PHY 712 Electrodynamics
9-9:50 AM MWF Olin 103

Plan for Lecture 29:

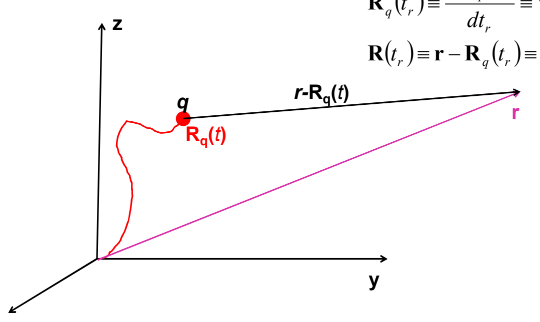
Continue reading Chap. 14 –

Radiation by moving charges

1. Motion in a line
2. Motion in a circle
3. Spectral analysis of radiation

	Fri: 03/17/2017		*APS Meeting - no class*		
23	Mon: 03/20/2017	Chap. 9	Sources of Radiation	#17	03/24/2017
24	Wed: 03/22/2017	Chap. 9 & 10	Radiation and Scattering		
25	Fri: 03/24/2017	Chap. 9 & 10	Radiation and Scattering	#18	03/27/2017
26	Mon: 03/27/2017	Chap. 11	Special relativity	#19	03/31/2017
27	Wed: 03/29/2017	Chap. 11	Special relativity		
28	Fri: 03/31/2017	Chap. 11	Special relativity	#20	04/3/2017
29	Mon: 04/03/2017	Chap. 14	Radiation from moving charges	#21	04/5/2017
30	Wed: 04/05/2017				
31	Fri: 04/07/2017				
32	Mon: 04/10/2017				
33	Wed: 04/12/2017				
	Fri: 04/14/2017		*Good Friday Holiday -- no class*		
34	Mon: 04/17/2017				
35	Wed: 04/19/2017				
36	Fri: 04/21/2017				
	Mon: 04/24/2017		Presentations I		
	Wed: 04/26/2017		Presentations II		

Radiation from a moving charged particle



Variables (notation) :

$$\dot{\mathbf{R}}_q(t_r) \equiv \frac{d\mathbf{R}_q(t_r)}{dt_r} \equiv \mathbf{v}$$

$$\mathbf{R}(t_r) \equiv \mathbf{r} - \mathbf{R}_q(t_r) \equiv \mathbf{R}$$

Liénard-Wiechert fields (cgs Gaussian units):

$$\mathbf{E}(\mathbf{r},t) = \frac{q}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left[\left(\mathbf{R} - \frac{\mathbf{v}R}{c}\right)\left(1 - \frac{v^2}{c^2}\right) + \left(\mathbf{R}\times\left\{\left(\mathbf{R} - \frac{\mathbf{v}R}{c}\right)\times\frac{\dot{\mathbf{v}}}{c^2}\right\}\right)\right]. \quad (19)$$

$$\mathbf{B}(\mathbf{r},t) = \frac{q}{c}\left[\frac{-\mathbf{R}\times\mathbf{v}}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left(1 - \frac{v^2}{c^2} + \frac{\dot{\mathbf{v}}\cdot\mathbf{R}}{c^2}\right) - \frac{\mathbf{R}\times\dot{\mathbf{v}}/c}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^2}\right]. \quad (20)$$

In this case, the electric and magnetic fields are related according to

$$\mathbf{B}(\mathbf{r},t) = \frac{\mathbf{R}\times\mathbf{E}(\mathbf{r},t)}{R}. \quad (21)$$

$$\dot{\mathbf{R}}_q(t_r) \equiv \frac{d\mathbf{R}_q(t_r)}{dt_r} \equiv \mathbf{v} \qquad \mathbf{R}(t_r) \equiv \mathbf{r} - \mathbf{R}_q(t_r) \equiv \mathbf{R} \qquad \dot{\mathbf{v}} \equiv \frac{d^2\mathbf{R}_q(t_r)}{dt_r^{\,2}}$$

