PHY 712 Electrodynamics
9-9:50 AM MWF Olin 103

Plan for Lecture 28:

Finish Chap. 11 and begin Chap. 14

A. **Electromagnetic field transformations & corresponding analysis using Liénard-Wiechert potentials for constant velocity sources**

B. **Radiation by moving charged particles**

	Fri: 03/17/2017		*APS Meeting - no class*		
23	Mon: 03/20/2017	Chap. 9	Sources of Radiation	#17	03/24/2017
24	Wed: 03/22/2017	Chap. 9 & 10	Radiation and Scattering		
25	Fri: 03/24/2017	Chap. 9 & 10	Radiation and Scattering	#18	03/27/2017
26	Mon: 03/27/2017	Chap. 11	Special relativity	#19	03/31/2017
27	Wed: 03/29/2017	Chap. 11	Special relativity		
28	Fri: 03/31/2017	Chap. 11	Special relativity	#20	04/3/2017
29	Mon: 04/03/2017				
30	Wed: 04/05/2017				
31	Fri: 04/07/2017				
32	Mon: 04/10/2017				
33	Wed: 04/12/2017				
	Fri: 04/14/2017		*Good Friday Holiday -- no class*		
34	Mon: 04/17/2017				
35	Wed: 04/19/2017				
36	Fri: 04/21/2017				
	Mon: 04/24/2017		Presentations I		
	Wed: 04/26/2017		Presentations II		

Inverse transformation of field strength tensor

$$F'^{\alpha\beta} = \mathcal{L}_v^{-1\alpha\gamma} F^{\gamma\delta} \mathcal{L}_v^{-1\delta\beta} \qquad \mathcal{L}_v^{-1} = \begin{pmatrix} \gamma_v & -\gamma_v\beta_v & 0 & 0 \\ -\gamma_v\beta_v & \gamma_v & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$F'^{\alpha\beta} = \begin{pmatrix} 0 & -E'_x & -\gamma_v(E'_y - \beta_v B'_z) & -\gamma_v(E'_z + \beta_v B'_y) \\ E'_x & 0 & -\gamma_v(B'_z - \beta_v E'_y) & \gamma_v(B'_y + \beta_v E'_z) \\ \gamma_v(E'_y - \beta_v B'_z) & \gamma_v(B'_z - \beta_v E'_y) & 0 & -B'_x \\ \gamma_v(E'_z + \beta_v B'_y) & -\gamma_v(B'_y + \beta_v E'_z) & B'_x & 0 \end{pmatrix}$$

Summary of results:

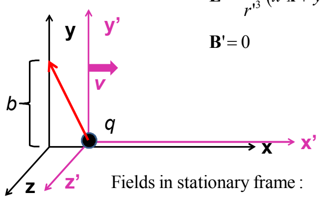
$$E_x = E'_x \qquad B_x = B'_x$$
$$E_y = \gamma_v(E'_y + \beta_v B'_z) \qquad B_y = \gamma_v(B'_y - \beta_v E'_z)$$
$$E_z = \gamma_v(E'_z - \beta_v B'_y) \qquad B_z = \gamma_v(B'_z + \beta_v E'_y)$$

Example:

Fields in moving frame :

$$\mathbf{E}' = \frac{q}{r'^3}\left(x'\hat{\mathbf{x}} + y'\hat{\mathbf{y}}\right) = \frac{q\left(-vt'\hat{\mathbf{x}} + b\hat{\mathbf{y}}\right)}{\left((-vt')^2 + b^2\right)^{3/2}}$$

$$\mathbf{B}' = 0$$



Fields in stationary frame :

$$E_x = E'_x \qquad B_x = B'_x$$
$$E_y = \gamma_v\left(E'_y + \beta_v B'_z\right) \qquad B_y = \gamma_v\left(B'_y - \beta_v E'_z\right)$$
$$E_z = \gamma_v\left(E'_z - \beta_v B'_y\right) \qquad B_z = \gamma_v\left(B'_z + \beta_v E'_y\right)$$

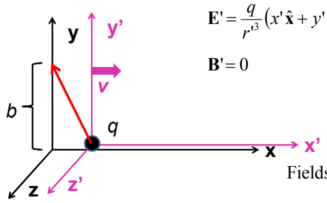
03/31/2017 PHY 712 Spring 2017 -- Lecture 28 4

