PHY 745 Group Theory
11-11:50 AM MWF Olin 102

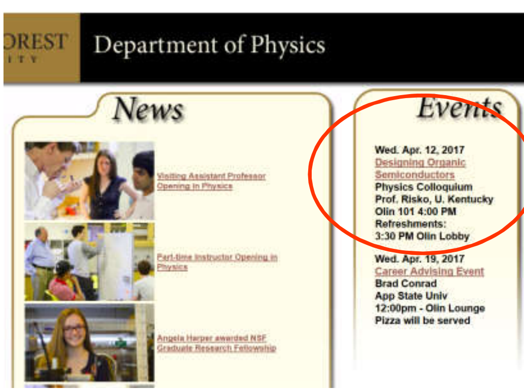
Plan for Lecture 33:

Introduction to linear Lie groups -- continued

1. Linear Lie group and real Lie algebra
2. Fundamental theorem
3. Examples

Ref. J. F. Cornwell, Group Theory in Physics, Vol I and II, Academic Press (1984)

23	Mon: 03/20/2017	Chap. 7.7	Jahn-Teller Effect	#15	03/24/2017
24	Wed: 03/22/2017	Chap. 7.7	Jahn-Teller Effect		
25	Fri: 03/24/2017		Spin 1/2	#16	03/27/2017
26	Mon: 03/27/2017		Dirac equation for H-like atoms	#17	03/29/2017
27	Wed: 03/29/2017	Chap. 14	Angular momenta	#18	03/31/2017
28	Fri: 03/31/2017	Chap. 16	Time reversal symmetry	#19	04/05/2017
29	Mon: 04/03/2017	Chap. 16	Magnetic point groups		
30	Wed: 04/05/2017	Literature	Topology and group theory in Bloch states	#20	04/07/2017
31	Fri: 04/07/2017		Introduction to Lie groups	#21	04/10/2017
32	Mon: 04/10/2017		Introduction to Lie groups		
33	Wed: 04/12/2017		Introduction to Lie groups		
	Fri: 04/14/2017		*Good Friday Holiday -- no class*		
34	Mon: 04/17/2017				
35	Wed: 04/19/2017				
36	Fri: 04/21/2017				
	Mon: 04/24/2017		Presentations I		
	Wed: 04/26/2017		Presentations II		



Mon. April 17, 2017 -- David Harrison, WFU (Ph. D. Thesis; Mentor: T. Thonhauser) Improving Existing and Discovering New Hydrogen Storage Materials Using Computational Materials Modeling Note: Public talk will begin at 12:30 PM in ZSR 204.

Mon. April 17, 2017 -- Ryan Godwin, WFU (Ph. D. Thesis; Mentor: F. Salsbury) Binding NEMO: Adventures in Molecular Dynamics Note: Public talk will begin at 3:00 PM in Olin 101.

Wed. April 19, 2017 -- Honors presentations Part I --

Fri. April 21, 2017 -- Larry Rush, WFU (MS. Thesis; Mentor: N. Holzwarth) Note: Public talk will begin at 12:30 PM in Scales 009.

Mon. April 24, 2017 -- Xiaohua (Nina) Liu, WFU (Ph. D. Thesis; Mentor: D. Kim-Shapiro) "Effects of Red Blood Cells on Nitric Oxide Bioactivity" Note: Public talk will begin at 10:00 AM in ZSR 204.

Wed. Apr. 26, 2017 -- Honors presentations Part II --

Thur. April 27, 2017 -- Crystal Bolden, WFU (Ph. D. Thesis; Mentor: D. Kim-Shapiro) "Interaction between RSNO and H_2S:The formation, stability, and NO-donating capacity of SSNO⁻ and the effects of SSNO⁻ on platelet activation" Note: Public talk will begin at 9:00 AM in ??

Definition of a linear Lie group

1. A linear Lie group is a group
   - Each element of the group *T* forms a member of the group *T"* when "multiplied" by another member of the group *T"=T·T'*
   - One of the elements of the group is the identity *E*
   - For each element of the group *T,* there is a group member a group member T^{-1} such that $T \cdot T^{-1} = E$.
   - Associative property: *T·(T'·T")= (T·T')·T"*
2. Elements of group form a "topological space"
3. Elements also constitute an "analytic manifold"

➔Non countable number elements lying in a region "near" its identity

Definition: Linear Lie group of dimension *n*

A group G is a linear Lie group of dimension *n* if it satisfied the following four conditions:

1. G must have at least one faithful finite-dimensional representation Γ which defines the notion of distance.

For represent Γ having dimension m, the distance between two group elements T and T' can be defined:

$$d(T,T') \equiv \left\{ \sum_{j=1}^{m} \sum_{k=1}^{m} \left| \Gamma(T)_{jk} - \Gamma(T')_{jk} \right|^2 \right\}^{1/2}$$

Note that $d(T,T')$ has the following properties

(i) $d(T,T') = d(T',T)$

(ii) $d(T,T) = 0$

(iii) $d(T,T') > 0$ if $T \neq T'$

(iv) For elements T, T', and T'',

$$d(T,T') \leq d(T,T'') + d(T',T'')$$

Definition: Linear Lie group of dimension *n* -- continued

2. Consider the distance between group elements *T* with respect to the identity *E* -- *d(T,E)*. It is possible to define a sphere M_δ that contains all elements *T'* such that $d(E,T') \leq \delta$.