Electric field far from source:

$$\mathbf{E}(\mathbf{r},t) = \frac{q}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left\{\mathbf{R}\times\left[\left(\mathbf{R} - \frac{\mathbf{v}R}{c}\right)\times\frac{\dot{\mathbf{v}}}{c^2}\right]\right\}$$

$$\mathbf{B}(\mathbf{r},t) = \frac{\mathbf{R}\times\mathbf{E}(\mathbf{r},t)}{R}$$

Let $\hat{\mathbf{R}} \equiv \frac{\mathbf{R}}{R} \qquad \boldsymbol{\beta} \equiv \frac{\mathbf{v}}{c} \qquad \dot{\boldsymbol{\beta}} \equiv \frac{\dot{\mathbf{v}}}{c}$

$$\mathbf{E}(\mathbf{r},t) = \frac{q}{cR(1 - \boldsymbol{\beta}\cdot\hat{\mathbf{R}})^3}\left\{\hat{\mathbf{R}}\times\left[(\hat{\mathbf{R}} - \boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\right]\right\}$$

$$\mathbf{B}(\mathbf{r},t) = \hat{\mathbf{R}}\times\mathbf{E}(\mathbf{r},t)$$

Poynting vector:

$$\mathbf{S}(\mathbf{r},t) = \frac{c}{4\pi}(\mathbf{E}\times\mathbf{B})$$

$$\mathbf{E}(\mathbf{r},t) = \frac{q}{cR(1 - \boldsymbol{\beta}\cdot\hat{\mathbf{R}})^3}\left\{\hat{\mathbf{R}}\times\left[(\hat{\mathbf{R}} - \boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\right]\right\}$$

$$\mathbf{B}(\mathbf{r},t) = \hat{\mathbf{R}}\times\mathbf{E}(\mathbf{r},t)$$

$$\mathbf{S}(\mathbf{r},t) = \frac{c}{4\pi}\hat{\mathbf{R}}\left|\mathbf{E}(\mathbf{r},t)\right|^2 = \frac{q^2}{4\pi cR^2}\hat{\mathbf{R}}\frac{\left|\hat{\mathbf{R}}\times\left[(\hat{\mathbf{R}} - \boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\right]\right|^2}{(1 - \boldsymbol{\beta}\cdot\hat{\mathbf{R}})^6}$$

Note: We have assumed that

$$\hat{\mathbf{R}}\cdot\mathbf{E}(\mathbf{r},t) = 0$$

Power radiated

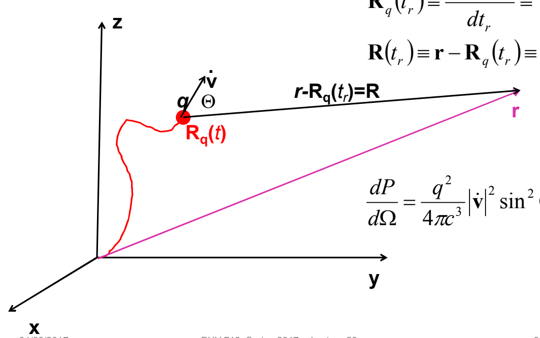
$$\mathbf{S}(\mathbf{r},t) = \frac{c}{4\pi}\hat{\mathbf{R}}\left|\mathbf{E}(\mathbf{r},t)\right|^2 = \frac{q^2}{4\pi c R^2}\hat{\mathbf{R}}\frac{\left|\hat{\mathbf{R}}\times\left[\left(\hat{\mathbf{R}}-\boldsymbol{\beta}\right)\times\dot{\boldsymbol{\beta}}\right]\right|^2}{\left(1-\boldsymbol{\beta}\cdot\hat{\mathbf{R}}\right)^6}$$

$$\frac{dP}{d\Omega} = \mathbf{S}\cdot\hat{\mathbf{R}}R^2 = \frac{q^2}{4\pi c}\frac{\left|\hat{\mathbf{R}}\times\left[\left(\hat{\mathbf{R}}-\boldsymbol{\beta}\right)\times\dot{\boldsymbol{\beta}}\right]\right|^2}{\left(1-\boldsymbol{\beta}\cdot\hat{\mathbf{R}}\right)^6}$$