Example:

Fields in moving frame :

$$\mathbf{E}' = \frac{q}{r'^3}\left(x'\hat{\mathbf{x}} + y'\hat{\mathbf{y}}\right) = \frac{q\left(-vt'\hat{\mathbf{x}} + b\hat{\mathbf{y}}\right)}{\left((-vt')^2 + b^2\right)^{3/2}}$$

$$\mathbf{B}' = 0$$



Fields in stationary frame :

$$E_x = E'_x = \frac{q(-vt')}{\left((-vt')^2 + b^2\right)^{3/2}}$$

$$E_y = \gamma_v\left(E'_y\right) = \frac{q(\gamma_v b)}{\left((-vt')^2 + b^2\right)^{3/2}}$$

$$B_z = \gamma_v\left(\beta_v E'_y\right) = \frac{q(\gamma_v \beta_v b)}{\left((-vt')^2 + b^2\right)^{3/2}}$$

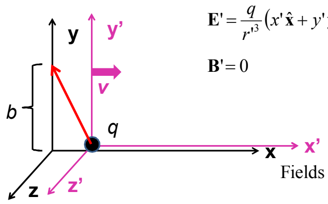
03/31/2017 PHY 712 Spring 2017 -- Lecture 28 5

Example:

Fields in moving frame :

$$\mathbf{E}' = \frac{q}{r'^3}\left(x'\hat{\mathbf{x}} + y'\hat{\mathbf{y}}\right) = \frac{q\left(-vt'\hat{\mathbf{x}} + b\hat{\mathbf{y}}\right)}{\left((-vt')^2 + b^2\right)^{3/2}}$$

$$\mathbf{B}' = 0$$



Fields in stationary frame :

$$E_x = E'_x = \frac{q(-v\gamma_v t)}{\left((-v\gamma_v t)^2 + b^2\right)^{3/2}}$$

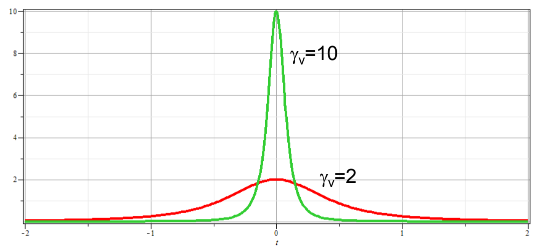
Expression in terms of consistent coordinates

$$E_y = \gamma_v\left(E'_y\right) = \frac{q(\gamma_v b)}{\left((-v\gamma_v t)^2 + b^2\right)^{3/2}}$$

$$B_z = \gamma_v\left(\beta_v E'_y\right) = \frac{q(\gamma_v \beta_v b)}{\left((-v\gamma_v t)^2 + b^2\right)^{3/2}}$$

03/31/2017 PHY 712 Spring 2017 -- Lecture 28 6

$$E_y = \frac{q(\gamma_v b)}{\left((-v\gamma_v t)^2 + b^2\right)^{3/2}}$$



Examination of this system from the viewpoint of the the Liènard-Wiechert potentials –(Gaussian units)

$$\mathbf{E}(\mathbf{r},t) = \frac{q}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left[\left(\mathbf{R} - \frac{\mathbf{v}R}{c}\right)\left(1 - \frac{v^2}{c^2}\right) + \left(\mathbf{R}\times\left\{\left(\mathbf{R} - \frac{\mathbf{v}R}{c}\right)\times\frac{\dot{\mathbf{v}}}{c^2}\right\}\right)\right]$$

$$\mathbf{B}(\mathbf{r},t) = \frac{q}{c}\left[\frac{-\mathbf{R}\times\mathbf{v}}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left(1 - \frac{v^2}{c^2} + \frac{\dot{\mathbf{v}}\cdot\mathbf{R}}{c^2}\right) - \frac{\mathbf{R}\times\dot{\mathbf{v}}/c}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^2}\right]$$

$$\mathbf{B}(\mathbf{r},t) = \frac{\mathbf{R}\times\mathbf{E}(\mathbf{r},t)}{R}.$$

