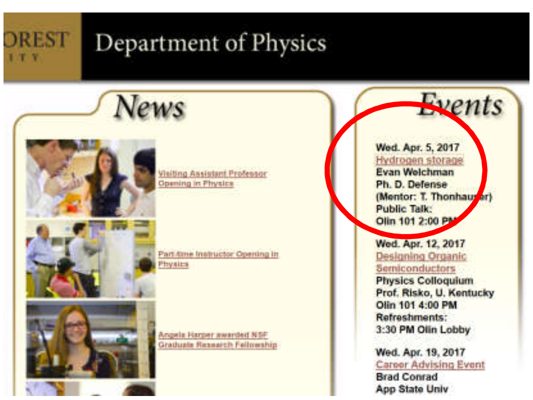
PHY 712 Electrodynamics
9-9:50 AM MWF Olin 103

Plan for Lecture 30:

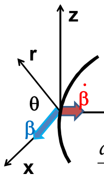
Continue reading Chap. 14 – Synchrotron radiation

1. **Radiation from electron synchrotron devices**
2. **Radiation from astronomical objects in circular orbits**

	Fri: 03/17/2017		*APS Meeting - no class*		
23	Mon: 03/20/2017	Chap. 9	Sources of Radiation	#17	03/24/2017
24	Wed: 03/22/2017	Chap. 9 & 10	Radiation and Scattering		
25	Fri: 03/24/2017	Chap. 9 & 10	Radiation and Scattering	#18	03/27/2017
26	Mon: 03/27/2017	Chap. 11	Special relativity	#19	03/31/2017
27	Wed: 03/29/2017	Chap. 11	Special relativity		
28	Fri: 03/31/2017	Chap. 11	Special relativity	#20	04/3/2017
29	Mon: 04/03/2017	Chap. 14	Radiation from moving charges	#21	04/5/2017
30	Wed: 04/05/2017	Chap. 14	Radiation from moving charges	#22	04/7/2017
31	Fri: 04/07/2017				
32	Mon: 04/10/2017				
33	Wed: 04/12/2017				
	Fri: 04/14/2017		*Good Friday Holiday -- no class*		
34	Mon: 04/17/2017				
35	Wed: 04/19/2017				
36	Fri: 04/21/2017				
	Mon: 04/24/2017		Presentations I		
	Wed: 04/26/2017		Presentations II		



Radiation from charged particle in circular path



Power distribution for circular acceleration

$$\frac{dP_r(t_r)}{d\Omega} = \frac{q^2}{4\pi c} \frac{\left|\hat{\mathbf{R}} \times \left[\left(\hat{\mathbf{R}} - \boldsymbol{\beta}\right) \times \dot{\boldsymbol{\beta}}\right]\right|^2}{\left(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}}\right)^5}\Bigg|_{t_r = t - R/c}$$

$$= \frac{q^2}{4\pi c} \frac{\left|\dot{\boldsymbol{\beta}}\right|^2 \left(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}}\right)^2 - \left(\hat{\mathbf{R}} \cdot \dot{\boldsymbol{\beta}}\right)^2 \left(1 - \beta^2\right)}{\left(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}}\right)^5}\Bigg|_{t_r = t - R/c}$$

$$P_r(t_r) = \int d\Omega \frac{dP_r(t_r)}{d\Omega} = \frac{2}{3} \frac{q^2}{c^3} \left|\dot{\mathbf{v}}\right|^2 \gamma^4$$

Spectral composition of electromagnetic radiation -- continued

When the dust clears, the spectral intensity depends on the following integral :

$$\frac{\partial^2 I}{\partial \omega \partial \Omega} = \frac{q^2 \omega^2}{4\pi^2 c} \left| \int_{-\infty}^{\infty} dt_r \left[\hat{\mathbf{r}} \times \left(\hat{\mathbf{r}} \times \boldsymbol{\beta}(t_r)\right)\right] e^{i\omega\left(t_r - \hat{\mathbf{r}} \cdot \mathbf{R}_q(t_r)/c\right)} \right|^2$$