In the non-relativistic limit: $\beta << 1$

$$\frac{dP}{d\Omega} = \frac{q^2}{4\pi c}\left|\hat{\mathbf{R}}\times\left[\hat{\mathbf{R}}\times\dot{\boldsymbol{\beta}}\right]\right|^2 = \frac{q^2}{4\pi c^3}\left|\dot{\mathbf{v}}\right|^2\sin^2\Theta$$

Radiation from a moving charged particle



Variables (notation):

$$\dot{\mathbf{R}}_q(t_r) \equiv \frac{d\mathbf{R}_q(t_r)}{dt_r} \equiv \mathbf{v}$$

$$\mathbf{R}(t_r) \equiv \mathbf{r} - \mathbf{R}_q(t_r) \equiv \mathbf{R}$$

$$\frac{dP}{d\Omega} = \frac{q^2}{4\pi c^3}\left|\dot{\mathbf{v}}\right|^2\sin^2\Theta$$

Radiation power in non-relativistic case -- continued

$$\frac{dP}{d\Omega} = \frac{q^2}{4\pi c^3}\left|\dot{\mathbf{v}}\right|^2\sin^2\Theta$$

$$P = \int d\Omega\frac{dP}{d\Omega} = \frac{2}{3}\frac{q^2}{c^3}\left|\dot{\mathbf{v}}\right|^2$$

Radiation distribution in the relativistic case

$$\frac{dP}{d\Omega} = \mathbf{S} \cdot \hat{\mathbf{R}} R^2 = \frac{q^2}{4\pi c} \left. \frac{\left| \hat{\mathbf{R}} \times \left[\left(\hat{\mathbf{R}} - \boldsymbol{\beta} \right) \times \dot{\boldsymbol{\beta}} \right] \right|^2}{\left(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}} \right)^6} \right|_{t_r = t - R/c}$$

This expression gives us the energy per unit field time *t*. We are often interested in the power per unit retarded time $t_r = t - R/c$:

$$\frac{dP(t)}{d\Omega} = \frac{dP_r(t_r)}{d\Omega} \frac{dt_t}{dt} \qquad \frac{dt}{dt_r} = 1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}}$$

$$\frac{dP_r(t_r)}{d\Omega} = \frac{q^2}{4\pi c} \left. \frac{\left| \hat{\mathbf{R}} \times \left[\left(\hat{\mathbf{R}} - \boldsymbol{\beta} \right) \times \dot{\boldsymbol{\beta}} \right] \right|^2}{\left(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}} \right)^5} \right|_{t_r = t - R/c}$$

Radiation distribution in the relativistic case -- continued

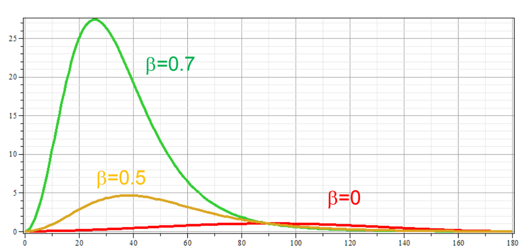
$$\frac{dP_r(t_r)}{d\Omega} = \frac{q^2}{4\pi c} \left. \frac{\left| \hat{\mathbf{R}} \times \left[\left(\hat{\mathbf{R}} - \boldsymbol{\beta} \right) \times \dot{\boldsymbol{\beta}} \right] \right|^2}{\left(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}} \right)^5} \right|_{t_r = t - R/c}$$