   It follows that there must exist a $\delta > 0$ such that every T' of G lying in the sphere M_δ can be parameterized by n real parameters $x_1, x_2, ...x_n$ such each T' has a different set of parameters and for E the parameters are $x_1 = 0, x_2 = 0, ...x_n = 0$

Definition: Linear Lie group of dimension *n* -- continued

3. There must exist $\eta > 0$ such that for every parameter set $\{x_1, x_2, ...x_n\}$ corresponding to T' in the sphere M_δ:

$$\sum_{j=1}^{n} x_j^2 < \eta^2$$

4. There is a requirement that the corresponding representation is analytic

   For element T' within M_δ, $\Gamma(T'(x_1, x_2, ..x_n))$ must be an analytic (polynomial) function of $x_1, x_2, ...x_n$.

Some more details

4. There is a requirement that the corresponding representation is analytic

   For element T' within M_δ, $\Gamma(T'(x_1, x_2, ..x_n))$ must be an analytic (polynomial) function of $x_1, x_2, ...x_n$.

   Because of the mapping to the n parameters $x_1, x_2 ... x_n$ to each group element T', $\Gamma(T'(x_1, x_2 ... x_n)) = \Gamma(x_1, x_2 ... x_n)$.

   The analytic property of $\Gamma(x_1, x_2 ... x_n)$ also means that derivatives

$$\frac{\partial^\alpha \Gamma_{jk}(x_1, x_2 ... x_n)}{\partial^\alpha x_p} \text{ must exist for all } \alpha = 1, 2, ...$$

   Define n $m \times m$ matrices:

$$\left(\mathbf{a}_p\right)_{jk} \equiv \left.\frac{\partial \Gamma_{jk}(x_1, x_2 ... x_n)}{\partial x_p}\right|_{x_1=0, x_2=0, ... x_n=0}$$

Definition: Real Lie algebra

A real Lie algebra of dimension $n \geq 1$ is a real vector space of dimension n which includes a comutator $[M, N]$ as follows:

1. For all M, N in algebra, $[M, N]$ is also in algebra
2. For real numbers α and β, and members M, N, O, $[\alpha M + \beta N, O] = \alpha[M, O] + \beta[N, O]$
3. $[M, N] = -[N, M]$
4. $[M,[N,O]] + [N,[O,M]] + [O,[M,N]] = 0$

Structure constants of Lie algebra

Consider the n basis matrices of the algebra $\mathbf{a}_1, \mathbf{a}_2, \ldots \mathbf{a}_n$:

$$[\mathbf{a}_p, \mathbf{a}_q] = \sum_{r=1}^{n} c_{pq}^r \mathbf{a}_r \quad \text{for } p,q\text{=1,2 ... n}$$

Example: G is the group SU(2) of all 2×2 unitary matrices having determinant 1

$$\mathbf{a}_1 = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad \mathbf{a}_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \mathbf{a}_3 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

Structure constants for this case:

$[\mathbf{a}_1, \mathbf{a}_2] = -\mathbf{a}_3 \qquad [\mathbf{a}_2, \mathbf{a}_3] = -\mathbf{a}_1 \qquad [\mathbf{a}_3, \mathbf{a}_1] = -\mathbf{a}_2$

Fundamental theorem –

For every linear Lie group there exisits a corresponding real Lie algebra of the same dimension. For example if the linear Lie group has dimension *n* and has *mxm* matrices $\mathbf{a}_1, \mathbf{a}_2, \ldots \mathbf{a}_n$ then these matrices form a basis for the real Lie algebra.

One-parameter subgroup of a linear Lie group

A "one-parameter subgroup" of a linear Lie group fullfills the requirements of a Lie group as a subgroup whose elements $T(t)$ depend on a real parameter t with $-\infty < t < \infty$ where

$$T(s)T(t) = T(s+t)$$

Note that $T(s)T(t) = T(t)T(s)$ so that the subgroup is Abelian. It follows that $T(t=0) = E$

Theorem: Every one-parameter subgroup of a linear Lie group of $m \times m$ matrices is formed by exponentiation of $m \times m$ matrices. $\mathbf{A}(t) = e^{t\mathbf{a}}$ $\mathbf{a} = \left.\frac{d\mathbf{A}}{dt}\right|_{t=0}$

Theorem: Every one-parameter subgroup of a linear Lie group of $m \times m$ matrices is formed by exponentiation of $m \times m$ matrices. $\mathbf{A}(t) = e^{\mathbf{a}t}$ $\mathbf{a} = \left.\frac{d\mathbf{A}}{dt}\right|_{t=0}$

$$\frac{d\mathbf{A}(t)}{dt} = \lim_{s\to 0}\left(\frac{\mathbf{A}(t+s)-\mathbf{A}(t)}{s}\right) = \lim_{s\to 0}\left(\mathbf{A}(t)\left(\frac{\mathbf{A}(s)-\mathbf{A}(0)}{s}\right)\right)$$

$$= \mathbf{A}(t)\mathbf{a}$$

$$\mathbf{A}(t) = e^{\mathbf{a}t}$$

Correspondence between each linear Lie group $\mathcal{G}$ and a real Lie algebra $\mathcal{L}$ Simplify the consideration to $\mathcal{G}$ consisting of *mxm* matrices **T=A** and Γ(**T**)=**A.**

