PHY 712 Electrodynamics
9-9:50 AM MWF Olin 103

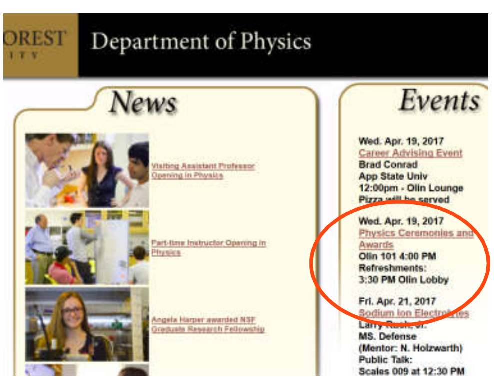
Plan for Lecture 35:

Special Topics in Electrodynamics:

Electromagnetic aspects of superconductivity

- Tunneling between two superconductors

	Fri: 03/17/2017		APS Meeting -- no class		
23	Mon: 03/20/2017	Chap. 9	Sources of Radiation	#17	03/24/2017
24	Wed: 03/22/2017	Chap. 9 & 10	Radiation and Scattering		
25	Fri: 03/24/2017	Chap. 9 & 10	Radiation and Scattering	#18	03/27/2017
26	Mon: 03/27/2017	Chap. 11	Special relativity	#19	03/31/2017
27	Wed: 03/29/2017	Chap. 11	Special relativity		
28	Fri: 03/31/2017	Chap. 11	Special relativity	#20	04/3/2017
29	Mon: 04/03/2017	Chap. 14	Radiation from moving charges	#21	04/5/2017
30	Wed: 04/05/2017	Chap. 14	Radiation from moving charges	#22	04/7/2017
31	Fri: 04/07/2017	Chap. 14	Radiation from moving charges	#23	04/10/2017
32	Mon: 04/10/2017	Chap. 15	Radiation from collisions		
33	Wed: 04/12/2017	Chap. 13	Cherenkov radiation		
	Fri: 04/14/2017		*Good Friday Holiday -- no class*		
34	Mon: 04/17/2017		Superconductivity		
	Wed: 04/19/2017		Superconductivity		
36	Fri: 04/21/2017		Review		
	Mon: 04/24/2017		Presentations I		
	Wed: 04/26/2017		Presentations II		



Review: London model of superconducting state

Recall Drude model of conductivity in "normal" materials having mobile charges of mass m and charge $q = e$:

$$m\frac{d\mathbf{v}}{dt} = -e\mathbf{E} - m\frac{\mathbf{v}}{\tau}$$

$$\mathbf{v}(t) = \mathbf{v}_0 e^{-t/\tau} - \frac{e\mathbf{E}\tau}{m}$$

$$\mathbf{J} = -ne\mathbf{v} \qquad \text{for } t >> \tau \qquad \mathbf{J} = \frac{ne^2\tau}{m}\mathbf{E} \equiv \sigma\mathbf{E}$$

Suppose $\tau \to 0$; London model of conductivity in superconducting materials

$$m\frac{d\mathbf{v}}{dt} = -e\mathbf{E}$$

$$\frac{d\mathbf{v}}{dt} = -\frac{e\mathbf{E}}{m} \qquad \frac{d\mathbf{J}}{dt} = -ne\frac{d\mathbf{v}}{dt} = \frac{ne^2\mathbf{E}}{m}$$

From Maxwell's equations:

$$\nabla\times\mathbf{B} = \frac{4\pi}{c}\mathbf{J} + \frac{1}{c}\frac{\partial\mathbf{E}}{\partial t} \qquad \nabla\times\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}$$

London model, continued

London model of conductivity in superconducting materials

$$\frac{d\mathbf{J}}{dt} = -ne\frac{d\mathbf{v}}{dt} = \frac{ne^2\mathbf{E}}{m}$$