For linear acceleration: $\boldsymbol{\beta} \times \dot{\boldsymbol{\beta}} = 0$

$$\frac{dP_r(t_r)}{d\Omega} = \frac{q^2}{4\pi c} \left. \frac{\left| \hat{\mathbf{R}} \times \left(\hat{\mathbf{R}} \times \dot{\boldsymbol{\beta}} \right) \right|^2}{\left(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}} \right)^5} \right|_{t_r = t - R/c} = \frac{q^2}{4\pi c^3} \left| \dot{\mathbf{v}} \right|^2 \frac{\sin^2\theta}{\left(1 - \beta \cos\theta \right)^5}$$

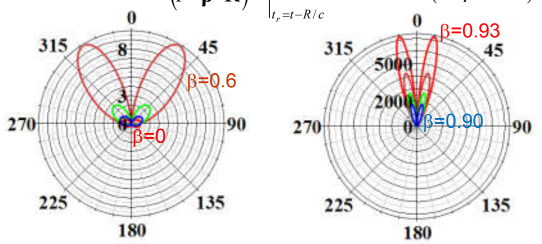
Power from linearly accelerating particle

$$\frac{dP_r(t_r)}{d\Omega} = \frac{q^2}{4\pi c} \left. \frac{\left| \hat{\mathbf{R}} \times \left(\hat{\mathbf{R}} \times \dot{\boldsymbol{\beta}} \right) \right|^2}{\left(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}} \right)^5} \right|_{t_r = t - R/c} = \frac{q^2}{4\pi c^3} \left| \dot{\mathbf{v}} \right|^2 \frac{\sin^2\theta}{\left(1 - \beta \cos\theta \right)^5}$$



Polar plots:

$$\frac{dP_r(t_r)}{d\Omega} = \frac{q^2}{4\pi c}\left.\frac{\left|\hat{\mathbf{R}}\times\left(\hat{\mathbf{R}}\times\dot{\boldsymbol{\beta}}\right)\right|^2}{\left(1-\boldsymbol{\beta}\cdot\hat{\mathbf{R}}\right)^5}\right|_{t_r=t-R/c} = \frac{q^2}{4\pi c^3}|\dot{\mathbf{v}}|^2\frac{\sin^2\theta}{(1-\beta\cos\theta)^5}$$

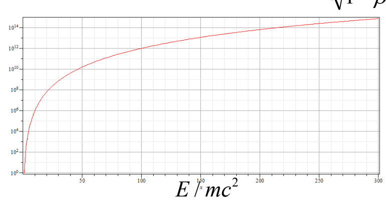


Power from linearly accelerating particle

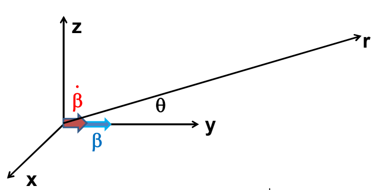
$$\frac{dP_r(t_r)}{d\Omega} = \frac{q^2}{4\pi c}\left.\frac{\left|\hat{\mathbf{R}}\times\left(\hat{\mathbf{R}}\times\dot{\boldsymbol{\beta}}\right)\right|^2}{\left(1-\boldsymbol{\beta}\cdot\hat{\mathbf{R}}\right)^5}\right|_{t_r=t-R/c} = \frac{q^2}{4\pi c^3}|\dot{\mathbf{v}}|^2\frac{\sin^2\theta}{(1-\beta\cos\theta)^5}$$

$$P_r(t_r) = \int\frac{dP_r(t_r)}{d\Omega}d\Omega = \frac{2}{3}\frac{q^2}{c^3}|\dot{\mathbf{v}}|^2\gamma^6 \quad \text{where } \gamma\equiv\frac{1}{\sqrt{1-\beta^2}}$$

$$\frac{P_r(\gamma)}{P_r(1)}$$



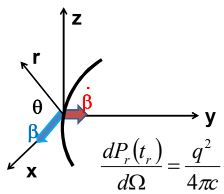
Power distribution for linear acceleration -- continued