Examination of this system from the viewpoint of the the Liènard-Wiechert potentials –(Gaussian units)

$$\mathbf{E}(\mathbf{r},t) = \frac{q}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left[\left(\mathbf{R} - \frac{\mathbf{v}R}{c}\right)\left(1 - \frac{v^2}{c^2}\right)\right]$$

$$\mathbf{B}(\mathbf{r},t) = \frac{q}{c}\left[\frac{-\mathbf{R}\times\mathbf{v}}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left(1 - \frac{v^2}{c^2}\right)\right]$$

For our example:

$\mathbf{R}_q(t_r) = vt_r\hat{\mathbf{x}}$ $\quad \mathbf{r} = b\hat{\mathbf{y}}$

$\mathbf{R} = b\hat{\mathbf{y}} - vt_r\hat{\mathbf{x}}$ $\quad R = \sqrt{v^2t_r^2 + b^2}$

$\mathbf{v} = v\hat{\mathbf{x}}$ $\quad t_r = t - \frac{R}{c}$

This should be equivalent to the result given in Jackson (11.152):

$$\mathbf{E}(x,y,z,t) = \mathbf{E}(0,b,0,t) = q\frac{-v\gamma t\hat{\mathbf{x}} + \gamma b\hat{\mathbf{y}}}{\left(b^2 + (v\gamma t)^2\right)^{3/2}}$$

$$\mathbf{B}(x,y,z,t) = \mathbf{B}(0,b,0,t) = q\frac{\gamma\beta b\hat{\mathbf{z}}}{\left(b^2 + (v\gamma t)^2\right)^{3/2}}$$

Some details

$$\mathbf{E}(\mathbf{r},t)=\frac{q}{\left(R-\frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left[\left(\mathbf{R}-\frac{\mathbf{v}R}{c}\right)\left(1-\frac{v^2}{c^2}\right)\right]$$

$$\mathbf{B}(\mathbf{r},t)=\frac{q}{c}\left[\frac{-\mathbf{R}\times\mathbf{v}}{\left(R-\frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left(1-\frac{v^2}{c^2}\right)\right]$$

For our example:

$$\mathbf{R}_q(t_r)=vt_r\hat{\mathbf{x}} \qquad \mathbf{r}=b\hat{\mathbf{y}}$$

$$\mathbf{R}=b\hat{\mathbf{y}}-vt_r\hat{\mathbf{x}} \qquad R=\sqrt{v^2t_r^2+b^2}$$

$$\mathbf{v}=v\hat{\mathbf{x}} \qquad t_r=t-\frac{R}{c}$$

t_r must be a solution to a quadradic equation:

$$t_r-t=-\frac{R}{c} \quad\Rightarrow\quad t_r^2-2\gamma^2tt_r+\gamma^2t^2-\gamma^2b^2/c^2=0$$

with the physical solution:

$$t_r=\gamma\left(\gamma t-\frac{\sqrt{(v\gamma t)^2+b^2}}{c}\right)$$

Some details continued:

Now we can express R as:

$$R=\gamma\left(-\beta v\gamma t+\sqrt{(v\gamma t)^2+b^2}\right)$$

and the related quantities:

$$\mathbf{R}-\mathbf{v}R/c=-vt\hat{\mathbf{x}}+b\hat{\mathbf{y}}$$

$$R-\mathbf{v}\cdot\mathbf{R}/c=\frac{\sqrt{(v\gamma t)^2+b^2}}{\gamma}$$

$$\mathbf{E}(\mathbf{r},t)=\frac{q}{\left(R-\frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left[\left(\mathbf{R}-\frac{\mathbf{v}R}{c}\right)\left(1-\frac{v^2}{c^2}\right)\right]=q\frac{-v\gamma t\hat{\mathbf{x}}+\gamma b\hat{\mathbf{y}}}{\left(b^2+(v\gamma t)^2\right)^{3/2}}$$