From Maxwell's equations:

$$\nabla\times\mathbf{B} = \frac{4\pi}{c}\mathbf{J} + \frac{1}{c}\frac{\partial\mathbf{E}}{\partial t} \qquad \nabla\times\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}$$

$$\nabla\times(\nabla\times\mathbf{B}) = -\nabla^2\mathbf{B} = \frac{4\pi}{c}\nabla\times\mathbf{J} - \frac{1}{c^2}\frac{\partial^2\mathbf{B}}{\partial t^2}$$

$$-\nabla^2\frac{\partial\mathbf{B}}{\partial t} = \frac{4\pi}{c}\nabla\times\frac{\partial\mathbf{J}}{\partial t} - \frac{1}{c^2}\frac{\partial^3\mathbf{B}}{\partial t^3}$$

$$-\nabla^2\frac{\partial\mathbf{B}}{\partial t} = \frac{4\pi ne^2}{mc}\nabla\times\mathbf{E} - \frac{1}{c^2}\frac{\partial^3\mathbf{B}}{\partial t^3}$$

$$-\nabla^2\frac{\partial\mathbf{B}}{\partial t} = -\frac{4\pi ne^2}{mc^2}\frac{\partial\mathbf{B}}{\partial t} - \frac{1}{c^2}\frac{\partial^3\mathbf{B}}{\partial t^3}$$

$$\frac{\partial}{\partial t}\left(\nabla^2 - \frac{1}{\lambda_L^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{B} = 0 \qquad \text{with } \lambda_L^2 \equiv \frac{mc^2}{4\pi ne^2}$$

London model – continued

London model of conductivity in superconducting materials

$$\frac{d\mathbf{J}}{dt} = -ne\frac{d\mathbf{v}}{dt} = \frac{ne^2\mathbf{E}}{m}$$

$$\frac{\partial}{\partial t}\left(\nabla^2 - \frac{1}{\lambda_L^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{B} = 0 \qquad \text{with } \lambda_L^2 \equiv \frac{mc^2}{4\pi ne^2}$$

For slowly varying solution:

$$\frac{\partial}{\partial t}\left(\nabla^2 - \frac{1}{\lambda_L^2}\right)\mathbf{B} = 0 \qquad \text{for } \frac{\partial\mathbf{B}}{\partial t} = \hat{\mathbf{z}}\frac{\partial B_z(x,t)}{\partial t}:$$

$$\Rightarrow \frac{\partial B_z(x,t)}{\partial t} = \frac{\partial B_z(0,t)}{\partial t}e^{-x/\lambda_L}$$

London leap: $B_z(x,t) = B_z(0,t)e^{-x/\lambda_L}$

Consistent results for current density:

$$\frac{4\pi}{c}\nabla\times\mathbf{J} = -\nabla^2\mathbf{B} = -\frac{1}{\lambda_L^2}\mathbf{B} \qquad \mathbf{J} = \hat{\mathbf{y}}J_y(x) \quad \Rightarrow \quad J_y(x) = \lambda_L\frac{ne^2}{mc}\mathrm{B}_z(0)\mathrm{e}^{-x/\lambda_L}$$

Behavior of superconducting material – exclusion of magnetic field according to the London model

Penetration length for superconductor: $\lambda_L^2 \equiv \frac{mc^2}{4\pi n e^2}$

$B_z(x,t) = B_z(0,t)e^{-x/\lambda_L}$

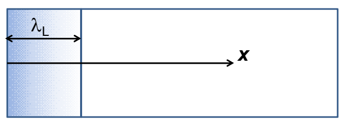
Vector potential for $\nabla \cdot \mathbf{A} = 0$:

$\mathbf{A} = \hat{\mathbf{y}} A_y(x) \qquad A_y(x) = -\lambda_L B_z(0) e^{-x/\lambda_L}$

Current density: $J_y(x) = \lambda_L \frac{ne^2}{mc} \mathrm{B}_z(0) \mathrm{e}^{-x/\lambda_L}$

$\Rightarrow \mathbf{J} + \frac{ne^2}{mc}\mathbf{A} = 0 \quad \text{or} \quad \frac{ne}{m}\left(m\mathbf{v} + \frac{e}{c}\mathbf{A}\right) = 0$

Typically, $\lambda_L \approx 10^{-7} m$



Quantization of current flux associated with the superconducting state (Ref: Ashcroft and Mermin, ***Solid State Physics***)

From the London equations for the interior of the superconductor:

$$\left(m\mathbf{v} + \frac{e}{c}\mathbf{A}\right) = 0$$

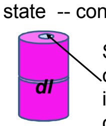
Now suppose that the current carrier is a pair of electrons characterized by a wavefunction of the form $\psi = |\psi| e^{i\phi}$