$$\frac{dP_r(t_r)}{d\Omega} = \frac{q^2}{4\pi c}\left.\frac{\left|\hat{\mathbf{R}}\times\left(\hat{\mathbf{R}}\times\dot{\boldsymbol{\beta}}\right)\right|^2}{\left(1-\boldsymbol{\beta}\cdot\hat{\mathbf{R}}\right)^5}\right|_{t_r=t-R/c} = \frac{q^2}{4\pi c^3}|\dot{\mathbf{v}}|^2\frac{\sin^2\theta}{(1-\beta\cos\theta)^5}$$

$$P_r(t_r) = \int\frac{dP_r(t_r)}{d\Omega}d\Omega = \frac{2}{3}\frac{q^2}{c^3}|\dot{\mathbf{v}}|^2\gamma^6 \quad \text{where } \gamma\equiv\frac{1}{\sqrt{1-\beta^2}}$$

Power distribution for circular acceleration

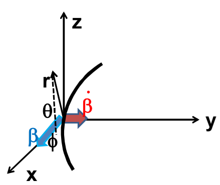


$$\frac{dP_r(t_r)}{d\Omega} = \frac{q^2}{4\pi c} \frac{\left|\hat{\mathbf{R}} \times \left[\left(\hat{\mathbf{R}} - \boldsymbol{\beta}\right) \times \dot{\boldsymbol{\beta}}\right]\right|^2}{\left(1 - \boldsymbol{\beta}\cdot\hat{\mathbf{R}}\right)^5}\Bigg|_{t_r = t - R/c}$$

$$= \frac{q^2}{4\pi c} \frac{\left|\dot{\boldsymbol{\beta}}\right|^2 \left(1 - \boldsymbol{\beta}\cdot\hat{\mathbf{R}}\right)^2 - \left(\hat{\mathbf{R}}\cdot\dot{\boldsymbol{\beta}}\right)^2 \left(1 - \beta^2\right)}{\left(1 - \boldsymbol{\beta}\cdot\hat{\mathbf{R}}\right)^5}\Bigg|_{t_r = t - R/c}$$

$$P_r(t_r) = \int d\Omega \frac{dP_r(t_r)}{d\Omega} = \frac{2}{3}\frac{q^2}{c^3}\left|\dot{\mathbf{v}}\right|^2 \gamma^4$$

Power distribution for circular acceleration



$$\frac{dP_r(t_r)}{d\Omega} = \frac{q^2}{4\pi c} \frac{\left|\dot{\boldsymbol{\beta}}\right|^2 \left(1 - \boldsymbol{\beta}\cdot\hat{\mathbf{R}}\right)^2 - \left(\hat{\mathbf{R}}\cdot\dot{\boldsymbol{\beta}}\right)^2 \left(1 - \beta^2\right)}{\left(1 - \boldsymbol{\beta}\cdot\hat{\mathbf{R}}\right)^5}\Bigg|_{t_r = t - R/c}$$

$$= \frac{q^2}{4\pi c^3} \frac{\left|\dot{\mathbf{v}}\right|^2}{\left(1 - \beta\cos(\theta)\right)^3} \left(1 - \frac{\cos^2\theta \sin^2\phi}{\gamma^2 \left(1 - \beta\cos(\theta)\right)^2}\right)$$

Spectral composition of electromagnetic radiation

Previously we determined the power distribution from a charged particle:

$$\frac{dP(t)}{d\Omega} = \mathbf{S}\cdot\hat{\mathbf{R}}R^2 = \frac{q^2}{4\pi c} \frac{\left|\hat{\mathbf{R}} \times \left[\left(\hat{\mathbf{R}} - \boldsymbol{\beta}\right) \times \dot{\boldsymbol{\beta}}\right]\right|^2}{\left(1 - \boldsymbol{\beta}\cdot\hat{\mathbf{R}}\right)^6}\Bigg|_{t_r = t - R/c}$$

$$\equiv \left|\mathbf{a}(t)\right|^2$$

where $$\mathbf{a}(t) \equiv \sqrt{\frac{q^2}{4\pi c}} \frac{\left|\hat{\mathbf{R}} \times \left[\left(\hat{\mathbf{R}} - \boldsymbol{\beta}\right) \times \dot{\boldsymbol{\beta}}\right]\right|}{\left(1 - \boldsymbol{\beta}\cdot\hat{\mathbf{R}}\right)^3}\Bigg|_{t_r = t - R/c}$$