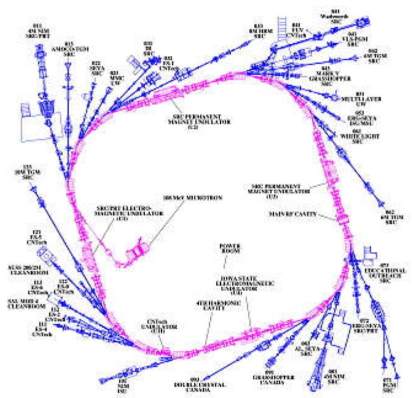
Synchrotron radiation light source installations

Synchrotron radiation center in Madison, Wisconsin

E_c=0.5 GeV and 1 GeV; λ_c = 20 Å and 10 Å

http://www.src.wisc.edu/

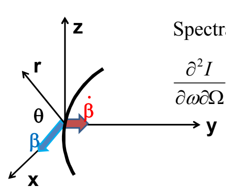
SRC –
“Aladdin” --
Madison
Wisconsin



Brookhaven National Laboratory – National Light Source

E_c=0.6 GeV; λ_c = 20 Å

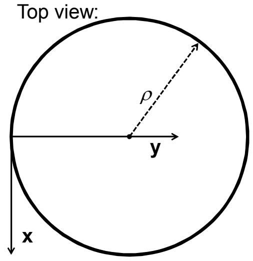
http://www.bnl.gov/ps/



Spectral intensity relationship:

$$\frac{\partial^2 I}{\partial\omega\partial\Omega} = \frac{q^2\omega^2}{4\pi^2 c}\left|\int_{-\infty}^{\infty} dt_r\left[\hat{\mathbf{r}}\times\left(\hat{\mathbf{r}}\times\boldsymbol{\beta}(t_r)\right)\right]e^{i\omega\left(t_r-\hat{\mathbf{r}}\cdot\mathbf{R}_q(t_r)/c\right)}\right|^2$$

Top view:

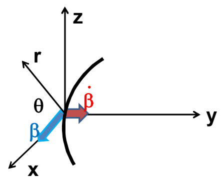


$$\mathbf{R}_q(t_r) = \rho\hat{\mathbf{x}}\sin(vt_r/\rho) + \rho\hat{\mathbf{y}}\left(1-\cos(vt_r/\rho)\right)$$

$$\boldsymbol{\beta}(t_r) = \beta\left(\hat{\mathbf{x}}\cos(vt_r/\rho)+\hat{\mathbf{y}}\sin(vt_r/\rho)\right)$$

For convenience, choose:

$$\hat{\mathbf{r}} = \hat{\mathbf{x}}\cos\theta + \hat{\mathbf{z}}\sin\theta$$



$$\mathbf{R}_q(t_r) = \rho\hat{\mathbf{x}}\sin(vt_r/\rho) + \rho\hat{\mathbf{y}}(1-\cos(vt_r/\rho))$$

$$\boldsymbol{\beta}(t_r) = \beta(\hat{\mathbf{x}}\cos(vt_r/\rho) + \hat{\mathbf{y}}\sin(vt_r/\rho))$$

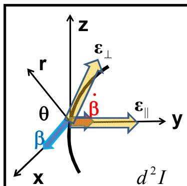
For convenience, choose:

$$\hat{\mathbf{r}} = \hat{\mathbf{x}}\cos\theta + \hat{\mathbf{z}}\sin\theta$$

Note that we have previous shown that in the radiation zone, the Poynting vector is in the $\hat{\mathbf{r}}$ direction; we can then choose to analyze two orthogonal polarization directions:

$$\boldsymbol{\varepsilon}_{\parallel} = \hat{\mathbf{y}} \qquad \boldsymbol{\varepsilon}_{\perp} = -\hat{\mathbf{x}}\sin\theta + \hat{\mathbf{z}}\cos\theta$$

$$\hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \boldsymbol{\beta}) = \beta\left(-\boldsymbol{\varepsilon}_{\parallel}\sin(vt_r/\rho) + \boldsymbol{\varepsilon}_{\perp}\sin\theta\cos(vt_r/\rho)\right)$$



$$\boldsymbol{\varepsilon}_{\parallel} = \hat{\mathbf{y}} \qquad \boldsymbol{\varepsilon}_{\perp} = -\hat{\mathbf{x}}\sin\theta + \hat{\mathbf{z}}\cos\theta$$

$$\hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \boldsymbol{\beta}) = \beta\left(-\boldsymbol{\varepsilon}_{\parallel}\sin(vt_r/\rho) + \boldsymbol{\varepsilon}_{\perp}\sin\theta\cos(vt_r/\rho)\right)$$