The quantum mechanical current associated with the electron pair is

$$\mathbf{j} = -\frac{e\hbar}{2mi}\left(\psi^* \nabla \psi - \psi \nabla \psi^*\right) - \frac{2e^2}{mc}\mathbf{A}|\psi|^2$$

$$= -\left(\frac{e\hbar}{m}\nabla\phi + \frac{2e^2}{mc}\mathbf{A}\right)|\psi|^2$$

Quantization of current flux associated with the superconducting state -- continued



Suppose a superconducting material has a cylindrical void. Evaluate the integral of the current in a closed path within the superconductor containing the void.

$$\oint \mathbf{j} \cdot d\mathbf{l} = 0 = -\oint \left(\frac{e\hbar}{m}\nabla\phi + \frac{2e^2}{mc}\mathbf{A}\right)|\psi|^2 \cdot d\mathbf{l}$$

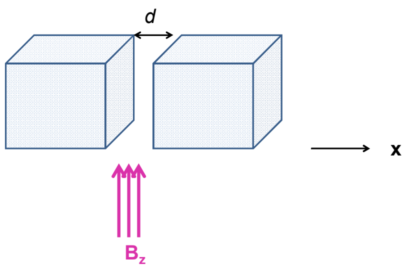
$$\oint \mathbf{A} \cdot d\mathbf{l} = \int (\nabla \times \mathbf{A}) \cdot d\mathbf{a} = \int \mathbf{B} \cdot d\mathbf{a} = \Phi \quad \text{magnetic flux}$$

$$\oint \nabla\phi \cdot d\mathbf{l} = 2\pi n \quad \text{for some integer } n$$

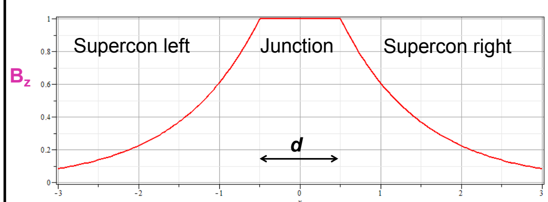
$$\Rightarrow \text{Quantization of flux in the void: } |\Phi| = n\frac{hc}{2e} \equiv n\Phi_0$$

Such "vortex" fields can exist within type II superconductors.

Josephson junction -- tunneling current between two superconductors (Ref. Teplitz, ***Electromagnetism*** (1982))

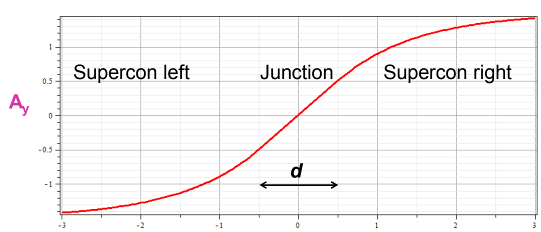


Josephson junction -- continued



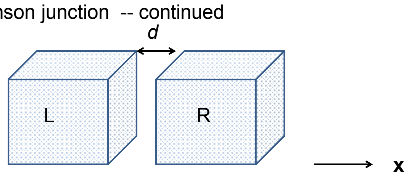
$$B_z(x) = \begin{cases} B_0 e^{(x+d/2)/\lambda_L} & x < -d/2 \\ B_0 & -d/2 < x < d/2 \\ B_0 e^{(-x+d/2)/\lambda_L} & x > d/2 \end{cases}$$

Josephson junction -- continued



$$A_y(x) = \begin{cases} B_0\left(\lambda_L e^{(x+d/2)/\lambda_L} - \left(\lambda_L + d/2\right)\right) & x < -d/2 \\ B_0 x & -d/2 < x < d/2 \\ B_0\left(-\lambda_L e^{(-x+d/2)/\lambda_L} + \left(\lambda_L + d/2\right)\right) & x > d/2 \end{cases}$$

Josephson junction -- continued



Quantum mechanical model of tunnelling current

Let $\Psi_L = \Psi_L{}^0 e^{i\phi_L}$ denote a wavefunction for a Cooper pair on left

Let $\Psi_R = \Psi_R{}^0 e^{i\phi_R}$ denote a wavefunction for a Cooper pair on right

$$-i\hbar \frac{\partial \Psi_L}{\partial t} = E_L \Psi_L + \varepsilon \Psi_R$$

$$-i\hbar \frac{\partial \Psi_R}{\partial t} = E_R \Psi_R + \varepsilon \Psi_L$$

Coupling parameter (ε)