Time integrated power per solid angle :

$$\frac{dW}{d\Omega} = \int_{-\infty}^{\infty} dt \frac{dP(t)}{d\Omega} = \int_{-\infty}^{\infty} dt \left|\mathbf{a}(t)\right|^2 = \int_{-\infty}^{\infty} d\omega \left|\tilde{\mathbf{a}}(\omega)\right|^2$$

Spectral composition of electromagnetic radiation -- continued

Time integrated power per solid angle:

$$\frac{dW}{d\Omega} = \int_{-\infty}^{\infty} dt \frac{dP(t)}{d\Omega} = \int_{-\infty}^{\infty} dt |\mathbf{a}(t)|^2 = \int_{-\infty}^{\infty} d\omega |\tilde{\mathbf{a}}(\omega)|^2$$

Fourier amplitude:

$$\tilde{\mathbf{a}}(\omega) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt\, \mathbf{a}(t) e^{i\omega t} \qquad \mathbf{a}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega\, \tilde{\mathbf{a}}(\omega) e^{-i\omega t}$$

Parseval's theorem

Marc-Antoine Parseval des Chênes 1755-1836

http://www-history.mcs.st-andrews.ac.uk/Biographies/Parseval.html

Spectral composition of electromagnetic radiation -- continued

Consequences of Parseval's analysis:

$$\frac{dW}{d\Omega} = \int_{-\infty}^{\infty} dt \frac{dP(t)}{d\Omega} = \int_{-\infty}^{\infty} dt |\mathbf{a}(t)|^2 = \int_{-\infty}^{\infty} d\omega |\tilde{\mathbf{a}}(\omega)|^2$$

Note that: $\tilde{\mathbf{a}}(\omega) = \tilde{\mathbf{a}}^*(-\omega)$

$$\frac{dW}{d\Omega} = \int_{-\infty}^{\infty} d\omega |\tilde{\mathbf{a}}(\omega)|^2 = \int_{0}^{\infty} d\omega \left(|\tilde{\mathbf{a}}(\omega)|^2 + |\tilde{\mathbf{a}}(-\omega)|^2 \right) \equiv \int_{0}^{\infty} d\omega \frac{\partial^2 I}{\partial\Omega\partial\omega}$$

$$\frac{\partial^2 I}{\partial\Omega\partial\omega} \equiv 2|\tilde{\mathbf{a}}(\omega)|^2$$

Spectral composition of electromagnetic radiation -- continued

For our case: $\mathbf{a}(t) \equiv \sqrt{\frac{q^2}{4\pi c}} \left. \frac{\left| \hat{\mathbf{R}} \times \left[(\hat{\mathbf{R}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}} \right] \right|}{(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}})^3} \right|_{t_r = t - R/c}$

Fourier amplitude:

$$\tilde{\mathbf{a}}(\omega) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt\, \mathbf{a}(t) e^{i\omega t}$$

$$= \sqrt{\frac{q^2}{8\pi^2 c}} \int_{-\infty}^{\infty} dt \left. \frac{\left| \hat{\mathbf{R}} \times \left[(\hat{\mathbf{R}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}} \right] \right|}{(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}})^3} \right|_{t_r = t - R/c} e^{i\omega t}$$