$$\frac{d^2 I}{d\omega d\Omega} = \frac{q^2\omega^2}{4\pi^2 c}\left|\int_{-\infty}^{\infty} \hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \beta) e^{i\omega(t - \hat{\mathbf{r}}\cdot\mathbf{R}_q(t)/c)} dt\right|^2$$

$$\frac{d^2 I}{d\omega d\Omega} = \frac{q^2\omega^2\beta^2}{4\pi^2 c}\left\{|C_{\parallel}(\omega)|^2 + |C_{\perp}(\omega)|^2\right\}$$

$$C_{\parallel}(\omega) = \int_{-\infty}^{\infty} dt \sin(vt/\rho) e^{i\omega(t - \frac{\rho}{c}\cos\theta\sin(vt/\rho))}$$

$$C_{\perp}(\omega) = \int_{-\infty}^{\infty} dt \sin\theta\cos(vt/\rho) e^{i\omega(t - \frac{\rho}{c}\cos\theta\sin(vt/\rho))}$$

We will analyze this expression for two different cases. The first case, is appropriate for man-made synchrotrons used as light sources. In this case, the light is produced by short bursts of electrons moving close to the speed of light ($v \approx c(1 - 1/(2\gamma^2))$) passing a beam line port. In addition, because of the design of the radiation ports, $\theta \approx 0$, and the relevant integration times t are close to $t \approx 0$. This results in the form shown in Eq. 14.79 of your text. It is convenient to rewrite this form in terms of a critical frequency $\omega_c \equiv \frac{3c\gamma^3}{2\rho}$.

$$\frac{d^2 I}{d\omega d\Omega} = \frac{3q^2\gamma^2}{4\pi^2 c}\left(\frac{\omega}{\omega_c}\right)^2 (1+\gamma^2\theta^2)^2 \left\{\left[K_{2/3}\left(\frac{\omega}{2\omega_c}(1+\gamma^2\theta^2)^{\frac{3}{2}}\right)\right]^2 + \frac{\gamma^2\theta^2}{1+\gamma^2\theta^2}\left[K_{1/3}\left(\frac{\omega}{2\omega_c}(1+\gamma^2\theta^2)^{\frac{3}{2}}\right)\right]^2\right\}$$

Some details:

Modified Bessel functions

$$K_{1/3}(\xi)=\sqrt{3}\int_0^\infty dx\cos\left[\tfrac{3}{2}\xi\left(x+\tfrac{1}{3}x^3\right)\right] \qquad K_{2/3}(\xi)=\sqrt{3}\int_0^\infty dx\, x\sin\left[\tfrac{3}{2}\xi\left(x+\tfrac{1}{3}x^3\right)\right]$$

Exponential factor

$$\omega\left(t_r-\hat{\mathbf{r}}\cdot\mathbf{R}_q(t_r)\right)=\omega\left(t_r-\frac{\rho}{c}\cos\theta\sin(vt_r/\rho)\right)$$

$$\text{In the limit of}\quad t_r\approx 0,\;\; \theta\approx 0,\;\; v\approx c\left(1-\frac{1}{2\gamma^2}\right)$$

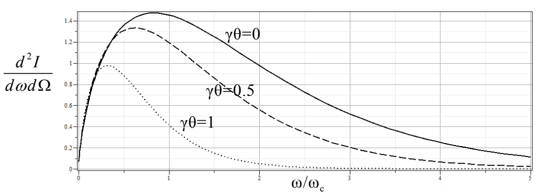
$$\omega\left(t_r-\hat{\mathbf{r}}\cdot\mathbf{R}_q(t_r)\right)\approx\frac{\omega t_r}{2\gamma^2}\left(1+\gamma^2\theta^2\right)+\frac{\omega c^2 t_r^{\,3}}{6\rho^2}=\frac{3}{2}\xi\left(x+\frac{1}{3}x^3\right)$$

$$\text{where}\quad \xi=\frac{\omega\rho}{3c\gamma^3}\left(1+\gamma^2\theta^2\right)^{3/2}\quad\text{and}\quad x=\frac{c\gamma t_r}{\rho\left(1+\gamma^2\theta^2\right)^{1/2}}$$

$$\frac{d^2I}{d\omega d\Omega}=\frac{3q^2\gamma^2}{4\pi^2c}\left(\frac{\omega}{\omega_c}\right)^2(1+\gamma^2\theta^2)^2\left\{\left[K_{2/3}\left(\frac{\omega}{2\omega_c}(1+\gamma^2\theta^2)^{\frac{3}{2}}\right)\right]^2\right.$$

$$\left.+\frac{\gamma^2\theta^2}{1+\gamma^2\theta^2}\left[K_{1/3}\left(\frac{\omega}{2\omega_c}(1+\gamma^2\theta^2)^{\frac{3}{2}}\right)\right]^2\right\}$$