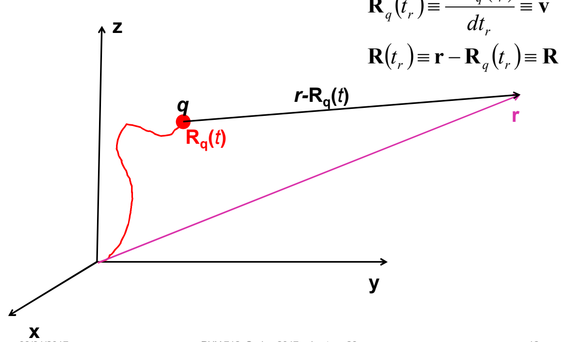
$$\mathbf{B}(\mathbf{r},t)=\frac{q}{c}\left[\frac{-\mathbf{R}\times\mathbf{v}}{\left(R-\frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left(1-\frac{v^2}{c^2}\right)\right]=q\frac{\gamma\beta b\hat{\mathbf{z}}}{\left(b^2+(v\gamma t)^2\right)^{3/2}}$$

Radiation from a moving charged particle

Variables (notation):

$$\dot{\mathbf{R}}_q(t_r)\equiv\frac{d\mathbf{R}_q(t_r)}{dt_r}\equiv\mathbf{v}$$

$$\mathbf{R}(t_r)\equiv\mathbf{r}-\mathbf{R}_q(t_r)\equiv\mathbf{R}$$



Liénard-Wiechert fields (cgs Gaussian units):

$$\mathbf{E}(\mathbf{r},t) = \frac{q}{(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c})^3}\left[\left(\mathbf{R} - \frac{\mathbf{v}R}{c}\right)\left(1 - \frac{v^2}{c^2}\right) + \left(\mathbf{R} \times \left\{\left(\mathbf{R} - \frac{\mathbf{v}R}{c}\right) \times \frac{\dot{\mathbf{v}}}{c^2}\right\}\right)\right]. \quad (19)$$

$$\mathbf{B}(\mathbf{r},t) = \frac{q}{c}\left[\frac{-\mathbf{R}\times\mathbf{v}}{(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c})^3}\left(1 - \frac{v^2}{c^2} + \frac{\dot{\mathbf{v}}\cdot\mathbf{R}}{c^2}\right) - \frac{\mathbf{R}\times\dot{\mathbf{v}}/c}{(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c})^2}\right]. \quad (20)$$

In this case, the electric and magnetic fields are related according to

$$\mathbf{B}(\mathbf{r},t) = \frac{\mathbf{R}\times\mathbf{E}(\mathbf{r},t)}{R}. \quad (21)$$

$$\dot{\mathbf{R}}_q(t_r) \equiv \frac{d\mathbf{R}_q(t_r)}{dt_r} \equiv \mathbf{v} \quad \mathbf{R}(t_r) \equiv \mathbf{r} - \mathbf{R}_q(t_r) \equiv \mathbf{R} \quad \dot{\mathbf{v}} \equiv \frac{d^2\mathbf{R}_q(t_r)}{dt_r^{\ 2}}$$

Electric field far from source:

$$\mathbf{E}(\mathbf{r},t) = \frac{q}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left\{\mathbf{R}\times\left[\left(\mathbf{R} - \frac{\mathbf{v}R}{c}\right)\times\frac{\dot{\mathbf{v}}}{c^2}\right]\right\}$$

$$\mathbf{B}(\mathbf{r},t) = \frac{\mathbf{R}\times\mathbf{E}(\mathbf{r},t)}{R}$$

$$\text{Let} \quad \hat{\mathbf{R}} \equiv \frac{\mathbf{R}}{R} \qquad \boldsymbol{\beta} \equiv \frac{\mathbf{v}}{c} \qquad \dot{\boldsymbol{\beta}} \equiv \frac{\dot{\mathbf{v}}}{c}$$

$$\mathbf{E}(\mathbf{r},t) = \frac{q}{cR(1 - \boldsymbol{\beta}\cdot\hat{\mathbf{R}})^3}\left\{\hat{\mathbf{R}}\times\left[(\hat{\mathbf{R}} - \boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\right]\right\}$$

$$\mathbf{B}(\mathbf{r},t) = \hat{\mathbf{R}}\times\mathbf{E}(\mathbf{r},t)$$

Poynting vector:

$$\mathbf{S}(\mathbf{r},t) = \frac{c}{4\pi}(\mathbf{E}\times\mathbf{B})$$

$$\mathbf{E}(\mathbf{r},t) = \frac{q}{cR(1 - \boldsymbol{\beta}\cdot\hat{\mathbf{R}})^3}\left\{\hat{\mathbf{R}}\times\left[(\hat{\mathbf{R}} - \boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\right]\right\}$$

$$\mathbf{B}(\mathbf{r},t) = \hat{\mathbf{R}}\times\mathbf{E}(\mathbf{r},t)$$

$$\mathbf{S}(\mathbf{r},t) = \frac{c}{4\pi}\hat{\mathbf{R}}|\mathbf{E}(\mathbf{r},t)|^2 = \frac{q^2}{4\pi c R^2}\hat{\mathbf{R}}\frac{\left|\hat{\mathbf{R}}\times\left[(\hat{\mathbf{R}} - \boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\right]\right|^2}{(1 - \boldsymbol{\beta}\cdot\hat{\mathbf{R}})^6}$$

Note: We have assumed that

$$\hat{\mathbf{R}}\cdot\mathbf{E}(\mathbf{r},t) = 0$$

Power radiated

$$\mathbf{S}(\mathbf{r},t) = \frac{c}{4\pi}\hat{\mathbf{R}}\left|\mathbf{E}(\mathbf{r},t)\right|^2 = \frac{q^2}{4\pi c R^2}\hat{\mathbf{R}}\frac{\left|\hat{\mathbf{R}}\times\left[\left(\hat{\mathbf{R}}-\boldsymbol{\beta}\right)\times\dot{\boldsymbol{\beta}}\right]\right|^2}{\left(1-\boldsymbol{\beta}\cdot\hat{\mathbf{R}}\right)^6}$$

$$\frac{dP}{d\Omega} = \mathbf{S}\cdot\hat{\mathbf{R}}R^2 = \frac{q^2}{4\pi c}\frac{\left|\hat{\mathbf{R}}\times\left[\left(\hat{\mathbf{R}}-\boldsymbol{\beta}\right)\times\dot{\boldsymbol{\beta}}\right]\right|^2}{\left(1-\boldsymbol{\beta}\cdot\hat{\mathbf{R}}\right)^6}$$

In the non-relativistic limit: $\beta << 1$

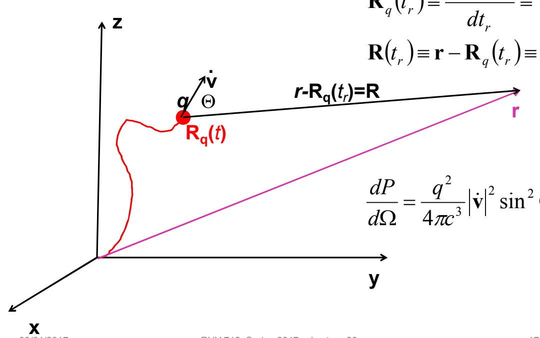
$$\frac{dP}{d\Omega} = \frac{q^2}{4\pi c}\left|\hat{\mathbf{R}}\times\left[\hat{\mathbf{R}}\times\dot{\boldsymbol{\beta}}\right]\right|^2 = \frac{q^2}{4\pi c^3}\left|\dot{\mathbf{v}}\right|^2\sin^2\Theta$$

Radiation from a moving charged particle

Variables (notation):

$$\dot{\mathbf{R}}_q(t_r) \equiv \frac{d\mathbf{R}_q(t_r)}{dt_r} \equiv \mathbf{v}$$

$$\mathbf{R}(t_r) \equiv \mathbf{r} - \mathbf{R}_q(t_r) \equiv \mathbf{R}$$



$$\frac{dP}{d\Omega} = \frac{q^2}{4\pi c^3}\left|\dot{\mathbf{v}}\right|^2\sin^2\Theta$$

Radiation power in non-relativistic case -- continued

$$\frac{dP}{d\Omega} = \frac{q^2}{4\pi c^3}\left|\dot{\mathbf{v}}\right|^2\sin^2\Theta$$

$$P = \int d\Omega\frac{dP}{d\Omega} = \frac{2}{3}\frac{q^2}{c^3}\left|\dot{\mathbf{v}}\right|^2$$