nu(E'_y + \beta_\nu B'_z) & \gamma_\nu(B'_z + \beta_\nu E'_y) & 0 & -B'_x \\ \gamma_\nu(E'_z - \beta_\nu B'_y) & -\gamma_\nu(B'_y - \beta_\nu E'_z) & B'_x & 0 \end{pmatrix}$

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**Inverse transformation of field strength tensor**

$$F^{\alpha\beta} = \Lambda^{\alpha\gamma} F^{\gamma\delta} \Lambda^{-1\delta\beta}$$

$$\Lambda_v^{-1} = \begin{pmatrix} \gamma_v & -\gamma_v \beta_v & 0 & 0 \\ -\gamma_v \beta_v & \gamma_v & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$F^{\alpha\beta} = \begin{pmatrix} 0 & -E'_x & -\gamma_v(E'_y - \beta_v B'_z) & -\gamma_v(E'_z + \beta_v B'_y) \\ E'_x & 0 & -\gamma_v(B'_z - \beta_v E'_y) & \gamma_v(B'_y + \beta_v E'_z) \\ \gamma_v(E'_y - \beta_v B'_z) & \gamma_v(B'_z - \beta_v E'_y) & 0 & -B'_x \\ \gamma_v(E'_z + \beta_v B'_y) & -\gamma_v(B'_y + \beta_v E'_z) & B'_x & 0 \end{pmatrix}$$

Summary of results :

$$\begin{aligned} E_x &= E'_x & B_x &= B'_x \\ E_y &= \gamma_v(E'_y + \beta_v B'_z) & B_y &= \gamma_v(B'_y - \beta_v E'_z) \\ E_z &= \gamma_v(E'_z - \beta_v B'_y) & B_z &= \gamma_v(B'_z + \beta_v E'_y) \end{aligned}$$

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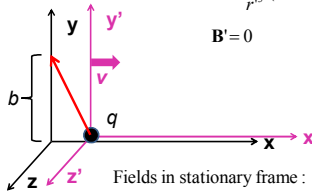
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**Example:**

Fields in moving frame :

$$\mathbf{E}' = \frac{q}{r'^3} (x' \hat{\mathbf{x}} + y' \hat{\mathbf{y}}) = \frac{q(-vt' \hat{\mathbf{x}} + b \hat{\mathbf{y}})}{((-vt')^2 + b^2)^{3/2}}$$

$$\mathbf{B}' = 0$$


Fields in stationary frame :

$$\begin{aligned} E_x &= E'_x & B_x &= B'_x \\ E_y &= \gamma_v(E'_y + \beta_v B'_z) & B_y &= \gamma_v(B'_y - \beta_v E'_z) \\ E_z &= \gamma_v(E'_z - \beta_v B'_y) & B_z &= \gamma_v(B'_z + \beta_v E'_y) \end{aligned}$$

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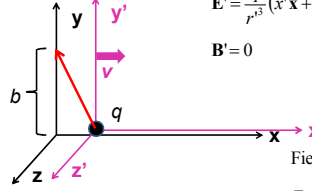
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**Example:**

Fields in moving frame :

$$\mathbf{E}' = \frac{q}{r'^3} (x' \hat{\mathbf{x}} + y' \hat{\mathbf{y}}) = \frac{q(-vt' \hat{\mathbf{x}} + b \hat{\mathbf{y}})}{((-vt')^2 + b^2)^{3/2}}$$

$$\mathbf{B}' = 0$$


Fields in stationary frame :

$$\begin{aligned} E_x &= E'_x = \frac{q(-vt')}{((-vt')^2 + b^2)^{3/2}} \\ E_y &= \gamma_v E'_y = \frac{q(y, b)}{((-vt')^2 + b^2)^{3/2}} \\ B_z &= \gamma_v (\beta_v E'_y) = \frac{q(\gamma_v \beta_v b)}{((-vt')^2 + b^2)^{3/2}} \end{aligned}$$

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Example:

Fields in moving frame:

$$\mathbf{E}' = \frac{q}{r'^3} (x' \hat{\mathbf{x}} + y' \hat{\mathbf{y}}) = \frac{q(-vt' \hat{\mathbf{x}} + b \hat{\mathbf{y}})}{((-vt')^2 + b^2)^{3/2}}$$

$$\mathbf{B}' = 0$$

Fields in stationary frame:

$$E_x = E'_x = \frac{q(-vt)}{((-vt)^2 + b^2)^{3/2}}$$

$$E_y = \gamma_v (E'_y) = \frac{q(\gamma_v b)}{((-v\gamma_v t)^2 + b^2)^{3/2}}$$

$$B_z = \gamma_v (\beta_v E'_y) = \frac{q(\gamma_v \beta_v b)}{((-v\gamma_v t)^2 + b^2)^{3/2}}$$

Expression in terms of consistent coordinates

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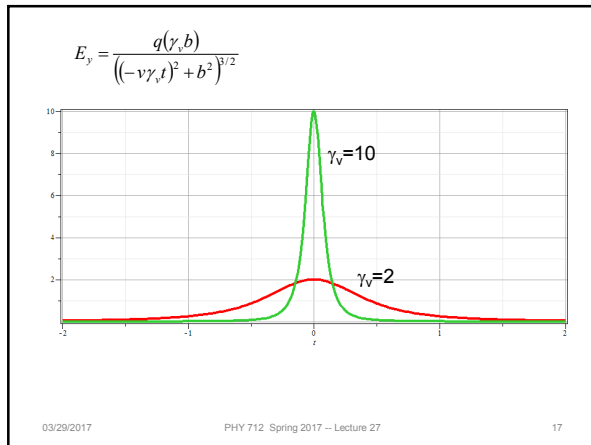
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Examination of this system from the viewpoint of the Liénard-Wiechert potentials (temporarily keeping SI units)

$$\rho(\mathbf{r}, t) = q \delta^3(\mathbf{r} - \mathbf{R}_q(t)) \quad \mathbf{J}(\mathbf{r}, t) = q \dot{\mathbf{R}}_q(t) \delta^3(\mathbf{r} - \mathbf{R}_q(t)) \quad \dot{\mathbf{R}}_q(t) = \frac{d\mathbf{R}_q(t)}{dt}$$

$$\Phi(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \int \int d^3r' dt' \frac{\rho(\mathbf{r}', t')}{|\mathbf{r} - \mathbf{r}'|} \delta(t' - (t - |\mathbf{r} - \mathbf{r}'|/c))$$

$$\mathbf{A}(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0 c^2} \int \int d^3r' dt' \frac{\mathbf{J}(\mathbf{r}', t')}{|\mathbf{r} - \mathbf{r}'|} \delta(t' - (t - |\mathbf{r} - \mathbf{r}'|/c))$$

Evaluating integral over  $t'$ :

$$\int_{-\infty}^{\infty} dt' f(t') \delta(t' - (t - |\mathbf{r} - \mathbf{R}_q(t')|/c)) = \frac{f(t_r)}{1 - \frac{\mathbf{R}_q(t_r) \cdot (\mathbf{r} - \mathbf{R}_q(t_r))}{c |\mathbf{r} - \mathbf{R}_q(t_r)|}}$$

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Examination of this system from the viewpoint of the Liénard-Wiechert potentials – continued (SI units)

$$\Phi(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0} \frac{1}{R - \frac{\mathbf{v} \cdot \mathbf{R}}{c}}$$

$$\mathbf{A}(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0 c^2} \frac{\mathbf{v}}{R - \frac{\mathbf{v} \cdot \mathbf{R}}{c}}$$

where  $\mathbf{R} = \mathbf{r} - \mathbf{R}_q(t_r)$      $\mathbf{v} = \frac{d\mathbf{R}_q(t_r)}{dt_r}$

$$\mathbf{E}(\mathbf{r}, t) = -\nabla\Phi(\mathbf{r}, t) - \frac{\partial\mathbf{A}(\mathbf{r}, t)}{\partial t}$$

$$\mathbf{B}(\mathbf{r}, t) = \nabla \times \mathbf{A}(\mathbf{r}, t)$$

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Examination of this system from the viewpoint of the Liénard-Wiechert potentials – continued (SI units)