Spectral composition of electromagnetic radiation -- continued

Fourier amplitude :

$$\tilde{\mathcal{a}}(\omega) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt\, \mathcal{a}(t) e^{i\omega t}$$

$$= \sqrt{\frac{q^2}{8\pi^2 c}} \int_{-\infty}^{\infty} dt \left. \frac{\left|\hat{\mathbf{R}} \times \left[(\hat{\mathbf{R}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}\right]\right|}{(1 - \boldsymbol{\beta}\cdot\hat{\mathbf{R}})^3} \right|_{t_r = t - R/c} e^{i\omega t}$$

$$= \sqrt{\frac{q^2}{8\pi^2 c}} \int_{-\infty}^{\infty} dt_r \frac{dt}{dt_r} \left. \frac{\left|\hat{\mathbf{R}} \times \left[(\hat{\mathbf{R}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}\right]\right|}{(1 - \boldsymbol{\beta}\cdot\hat{\mathbf{R}})^3} \right|_{t_r = t - R/c} e^{i\omega(t_r + R(t_r)/c)}$$

$$= \sqrt{\frac{q^2}{8\pi^2 c}} \int_{-\infty}^{\infty} dt_r \left. \frac{\left|\hat{\mathbf{R}} \times \left[(\hat{\mathbf{R}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}\right]\right|}{(1 - \boldsymbol{\beta}\cdot\hat{\mathbf{R}})^2} \right|_{t_r = t - R/c} e^{i\omega(t_r + R(t_r)/c)}$$

Spectral composition of electromagnetic radiation -- continued

Exact expression :

$$\tilde{\mathcal{a}}(\omega) = \sqrt{\frac{q^2}{8\pi^2 c}} \int_{-\infty}^{\infty} dt_r \left. \frac{\left|\hat{\mathbf{R}} \times \left[(\hat{\mathbf{R}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}\right]\right|}{(1 - \boldsymbol{\beta}\cdot\hat{\mathbf{R}})^2} \right|_{t_r = t - R/c} e^{i\omega(t_r + R(t_r)/c)}$$

Recall : $\dot{\mathbf{R}}_q(t_r) \equiv \frac{d\mathbf{R}_q(t_r)}{dt_r} \equiv \mathbf{v} \quad \mathbf{R}(t_r) \equiv \mathbf{r} - \mathbf{R}_q(t_r) \equiv \mathbf{R}$

For $r >> R_q(t_r) \qquad R(t_r) \approx r - \hat{\mathbf{r}}\cdot\mathbf{R}_q(t_r)$ where $\hat{\mathbf{r}} \equiv \frac{\mathbf{r}}{r}$

At the same level of approximation : $\hat{\mathbf{R}} \approx \hat{\mathbf{r}}$

Spectral composition of electromagnetic radiation -- continued

Exact expression:

$$\tilde{\mathcal{a}}(\omega) = \sqrt{\frac{q^2}{8\pi^2 c}} \int_{-\infty}^{\infty} dt_r \left. \frac{\left|\hat{\mathbf{R}} \times \left[(\hat{\mathbf{R}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}\right]\right|}{(1 - \boldsymbol{\beta}\cdot\hat{\mathbf{R}})^2} \right|_{t_r = t - R/c} e^{i\omega(t_r + R(t_r)/c)}$$

Approximate expression:

$$\tilde{\mathcal{a}}(\omega) = \sqrt{\frac{q^2}{8\pi^2 c}} e^{i\omega(r/c)} \int_{-\infty}^{\infty} dt_r \left. \frac{\left|\hat{\mathbf{r}} \times \left[(\hat{\mathbf{r}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}\right]\right|}{(1 - \boldsymbol{\beta}\cdot\hat{\mathbf{r}})^2} \right|_{t_r = t - R/c} e^{i\omega(t_r - \hat{\mathbf{r}}\cdot\mathbf{R}_q(t_r)/c)}$$

Resulting spectral intensity expression:

$$\frac{\partial^2 I}{\partial\omega\partial\Omega} = \frac{q^2}{4\pi^2 c} \left| \int_{-\infty}^{\infty} dt_r \left. \frac{\left|\hat{\mathbf{r}} \times \left[(\hat{\mathbf{r}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}\right]\right|}{(1 - \boldsymbol{\beta}\cdot\hat{\mathbf{r}})^2} \right|_{t_r = t - R/c} e^{i\omega(t_r - \hat{\mathbf{r}}\cdot\mathbf{R}_q(t_r)/c)} \right|^2$$