Josephson junction -- continued

Solving for wavefunctions

$$\frac{1}{2}\frac{\partial \left|\Psi_L{}^0\right|^2}{\partial t} + i\left|\Psi_L{}^0\right|^2 \frac{\partial \phi_L}{\partial t} = -\frac{i}{\hbar}\left(E_L \left|\Psi_L{}^0\right|^2 + \varepsilon \Psi_L{}^0 \Psi_R{}^0 e^{i(\phi_R - \phi_L)}\right)$$

$$\frac{1}{2}\frac{\partial \left|\Psi_R{}^0\right|^2}{\partial t} + i\left|\Psi_R{}^0\right|^2 \frac{\partial \phi_R}{\partial t} = -\frac{i}{\hbar}\left(E_R \left|\Psi_R{}^0\right|^2 + \varepsilon \Psi_L{}^0 \Psi_R{}^0 e^{-i(\phi_R - \phi_L)}\right)$$

$$\left|\Psi_L{}^0\right|^2 \equiv n_L \qquad \left|\Psi_R{}^0\right|^2 \equiv n_R \qquad \phi_{LR} \equiv \phi_L - \phi_R$$

$$\frac{\partial n_L}{\partial t} = -\frac{\partial n_R}{\partial t} = -\frac{2\varepsilon}{\hbar}\sqrt{(n_L n_R)}\sin\phi_{LR}$$

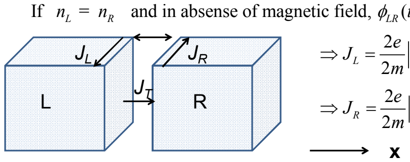
$$\frac{\partial \phi_L}{\partial t} = -\frac{E_L}{\hbar} - \varepsilon\sqrt{\frac{n_R}{n_L}}\cos\phi_{LR}$$

$$\frac{\partial \phi_R}{\partial t} = -\frac{E_R}{\hbar} - \varepsilon\sqrt{\frac{n_L}{n_R}}\cos\phi_{LR}$$

Josephson junction -- continued

Tunneling current: $J_T = 2e\frac{\partial n_L}{\partial t} = -\frac{4e\varepsilon}{\hbar}\sqrt{(n_L n_R)}\sin\phi_{LR}$

If $n_L = n_R$ and in absense of magnetic field, $\phi_{LR}(t) = \phi_{LR}(0) + \frac{E_R - E_L}{\hbar}t$



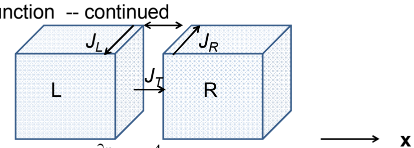
$$\Rightarrow J_L = \frac{2e}{2m}\left|\Psi_L{}^0\right|^2\left(\hbar\nabla\phi_L - \frac{2e}{c}\mathbf{A}\right)$$

$$\Rightarrow J_R = \frac{2e}{2m}\left|\Psi_R{}^0\right|^2\left(\hbar\nabla\phi_R - \frac{2e}{c}\mathbf{A}\right)$$

Relationship between superconductor currents J_L and J_R and tunneling current. Within the superconductor, denote the generalize current operator acting on pair wavefunction $\Psi = \Psi^0 e^{i\phi}$

$$\hat{\mathbf{v}} \equiv \frac{1}{2m}\left(-i\hbar\nabla - \frac{2e}{c}\mathbf{A}\right) \quad \text{with current } J = \frac{2e}{2}\left(\Psi^*\left(\hat{\mathbf{v}}\Psi\right) + \Psi\left(\hat{\mathbf{v}}\Psi\right)^*\right)$$

Josephson junction -- continued



Tunneling current: $J_T = 2e\frac{\partial n_L}{\partial t} = -\frac{4e\varepsilon}{\hbar}\sqrt{(n_L n_R)}\sin\phi_{LR}$

If $n_L = n_R = n$ and in absense of magnetic field, $\phi_{LR}(t) = \phi_{LR}(0) + \frac{E_R - E_L}{\hbar}t$