$$\mathbf{E}(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0} \frac{1}{\left(R - \frac{\mathbf{v} \cdot \mathbf{R}}{c}\right)^3} \left[ \left(\mathbf{R} - \frac{\mathbf{v}R}{c}\right) \left(1 - \frac{v^2}{c^2}\right) + \left(\mathbf{R} \times \left\{ \left(\mathbf{R} - \frac{\mathbf{v}R}{c}\right) \times \frac{\dot{\mathbf{v}}}{c^2} \right\} \right) \right]$$

$$\mathbf{B}(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0 c^2} \left[ \frac{-\mathbf{R} \times \mathbf{v}}{\left(R - \frac{\mathbf{v} \cdot \mathbf{R}}{c}\right)^3} \left(1 - \frac{v^2}{c^2} + \frac{\dot{\mathbf{v}} \cdot \mathbf{R}}{c^2}\right) - \frac{\mathbf{R} \times \dot{\mathbf{v}} / c}{\left(R - \frac{\mathbf{v} \cdot \mathbf{R}}{c}\right)^2} \right]$$

$$\mathbf{B}(\mathbf{r}, t) = \frac{\mathbf{R} \times \mathbf{E}(\mathbf{r}, t)}{cR}$$

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Examination of this system from the viewpoint of the Liénard-Wiechert potentials – continued (Gaussian units)

$$\mathbf{E}(\mathbf{r}, t) = \frac{q}{\left(R - \frac{\mathbf{v} \cdot \mathbf{R}}{c}\right)^3} \left[ \left(\mathbf{R} - \frac{\mathbf{v}R}{c}\right) \left(1 - \frac{v^2}{c^2}\right) + \left(\mathbf{R} \times \left\{ \left(\mathbf{R} - \frac{\mathbf{v}R}{c}\right) \times \frac{\dot{\mathbf{v}}}{c^2} \right\} \right) \right]$$

$$\mathbf{B}(\mathbf{r}, t) = \frac{q}{c} \left[ \frac{-\mathbf{R} \times \mathbf{v}}{\left(R - \frac{\mathbf{v} \cdot \mathbf{R}}{c}\right)^3} \left(1 - \frac{v^2}{c^2} + \frac{\dot{\mathbf{v}} \cdot \mathbf{R}}{c^2}\right) - \frac{\mathbf{R} \times \dot{\mathbf{v}} / c}{\left(R - \frac{\mathbf{v} \cdot \mathbf{R}}{c}\right)^2} \right]$$

$$\mathbf{B}(\mathbf{r}, t) = \frac{\mathbf{R} \times \mathbf{E}(\mathbf{r}, t)}{R}$$

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Examination of this system from the viewpoint of the  
the Liénard-Wiechert potentials – continued (Gaussian units)

$$\mathbf{E}(\mathbf{r}, t) = \frac{q}{\left(R - \frac{\mathbf{v} \cdot \mathbf{R}}{c}\right)^3} \left[ \left( \mathbf{R} - \frac{\mathbf{v}R}{c} \right) \left( 1 - \frac{v^2}{c^2} \right) \right]$$

For our example:

$$\mathbf{B}(\mathbf{r}, t) = \frac{q}{c} \left[ \frac{-\mathbf{R} \times \mathbf{v}}{\left(R - \frac{\mathbf{v} \cdot \mathbf{R}}{c}\right)^3} \left( 1 - \frac{v^2}{c^2} \right) \right]$$

$$\begin{aligned} \mathbf{R}_y(t_r) &= vt_r \hat{\mathbf{x}} & \mathbf{r} &= b \hat{\mathbf{y}} \\ \mathbf{R} &= b \hat{\mathbf{y}} - vt_r \hat{\mathbf{x}} & R &= \sqrt{v^2 t_r^2 + b^2} \\ \mathbf{v} &= v \hat{\mathbf{x}} & t_r &= t - \frac{R}{c} \end{aligned}$$

This should be equivalent to the result given in Jackson (11.152):

$$\mathbf{E}(x, y, z, t) = \mathbf{E}(0, b, 0, t) = q \frac{-v\gamma t \hat{\mathbf{x}} + \gamma b \hat{\mathbf{y}}}{(b^2 + (v\gamma t)^2)^{3/2}}$$

$$\mathbf{B}(x, y, z, t) = \mathbf{B}(0, b, 0, t) = q \frac{\gamma \beta b \hat{\mathbf{z}}}{(b^2 + (v\gamma t)^2)^{3/2}}$$

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