PHY 745 Group Theory
11-11:50 AM MWF Olin 102

Plan for Lecture 32:

Introduction to linear Lie groups

1. **Notion of linear Lie group**
2. **Notion of corresponding Lie algebra**
3. **Examples**

Ref. J. F. Cornwell, Group Theory in Physics, Vol I and II, Academic Press (1984)

23	Mon: 03/20/2017	Chap. 7.7	Jahn-Teller Effect	#15	03/24/2017
24	Wed: 03/22/2017	Chap. 7.7	Jahn-Teller Effect		
25	Fri: 03/24/2017		Spin 1/2	#16	03/27/2017
26	Mon: 03/27/2017		Dirac equation for H-like atoms	#17	03/29/2017
27	Wed: 03/29/2017	Chap. 14	Angular momenta	#18	03/31/2017
28	Fri: 03/31/2017	Chap. 16	Time reversal symmetry	#19	04/05/2017
29	Mon: 04/03/2017	Chap. 16	Magnetic point groups		
30	Wed: 04/05/2017	Literature	Topology and group theory in Bloch states	#20	04/07/2017
31	Fri: 04/07/2017		Introduction to Lie groups	#21	04/10/2017
32	Mon: 04/10/2017		Introduction to Lie groups		
33	Wed: 04/12/2017		Introduction to Lie groups		
	Fri: 04/14/2017		*Good Friday Holiday -- no class*		
34	Mon: 04/17/2017				
35	Wed: 04/19/2017				
36	Fri: 04/21/2017				
	Mon: 04/24/2017		Presentations I		
	Wed: 04/26/2017		Presentations II		

Definition of a linear Lie group

1. A linear Lie group is a group
   - Each element of the group T forms a member of the group T'' when "multiplied" by another member of the group $T''=T\cdot T'$
   - One of the elements of the group is the identity E
   - For each element of the group T, there is a group member a group member T^{-1} such that $T\cdot T^{-1}=E$.
   - Associative property: $T\cdot(T'\cdot T'')= (T\cdot T')\cdot T''$
2. Elements of group form a "topological space"
3. Elements also constitute an "analytic manifold"

➔Non countable number elements lying in a region "near" its identity

Definition: Linear Lie group of dimension *n*

A group G is a linear Lie group of dimension *n* if it satisfied the following four conditions:

1. G must have at least one faithful finite-dimensional representation Γ which defines the notion of distance.

For represent Γ having dimension m, the distance between two group elements T and T' can be defined:

$$d(T,T') \equiv \left\{ \sum_{j=1}^{m} \sum_{k=1}^{m} \left| \Gamma(T)_{jk} - \Gamma(T')_{jk} \right|^2 \right\}^{1/2}$$

Note that $d(T,T')$ has the following properties

(i) $d(T,T') = d(T',T)$

(ii) $d(T,T) = 0$

(iii) $d(T,T') > 0$ if $T \neq T'$

(iv) For elements T, T', and T'',

$$d(T,T') \leq d(T,T'') + d(T',T'')$$

Definition: Linear Lie group of dimension *n* -- continued

2. Consider the distance between group elements *T* with respect to the identity *E* -- *d(T,E)*. It is possible to define a sphere M_δ that contains all elements *T'* such that $d(E,T') \leq \delta$.

It follows that there must exist a $\delta > 0$ such that every T' of G lying in the sphere M_δ can be parameterized by n real parameters $x_1, x_2, \ldots x_n$ such each T' has a different set of parameters and for E the parameters are $x_1 = 0, x_2 = 0, \ldots x_n = 0$

Definition: Linear Lie group of dimension *n* -- continued

3. There must exist $\eta > 0$ such that for every parameter set $\{x_1, x_2, \ldots x_n\}$ corresponding to T' in the sphere M_δ:

$$\sum_{j=1}^{n} x_j^2 < \eta^2$$

4. There is a requirement that the corresponding representation is analytic

For element T' within M_δ, $\Gamma(T'(x_1, x_2, \ldots x_n))$ must be an analytic (polynomial) function of $x_1, x_2, \ldots x_n$.

Example: G is the group SU(2) of all 2×2 unitary matrices having determinant 1.

