PHY 745 Group Theory
11-11:50 AM MWF Olin 102

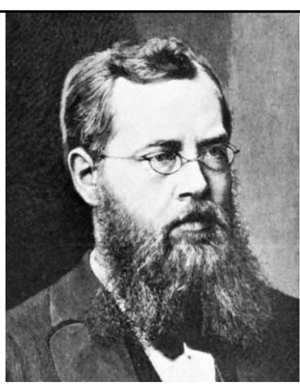
Plan for Lecture 31:

Introduction to linear Lie groups

1. Definitions
2. Properties
3. Examples

Ref. J. F. Cornwell, Group Theory in Physics, Vol I and II, Academic Press (1984)

23	Mon: 03/20/2017	Chap. 7.7	Jahn-Teller Effect	#15	03/24/2017
24	Wed: 03/22/2017	Chap. 7.7	Jahn-Teller Effect		
25	Fri: 03/24/2017		Spin 1/2	#16	03/27/2017
26	Mon: 03/27/2017		Dirac equation for H-like atoms	#17	03/29/2017
27	Wed: 03/29/2017	Chap. 14	Angular momenta	#18	03/31/2017
28	Fri: 03/31/2017	Chap. 16	Time reversal symmetry	#19	04/05/2017
29	Mon: 04/03/2017	Chap. 16	Magnetic point groups		
30	Wed: 04/05/2017	Literature	Topology and group theory in Bloch states	#20	04/07/2017
31	Fri: 04/07/2017		Introduction to Lie groups	#21	04/10/2017
32	Mon: 04/10/2017				
33	Wed: 04/12/2017				
	Fri: 04/14/2017		*Good Friday Holiday -- no class*		
34	Mon: 04/17/2017				
35	Wed: 04/19/2017				
36	Fri: 04/21/2017				
	Mon: 04/24/2017		Presentations I		
	Wed: 04/26/2017		Presentations II		



Sophus Lie
1842-1899
Norweigian mathematician

https://www.britannica.com/biography/Sophus-Lie

Definition of a linear Lie group

1. A linear Lie group is a group
   - Each element of the group *T* forms a member of the group *T"* when "multiplied" by another member of the group *T"=T·T'*
   - One of the elements of the group is the identity *E*
   - For each element of the group *T,* there is a group member a group member T^{-1} such that $T \cdot T^{-1}=E$.
   - Associative property: *T·(T'·T")= (T·T')·T"*
2. Elements of group form a "topological space"
3. Elements also constitute an "analytic manifold"

➔Non countable number elements lying in a region "near" its identity

Definition: Linear Lie group of dimension *n*
A group G is a linear Lie group of dimension *n* if it satisfied the following four conditions:

1. G must have at least one faithful finite-dimensional represention Γ which defines the notion of distance.

For represent Γ having dimension m, the distance between two group elements T and T' can be defined:

$$d(T,T') \equiv \left\{ \sum_{j=1}^{m} \sum_{k=1}^{m} \left| \Gamma(T)_{jk} - \Gamma(T')_{jk} \right|^2 \right\}^{1/2}$$

Note that $d(T,T')$ has the following properties

(i) $d(T,T') = d(T',T)$

(ii) $d(T,T) = 0$

(iii) $d(T,T') > 0$ if $T \neq T'$

(iv) For elements T, T', and T'',

$$d(T,T') \leq d(T,T'') + d(T',T'')$$

Definition: Linear Lie group of dimension *n* -- continued

2. Consider the distance between group elements *T* with respect to the identity *E* -- *d(T,E)*. It is possible to define a sphere M_δ that constains all elements *T'* such that $d(E,T') \leq \delta$.

It follows that there must exist a $\delta > 0$ such that every T' of G lying in the sphere M_δ can be parameterized by n real parameters $x_1, x_2, \ldots x_n$ such each T' has a different set of parameters and for E the parameters are $x_1 = 0, x_2 = 0, \ldots x_n = 0$

Definition: Linear Lie group of dimension *n* -- continued

3. There must exist

$\eta > 0$ such that for every parameter set $\{x_1, x_2, ...x_n\}$ corresponding to T' in the sphere M_δ :

$$\sum_{j=1}^{n} x_j^2 < \eta^2$$

4. There is a requirement that the corresponding representation is analytic

For element T' within M_δ, $\Gamma(T'(x_1, x_2, ..x_n))$ must be an analytic (polynomial) function of $x_1, x_2, ...x_n$.

Example: G is the group of all real numbers $t \neq 0$ under multiplication. $E = 1$

$\Rightarrow$ Faithful representation $\Gamma(t) = t \quad n = 1$

$\Rightarrow d(t, t') = |t - t'|$

For $\delta = \frac{1}{2}$ and parameterization $t = e^{x_1}$

$d(t, 1) = |t - 1|$

$M_\delta \quad \Rightarrow \frac{1}{2} < t < \frac{3}{2}$

For $\eta = \ln\left(\frac{3}{2}\right) \qquad x_1^2 < \left(\ln\left(\frac{3}{2}\right)\right)^2$

Representation:

$$\Gamma(t(x_1)) = e^{x_1}$$

Example: G is the group SU(2) of all 2×2 unitary matrices having determinant 1.