$\Rightarrow$ Constant Josephson tunneling current for $E_R - E_L = 0$

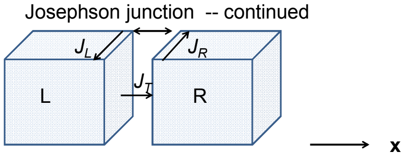
$$\mathrm{J}_T = -\frac{4e\varepsilon}{\hbar}n\sin\phi_{LR}(0)$$

$\Rightarrow$ Oscillatory Josephson tunneling current for $E_R - E_L = 2eV$

$$\mathrm{J}_T = -\frac{4e\varepsilon}{\hbar}n\sin\left(\phi_{LR}(0) + \frac{2eV}{\hbar}t\right)$$

Method for precise measurement of $e/\hbar$

Josephson junction -- continued



$$\Rightarrow J_L = \frac{2e}{2m}\left|\Psi_L{}^0\right|^2\left(\hbar\nabla\phi_L - \frac{2e}{c}\mathbf{A}\right) \equiv 2en_L\mathbf{v}_L$$

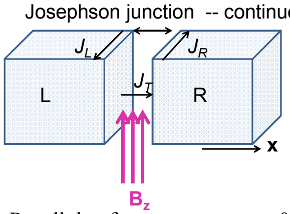
$$\Rightarrow J_R = \frac{2e}{2m}\left|\Psi_R{}^0\right|^2\left(\hbar\nabla\phi_R - \frac{2e}{c}\mathbf{A}\right) \equiv 2en_R\mathbf{v}_R$$

$$\nabla\phi_L = \frac{2m\mathbf{v}_L}{\hbar} + \frac{2e}{\hbar c}\mathbf{A} \qquad \nabla\phi_R = \frac{2m\mathbf{v}_R}{\hbar} + \frac{2e}{\hbar c}\mathbf{A}$$

Tunneling current: $J_T = 2e\frac{\partial n_L}{\partial t} = -\frac{4e\varepsilon}{\hbar}\sqrt{(n_L n_R)}\sin\phi_{LR}$

Need to evaluate ϕ_{LR} in presence of magnetic field

Josephson junction -- continued



Tunneling current:

$$J_T = -\frac{4e\varepsilon}{\hbar}\sqrt{(n_L n_R)}\sin\phi_{LR}$$

$$\mathbf{v}_L = \left(\frac{\hbar}{2m}\nabla\phi_L - \frac{2e}{2mc}\mathbf{A}\right)$$

$$\mathbf{v}_R = \left(\frac{\hbar}{2m}\nabla\phi_R - \frac{2e}{2mc}\mathbf{A}\right)$$

Recall that for $x \to -\infty$ $\mathbf{v}_L \to 0$ and $\mathbf{A} \to -(\lambda_L + d/2)B_0\hat{\mathbf{y}}$

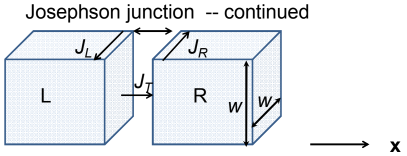
for $x \to \infty$ $\mathbf{v}_R \to 0$ and $\mathbf{A} \to (\lambda_L + d/2)B_0\hat{\mathbf{y}}$

Integrating the difference of the phase angles along y:

$$\phi_{LR} \equiv \phi_L\left(-\tfrac{d}{2}, y\right) - \phi_L\left(-\tfrac{d}{2}, 0\right) - \phi_R\left(\tfrac{d}{2}, y\right) + \phi_R\left(\tfrac{d}{2}, 0\right)$$

$$= \phi_{LR}^0 + \frac{2e}{\hbar c}B_0(2\lambda_L + d)y$$

Josephson junction -- continued



Integrating the difference of the phase angles along y:

$$\phi_{LR} = \phi_{LR}^0 + \frac{2e}{\hbar c} B_0 (2\lambda_L + d) y$$

Tunneling current density: $J_T = \frac{4e\varepsilon}{\hbar} n_L \sin \phi_{LR}$