An element of the group has the form:

$$\mathbf{u}=\begin{pmatrix}\alpha & \beta \\ -\beta^* & \alpha^*\end{pmatrix} \quad \text{with } |\alpha|^2+|\beta|^2=1$$

In terms of the real numbers $\alpha_1,\alpha_2,\beta_1,\beta_2$:

$$\mathbf{u}=\begin{pmatrix}\alpha_1+i\alpha_2 & \beta_1+i\beta_2 \\ -(\beta_1-i\beta_2) & \alpha_1-i\alpha_2\end{pmatrix}$$

3-dimensional mapping:

$$\beta_2=\tfrac{1}{2}x_1 \quad \beta_1=\tfrac{1}{2}x_2 \quad \alpha_2=\tfrac{1}{2}x_3 \quad \alpha_1=\left(1-\tfrac{1}{4}\left(x_1^2+x_2^2+x_3^2\right)\right)^{1/2}$$

Example: G is the group SU(2) -- continued

It can be shown that

$$d(\mathbf{u},1)=2\left(1-\left(1-\tfrac{1}{4}\left(x_1^2+x_2^2+x_3^2\right)\right)^{1/2}\right)^{1/2}$$

$$d(\mathbf{u},1)<\delta$$

$$\left(x_1^2+x_2^2+x_3^2\right)^{1/2}<\left(2\delta^2-\tfrac{1}{4}\delta^4\right)^{1/2}\equiv\eta$$

Note that $\delta<\sqrt{8}$

Alternatively define angles:

$0\le\theta\le\pi \quad 0\le\psi\le 4\pi \quad 0\le\phi\le 2\pi$

$$\mathbf{u}=\begin{pmatrix}\cos\left(\tfrac{1}{2}\theta\right)e^{i\frac{1}{2}(\psi+\phi)} & \sin\left(\tfrac{1}{2}\theta\right)e^{i\frac{1}{2}(\psi-\phi)} \\ -\sin\left(\tfrac{1}{2}\theta\right)e^{-i\frac{1}{2}(\psi-\phi)} & \cos\left(\tfrac{1}{2}\theta\right)e^{-i\frac{1}{2}(\psi+\phi)}\end{pmatrix}$$

Some more details

4. There is a requirement that the corresponding representation is analytic

For element T' within M_δ, $\Gamma(T'(x_1,x_2,..x_n))$ must be an analytic (polynomial) function of $x_1,x_2,....x_n$.

Because of the mapping to the n parameters $x_1,x_2...x_n$ to each group element T', $\Gamma(T'(x_1,x_2...x_n))=\Gamma(x_1,x_2...x_n)$.

The analytic property of $\Gamma(x_1,x_2...x_n)$ also means that derivatives

$$\frac{\partial^\alpha\Gamma_{jk}(x_1,x_2...x_n)}{\partial^\alpha x_p} \text{ must exist for all } \alpha=1,2,...$$

Define n $m\times m$ matrices:

$$\left(\mathbf{a}_p\right)_{jk}\equiv\left.\frac{\partial\Gamma_{jk}(x_1,x_2...x_n)}{\partial x_p}\right|_{x_1=0,x_2=0,...x_n=0}$$

Example: G is the group SU(2) of all 2×2 unitary matrices having determinant 1.

In terms of the real numbers $\alpha_1, \alpha_2, \beta_1, \beta_2$:

$$\mathbf{u} = \begin{pmatrix} \alpha_1 + i\alpha_2 & \beta_1 + i\beta_2 \\ -(\beta_1 - i\beta_2) & \alpha_1 - i\alpha_2 \end{pmatrix}$$

3-dimensional mapping:

$$\beta_2 = \tfrac{1}{2}x_1 \quad \beta_1 = \tfrac{1}{2}x_2 \quad \alpha_2 = \tfrac{1}{2}x_3 \quad \alpha_1 = \left(1 - \tfrac{1}{4}\left(x_1^2 + x_2^2 + x_3^2\right)\right)^{1/2}$$

$$\Gamma(x_1,x_2,x_3) = \begin{pmatrix} \left(1-\frac{1}{4}\left(x_1^2+x_2^2+x_3^2\right)\right)^{1/2} + \frac{1}{2}ix_3 & \frac{1}{2}(x_2 + ix_1) \\ -\frac{1}{2}(x_2 - ix_1) & \left(1-\frac{1}{4}\left(x_1^2+x_2^2+x_3^2\right)\right)^{1/2} - \frac{1}{2}ix_3 \end{pmatrix}$$