An element of the group has the form:

$$\mathbf{u} = \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix} \quad \text{with } |\alpha|^2 + |\beta|^2 = 1$$

In terms of the real numbers $\alpha_1, \alpha_2, \beta_1, \beta_2$:

$$\mathbf{u} = \begin{pmatrix} \alpha_1 + i\alpha_2 & \beta_1 + i\beta_2 \\ -(\beta_1 - i\beta_2) & \alpha_1 - i\alpha_2 \end{pmatrix}$$

3-dimensional mapping:

$$\beta_2 = \tfrac{1}{2}x_1 \quad \beta_1 = \tfrac{1}{2}x_2 \quad \alpha_2 = \tfrac{1}{2}x_3 \quad \alpha_1 = \left(1 - \tfrac{1}{4}\left(x_1^2 + x_2^2 + x_3^2\right)\right)^{1/2}$$

Example: G is the group SU(2) -- continued

It can be shown that

$$d(\mathbf{u},1) = 2\left(1-\left(1-\tfrac{1}{4}\left(x_1^2+x_2^2+x_3^2\right)\right)^{1/2}\right)^{1/2}$$

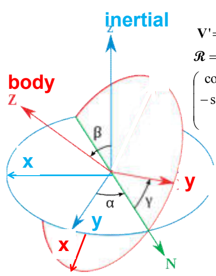
$$d(\mathbf{u},1) < \delta$$

$$\left(x_1^2+x_2^2+x_3^2\right)^{1/2} < \left(2\delta^2 - \tfrac{1}{4}\delta^4\right)^{1/2} \equiv \eta$$

Note that $\delta < \sqrt{8}$

Example: G is the group SO(3), describing rotations in 3-dimensional space

General transformation between rotated coordinates – Euler angles



$$\mathbf{V}' = \mathcal{R}\mathbf{V} = \mathcal{R}_\alpha \mathcal{R}_\beta \mathcal{R}_\gamma \mathbf{V}$$

$$\mathcal{R} = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{pmatrix} \begin{pmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

http://en.wikipedia.org/wiki/Euler_angles

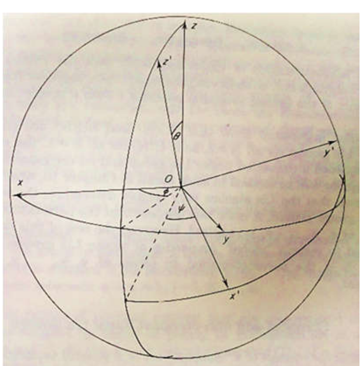


Figure 3.1 The Eulerian angles θ, ψ and φ.

Homomorphic mapping of SU(2) onto SO(3)

Suppose $\mathbf{u}$ denotes an element of the group SU(2)

It is possible to find a corresponding element $\mathbf{R}(\mathbf{u})$=$\mathbf{R}(-\mathbf{u})$ of the group SO(3).

Result: $\mathbf{R}(\mathbf{u})_{jk}=\frac{1}{2}\,\mathrm{tr}\left(\sigma_j\mathbf{u}\sigma_k\mathbf{u}^{-1}\right)$

where $\sigma_1=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ $\sigma_2=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}$ $\sigma_3=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$

Some justification

Consider $\mathbf{m}(\mathbf{r})\equiv\begin{pmatrix}z & x-iy\\ x+iy & -z\end{pmatrix}$

Note that for a 2×2 unitary matrix $\mathbf{u}$ (which is an element of SU(2))

$$\mathbf{u}\mathbf{m}(\mathbf{r})\mathbf{u}^{-1}=\mathbf{m}(\mathbf{r}')=\begin{pmatrix}z' & x'-iy'\\ x'+iy' & -z'\end{pmatrix}$$

Need to show that $\mathbf{r}\leftrightarrow\mathbf{r}'$ corresponding to $\mathbf{r}'=R(\mathbf{u})\mathbf{r}$

Recall:

In terms of the real numbers $\alpha_1,\alpha_2,\beta_1,\beta_2$:

$$\mathbf{u}=\begin{pmatrix}\alpha_1+i\alpha_2 & \beta_1+i\beta_2\\ -(\beta_1-i\beta_2) & \alpha_1-i\alpha_2\end{pmatrix}$$

3-dimensional mapping:

$$\beta_2=\tfrac{1}{2}x_1 \quad \beta_1=\tfrac{1}{2}x_2 \quad \alpha_2=\tfrac{1}{2}x_3 \quad \alpha_1=\left(1-\tfrac{1}{4}\left(x_1^2+x_2^2+x_3^2\right)\right)^{1/2}$$

Alternatively define angles:

$0\le\theta\le\pi \quad 0\le\psi\le 4\pi \quad 0\le\phi\le 2\pi$

$$\mathbf{u}=\begin{pmatrix}\cos\left(\tfrac{1}{2}\theta\right)e^{i\frac{1}{2}(\psi+\phi)} & \sin\left(\tfrac{1}{2}\theta\right)e^{i\frac{1}{2}(\psi-\phi)}\\ -\sin\left(\tfrac{1}{2}\theta\right)e^{-i\frac{1}{2}(\psi-\phi)} & \cos\left(\tfrac{1}{2}\theta\right)e^{-i\frac{1}{2}(\psi+\phi)}\end{pmatrix}$$

Mapping the 3 angles to the Euler angles, find rotation matrix

$$\mathrm{R}(\theta,\psi,\phi)=\begin{pmatrix}-\sin\phi\sin\psi+\cos\theta\cos\phi\cos\psi & \cos\phi\sin\psi+\cos\theta\sin\phi\cos\psi & -\sin\theta\cos\psi\\ -\sin\phi\cos\psi-\cos\theta\cos\phi\sin\psi & \cos\phi\cos\psi-\cos\theta\sin\phi\sin\psi & \sin\theta\sin\psi\\ \sin\theta\cos\phi & \sin\theta\sin\psi & \cos\theta\end{pmatrix}$$

Recall: $\mathbf{R}(\mathbf{u})_{jk}=\frac{1}{2}\,\mathrm{tr}\left(\sigma_j\mathbf{u}\sigma_k\mathbf{u}^{-1}\right)$

where $\sigma_1=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ $\sigma_2=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}$ $\sigma_3=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$

$$\mathbf{u}=\begin{pmatrix}\cos\left(\tfrac{1}{2}\theta\right)e^{i\frac{1}{2}(\psi+\phi)} & \sin\left(\tfrac{1}{2}\theta\right)e^{i\frac{1}{2}(\psi-\phi)}\\ -\sin\left(\tfrac{1}{2}\theta\right)e^{-i\frac{1}{2}(\psi-\phi)} & \cos\left(\tfrac{1}{2}\theta\right)e^{-i\frac{1}{2}(\psi+\phi)}\end{pmatrix}$$

Check:

$$\mathbf{R}(\mathbf{u})_{33}=\frac{1}{2}\mathrm{tr}\left(\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\cos\left(\frac{1}{2}\theta\right)e^{i\frac{1}{2}(\psi+\phi)} & \sin\left(\frac{1}{2}\theta\right)e^{i\frac{1}{2}(\psi-\phi)}\\ -\sin\left(\frac{1}{2}\theta\right)e^{-i\frac{1}{2}(\psi-\phi)} & \cos\left(\frac{1}{2}\theta\right)e^{-i\frac{1}{2}(\psi+\phi)}\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\cos\left(\frac{1}{2}\theta\right)e^{-i\frac{1}{2}(\psi+\phi)} & -\sin\left(\frac{1}{2}\theta\right)e^{i\frac{1}{2}(\psi-\phi)}\\ \sin\left(\frac{1}{2}\theta\right)e^{-i\frac{1}{2}(\psi-\phi)} & \cos\left(\frac{1}{2}\theta\right)e^{i\frac{1}{2}(\psi+\phi)}\end{pmatrix}\right)$$

$$=\frac{1}{2}\mathrm{tr}\left(\begin{pmatrix}\cos\left(\frac{1}{2}\theta\right)e^{i\frac{1}{2}(\psi+\phi)} & \sin\left(\frac{1}{2}\theta\right)e^{i\frac{1}{2}(\psi-\phi)}\\ \sin\left(\frac{1}{2}\theta\right)e^{-i\frac{1}{2}(\psi-\phi)} & -\cos\left(\frac{1}{2}\theta\right)e^{-i\frac{1}{2}(\psi+\phi)}\end{pmatrix}\begin{pmatrix}\cos\left(\frac{1}{2}\theta\right)e^{-i\frac{1}{2}(\psi+\phi)} & -\sin\left(\frac{1}{2}\theta\right)e^{i\frac{1}{2}(\psi-\phi)}\\ -\sin\left(\frac{1}{2}\theta\right)e^{-i\frac{1}{2}(\psi-\phi)} & -\cos\left(\frac{1}{2}\theta\right)e^{i\frac{1}{2}(\psi+\phi)}\end{pmatrix}\right)$$

$$\mathbf{R}(\mathbf{u})_{33}=\cos\theta$$

Note that $R(\mathbf{u}) \leftrightarrow \mathbf{u}(R)$

$\mathbf{u}(R)$ is a "two-valued" two-dimensional representation of SO(3) since $\mathbf{u}(R)$ and $-\mathbf{u}(R)$ are both valid mappings.

$\mathbf{u}(R_1)\mathbf{u}(R_2) = \pm\mathbf{u}(R_1 R_2)$