By plotting the intensity as a function of ω, we see that the intensity is largest near $\omega \approx \omega_c$. The plot below shows the intensity as a function of ω/ω_c for $\gamma\theta=0$, 0.5 and 1:



The second example of synchroton radiation comes from a distant charged particle moving in a circular trajectory such that the spectrum represents a superposition of light generated over many complete circles. In this case, there is an interference effect which results in the spectrum consisting of discrete multiples of v/ρ. For this case we need to reconsider the analysis. There is a very convenient Bessel function identity of the form:

$$e^{-ia\sin\alpha}=\sum_{m=-\infty}^{\infty}J_m(a)e^{-im\alpha}\quad \text{Here } J_m(a) \text{ is a Bessel function of integer order } m.$$

In our case $a=\frac{\omega\rho}{c}\cos\theta$ and $\alpha=\frac{vt}{\rho}$.

$$C_{\parallel}(\omega)=\int_{-\infty}^{\infty}dt\sin(vt/\rho)e^{i\omega(t-\frac{\rho}{c}\cos\theta\sin(vt/\rho))}=\frac{c}{-i\omega\rho}\frac{\partial}{\partial\cos\theta}\int_{-\infty}^{\infty}dt e^{i\omega(t-\frac{\rho}{c}\cos\theta\sin(vt/\rho))}$$

$$=\frac{c}{-i\omega\rho}\frac{\partial}{\partial\cos\theta}\sum_{m=-\infty}^{\infty}J_m\left(\frac{\omega\rho}{c}\cos\theta\right)2\pi\delta(\omega-m\frac{v}{\rho}).$$

Astronomical synchrotron radiation -- continued:

Note that:

$$\int_{-\infty}^{\infty} dt e^{i(\omega - m\frac{v}{\rho})t} = 2\pi\delta(\omega - m\frac{v}{\rho}).$$

$$\Rightarrow C_{\parallel}(\omega) = 2\pi i \sum_{m=-\infty}^{\infty} J_m'\left(\frac{\omega\rho}{c}\cos\theta\right)\delta(\omega - m\frac{v}{\rho}),$$

where $J_m'(a) \equiv \frac{dJ_m(a)}{da}$

Similarly:

$$C_{\perp}(\omega) = \int_{-\infty}^{\infty} dt \sin\theta\cos(vt/\rho) e^{i\omega(t - \frac{\rho}{c}\cos\theta\sin(vt/\rho))}$$

$$= 2\pi\frac{\tan\theta}{v/c}\sum_{m=-\infty}^{\infty} J_m\left(\frac{\omega\rho}{c}\cos\theta\right)\delta(\omega - m\frac{v}{\rho}).$$

Astronomical synchrotron radiation -- continued:

In both of the expressions, the sum over *m* includes both negative and positive values. However, only the positive values of ω and therefore positive values of *m* are of interest. Using the identity: $J_{-m}(a) = (-1)^m J_m(a),$ the result becomes:

$$\frac{d^2 I}{d\omega d\Omega} =$$

$$\frac{q^2\omega^2\beta^2}{c}\sum_{m=0}^{\infty}\delta(\omega - m\frac{v}{\rho})\left\{\left[J_m'\left(\frac{\omega\rho}{c}\cos\theta\right)\right]^2 + \frac{\tan^2\theta}{v^2/c^2}\left[J_m\left(\frac{\omega\rho}{c}\cos\theta\right)\right]^2\right\}.$$

These results were derived by Julian Schwinger (Phys. Rev. **75**, 1912-1925 (1949)). The discrete case is similar to the result quoted in Problem 14.15 in Jackson's text. For information on man-made synchrotron sources, the following web page is useful: http://www.als.lbl.gov/als/synchrotron_sources.html.