Example: G is the group SU(2) of all 2×2 unitary matrices having determinant 1 -- continued

$$\Gamma(x_1,x_2,x_3) = \begin{pmatrix} \left(1-\frac{1}{4}\left(x_1^2+x_2^2+x_3^2\right)\right)^{1/2} + \frac{1}{2}ix_3 & \frac{1}{2}(x_2 + ix_1) \\ -\frac{1}{2}(x_2 - ix_1) & \left(1-\frac{1}{4}\left(x_1^2+x_2^2+x_3^2\right)\right)^{1/2} - \frac{1}{2}ix_3 \end{pmatrix}$$

$$\mathbf{a}_1 \equiv \left.\frac{\partial\Gamma}{\partial x_1}\right|_{x_1=0,x_2=0,x_3=0} = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad \mathbf{a}_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \mathbf{a}_3 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

It can be shown that the matrices $\mathbf{a}_p$ form the basis for an *n*-dimensional vector space.

Correspondence between a linear Lie group and its corresponding Lie algebra

Definition: For any matrix M, the matrix exponential function is defined as follows: $e^M \equiv \sum_{j=0}^{\infty} \frac{M^j}{j!}$

Some theorems

If M and N are $m\times m$ which commute $(MN = NM)$ then $e^M e^N = e^N e^M = e^{(M+N)}$

If M and N are $m\times m$ where $MN \neq NM$, but the entries are sufficiently small, then

$e^M e^N = e^O$ where

$$O = M + N + \tfrac{1}{2}[M,N] + \tfrac{1}{12}\left([M,[M,N]] + [N,[N,M]]\right) + \ldots$$

If M and N are $m \times m$ where $MN \neq NM$, but the entries are sufficiently small, then

$e^M e^N = e^O$ where

$$O = M + N + \tfrac{1}{2}[M,N] + \tfrac{1}{12}\left([M,[M,N]] + [N,[N,M]]\right) + ...$$

Note that the last result is attributed to Campbell-Baker-Hausdorff formula.

Note that the matrix exponential function has some very convenient properties:

Inverse: $\left(e^M\right)^{-1} = e^{-M}$

Similarity transformations: $Se^M S^{-1} = e^{SMS^{-1}}$

Definition: Real Lie algebra

A real Lie algebra of dimension $n \geq 1$ is a real vector space of dimension n which includes a comutator $[M,N]$ as follows:

1. For all M, N in algebra, $[M,N]$ is also in algebra
2. For real numbers α and β, and members M, N, O, $[\alpha M + \beta N, O] = \alpha[M,O] + \beta[N,O]$
3. $[M,N] = -[N,M]$
4. $[M,[N,O]] + [N,[O,M]] + [O,[M,N]] = 0$

Structure constants of Lie algebra

Consider the n basis matrices of the algebra $\mathbf{a}_1, \mathbf{a}_2, ...\mathbf{a}_n$:

$$[\mathbf{a}_p, \mathbf{a}_q] = \sum_{r=1}^{n} c_{pq}^r \mathbf{a}_r \quad \text{for } p,q=1,2 \text{ ... n}$$

Example: G is the group SU(2) of all 2×2 unitary matrices having determinant 1

$$\mathbf{a}_1 = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad \mathbf{a}_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \mathbf{a}_3 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

Structure constants for this case:

$[\mathbf{a}_1, \mathbf{a}_2] = -\mathbf{a}_3$ $\quad [\mathbf{a}_2, \mathbf{a}_3] = -\mathbf{a}_1$ $\quad [\mathbf{a}_3, \mathbf{a}_1] = -\mathbf{a}_2$

Fundamental theorem –

For every linear Lie group there exisits a corresponding real Lie algebra of the same dimension. For example if the linear Lie group has dimension *n* and has *mxm* matrices $\mathbf{a}_1, \mathbf{a}_2, \ldots \mathbf{a}_n$ then these matrices form a basis for the real Lie algebra.