Example – radiation from a collinear acceleration burst

$$\frac{\partial^2 I}{\partial\omega\partial\Omega} = \frac{q^2}{4\pi^2 c}\left|\int_{-\infty}^{\infty} dt_r \left.\frac{\left|\hat{\mathbf{r}}\times\left[(\hat{\mathbf{r}}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\right]\right|}{(1-\boldsymbol{\beta}\cdot\hat{\mathbf{r}})^2}\right|_{t_r=t-R/c} e^{i\omega\left(t_r-\hat{\mathbf{r}}\cdot\mathbf{R}_q(t_r)/c\right)}\right|^2$$

Suppose that $\dot{\boldsymbol{\beta}} = \begin{cases} \dfrac{\hat{\boldsymbol{\beta}}\Delta v}{c\tau} & 0 < t_r < \tau \\ 0 & \text{otherwise} \end{cases}$

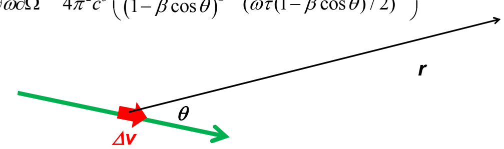
$$\frac{\partial^2 I}{\partial\omega\partial\Omega} = \frac{q^2}{4\pi^2 c^3}\left|\frac{\left|\hat{\mathbf{r}}\times\left[\hat{\mathbf{r}}\times\hat{\boldsymbol{\beta}}\right]\right|\Delta v}{(1-\boldsymbol{\beta}\cdot\hat{\mathbf{r}})^2\,\tau}\right|^2 \left|\int_0^{\tau} dt_r e^{i\omega(t_r-\hat{\mathbf{r}}\cdot\boldsymbol{\beta}t_r)}\right|^2 \quad \text{Let } \boldsymbol{\beta}\cdot\hat{\mathbf{r}} = \beta\cos\theta$$

$$\frac{\partial^2 I}{\partial\omega\partial\Omega} = \frac{q^2}{4\pi^2 c^3}\left(\frac{\Delta v\sin\theta}{(1-\beta\cos\theta)^2}\,\frac{\sin(\omega\tau(1-\beta\cos\theta)/2)}{(\omega\tau(1-\beta\cos\theta)/2)}\right)^2$$

Example:

Suppose that $\dot{\boldsymbol{\beta}} = \begin{cases} \dfrac{\hat{\boldsymbol{\beta}}\Delta v}{c\tau} & 0 < t_r < \tau \\ 0 & \text{otherwise} \end{cases}$

$$\frac{\partial^2 I}{\partial\omega\partial\Omega} = \frac{q^2}{4\pi^2 c^3}\left(\frac{\Delta v\sin\theta}{(1-\beta\cos\theta)^2}\,\frac{\sin(\omega\tau(1-\beta\cos\theta)/2)}{(\omega\tau(1-\beta\cos\theta)/2)}\right)^2$$



Spectral composition of electromagnetic radiation -- continued

Alternative expression --
It can be shown that:

$$\frac{\hat{\mathbf{r}}\times\left[(\hat{\mathbf{r}}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\right]}{(1-\boldsymbol{\beta}\cdot\hat{\mathbf{r}})^2} = \frac{d}{dt_r}\left(\frac{\hat{\mathbf{r}}\times(\hat{\mathbf{r}}\times\boldsymbol{\beta})}{(1-\boldsymbol{\beta}\cdot\hat{\mathbf{r}})}\right)$$

Integration by parts and assumptions about the integration limit behavior shows that the spectral intensity depends on the following integral:

$$\frac{\partial^2 I}{\partial\omega\partial\Omega} = \frac{q^2\omega^2}{4\pi^2 c}\left|\int_{-\infty}^{\infty} dt_r \left[\hat{\mathbf{r}}\times\left(\hat{\mathbf{r}}\times\boldsymbol{\beta}(t_r)\right)\right] e^{i\omega\left(t_r-\hat{\mathbf{r}}\cdot\mathbf{R}_q(t_r)/c\right)}\right|^2$$