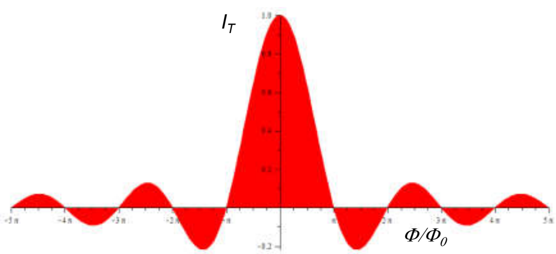
Integrating current density throughout width w of superconductors

$$I_T = w \int_{-w/2}^{w/2} J_T dy = w^2 J_{T0} \sin(\phi_{LR}^0) \frac{\sin(\pi\Phi / \Phi^0)}{\pi\Phi / \Phi^0}$$

where $\Phi = B_0 w (2\lambda_L + d)$ and $\Phi^0 = \frac{2\pi\hbar c}{2e}$

Josephson junction -- continued

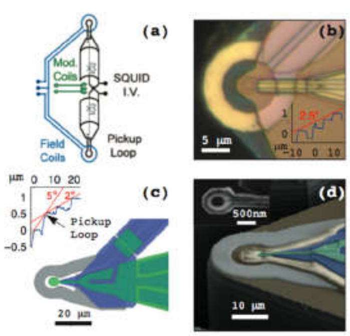
SQUID =*superconducting quantum interference device*



Note: This very sensitive "SQUID" technology has been used in scanning probe techniques. See for example, J. R. Kirtley, Rep. Prog. Physics 73, 126501 (2010).

Scanning SQIUD microscopy

Ref. J. R. Kirtley, Rep. Prog. Phys. **73** 126501 (2010)



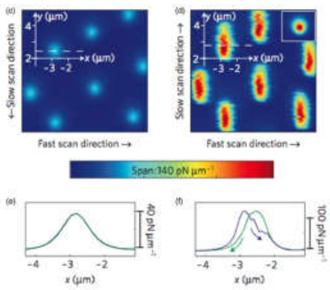


Figure 31. MFM imaging and manipulation of individual vortices in YBCO at $T = 22.3$ K. (a), (b) Schematic drawings of an MFM tip (triangle) that attracts a vortex (thick lines) in a sample with randomly distributed pinning sites (dots): at large heights (a) the force between tip and vortex is too weak to move the vortex; at small heights (b) the vortex moves right and then left as the tip rasters over it. (c) MFM image taken at $z = 420$ nm (maximum applied lateral force $F^{lat}_{max} \approx 6$ pN). (d) $z = 170$ nm ($F^{lat}_{max} \approx 12$ pN). Inset: scan taken at a comparable height at $T = 5.2$ K. (e) Line cut through the data in (c) along the dashed line, showing the signal from a stationary vortex (solid curve with peak to the left, blue online). Overlapping it is a line cut from the reverse scan (solid curve with peak to the right, green online). (f) Line cut through the data in (d) along the dashed line, showing a typical signal from a dragged vortex. Figure reprinted by permission from [312]. Copyright 2008 Macmillan Publishers Ltd.

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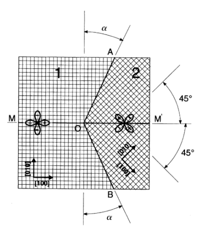

Figure 1 Schematic diagram illustrating the geometrical configuration of a tetracrystal (100) $SrTiO_3$ substrate, which effectively consists of two crystals rotated about the normal to the substrate plane by $\pi/4$ with respect to each other. For a given total misorientation angle $\theta = \theta_1 + \theta_2$, the pair tunnelling current is maximized[9] for a symmetric grain boundary $\theta_1 = \theta_2 = \pi/8$. We chose the angle $\alpha = \theta_1$ between the vertical ([010] in grain 1) and the grain boundaries (OA and OB) as 25°. Also shown are the polar plots of the assumed $d_{x^2-y^2}$ gap functions aligned with the crystallographic axes in the substrate.

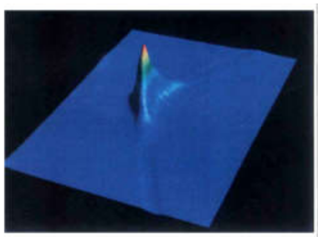

Figure 3 A three-dimensional presentation of the SSM image of the Josephson vortex trapped at the wedge tip in a Tl2201 blanket film deposited on a tetracrystal (100) $SrTiO_3$ substrate of the geometry shown in Fig. 1. The sample was cooled in a field of <1 mG and imaged at 4.2 K.

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