As part of the definition of the linear Lie group, there are n parameters $x_1, x_2, \ldots x_n$ such that all $\mathbf{A}(x_1, x_2, \ldots x_n)$ are analytic functions of the parameters, and the n $m \times m$ matrices

$$\left(\mathbf{a}_p\right)_{jk} = \left.\frac{\partial \mathbf{A}}{\partial x_p}\right|_{x_1 = x_2 = \ldots x_n = 0}$$

form the basis of an n-dimensional real vector space.

Definition: Analytic curve in $\mathcal{G}$

Suppose $x_1(t), x_2(t)....x_n(t)$ are analytic functions of $0 \le t \le t_0$ within which $\sum_{j=1}^{n} x_j^{\,2}(t) \le \eta^2$, then the set of $m \times m$ matrices $\mathbf{A}(\mathrm{t})=\mathbf{A}(x_1(t), x_2(t)....x_n(t))$ are said to form an analytic curve in $\mathcal{G}$.

Definition: Tangent vector of an analytic curve in $\mathcal{G}$

The tangent vector of an analytic curve $\mathbf{A}(t)$ in $\mathcal{G}$ is defined to be the $m \times m$ matrix $\mathbf{a}=\left.\frac{d\mathbf{A}(t)}{dt}\right|_{t=0}$.

Theorem The tangent vector of any analytic curve in $\mathcal{G}$ is a member of the real vector space having the matrices $\mathbf{a}_1, \mathbf{a}_2, ...\mathbf{a}_n$ as its basis. Conversely, every member of the real vector space is the tangent vector of some analytic curve in $\mathcal{G}$.

Theorem If $\mathbf{a}$ and $\mathbf{b}$ are the tangent vectors of the analytic curves $\mathbf{A}(t)$ and $\mathbf{B}(t)$ in $\mathcal{G}$, then $\mathbf{c}=[\mathbf{a},\mathbf{b}]$ is the tangent vector of the analytic curve $\mathbf{C}(t)$ in $\mathcal{G}$, where

$$\mathbf{C}(t) = \mathbf{A}(\sqrt{t})\mathbf{B}(\sqrt{t})\left(\mathbf{A}(\sqrt{t})\right)^{-1}\left(\mathbf{B}(\sqrt{t})\right)^{-1}$$

Note that $\mathbf{A}(\sqrt{t}) \approx 1+\mathbf{a}\sqrt{t} + O(t)$

$$\mathbf{C}(t) \approx \left(1+\mathbf{a}\sqrt{t}...\right)\left(1+\mathbf{b}\sqrt{t}...\right)\left(1-\mathbf{a}\sqrt{t}...\right)\left(1-\mathbf{b}\sqrt{t}...\right)$$

$$\approx 1+\mathbf{ab}t - \mathbf{ba}t.... \quad \Rightarrow \quad \left.\frac{d\mathbf{C}(t)}{dt}\right|_{t=0} = [\mathbf{a},\mathbf{b}]$$

Fundamental theorem: For every linear Lie group $\mathcal{G}$ there exists a corresponding real Lie algebra $\mathcal{L}$ of the same dimension. More precisely, if $\mathcal{G}$ has dimenison n then the $m \times m$ matrices $\mathbf{a}_1, \mathbf{a}_2, ...\mathbf{a}_n$ form a basis for $\mathcal{L}$.

Converse to fundamental theorem: Every real Lie algebra is isomorphic to the real Lie algebra of some linear Lie group.

Example: SO(3)

In this case, it is convenient to consider rotations about the 3 orthogonal directions:

$$R(\alpha,\beta,\gamma) = R(\alpha,\hat{\mathbf{x}})R(\beta,\hat{\mathbf{y}})R(\gamma,\hat{\mathbf{z}})$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{pmatrix} \begin{pmatrix} \cos\gamma & -\sin\beta & - \\ \sin\beta & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\left.\frac{\partial R}{\partial \alpha}\right|_{\alpha=\beta=\gamma=0} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \equiv -\mathbf{a}_1$$

$$\left.\frac{\partial R}{\partial \beta}\right|_{\alpha=\beta=\gamma=0} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \equiv -\mathbf{a}_2$$

$$\left.\frac{\partial R}{\partial \gamma}\right|_{\alpha=\beta=\gamma=0} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \equiv -\mathbf{a}_3$$

Note that:

$$[\mathbf{a}_1, \mathbf{a}_2] = -\mathbf{a}_3$$

$$[\mathbf{a}_2, \mathbf{a}_3] = -\mathbf{a}_1$$

$$[\mathbf{a}_3, \mathbf{a}_1] = -\mathbf{a}_2$$

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