PHY 745 Group Theory
11-11:50 AM MWF Olin 102

Plan for Lecture 34:

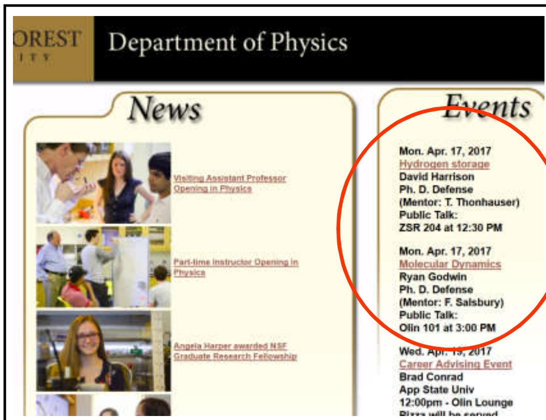
Introduction to linear Lie groups

1. Definitions and properties
2. Comparison with finite groups
3. Usefulness for physics and mathematics

Ref. J. F. Cornwell, *Group Theory in Physics*, Vol I and II, Academic Press (1984)

Robert Gilmore, *Lie Groups, Physics, and Geometry*, Cambridge U. Press (2008)

23	Mon: 03/20/2017	Chap. 7.7	Jahn-Teller Effect	#15	03/24/2017
24	Wed: 03/22/2017	Chap. 7.7	Jahn-Teller Effect		
25	Fri: 03/24/2017		Spin 1/2	#16	03/27/2017
26	Mon: 03/27/2017		Dirac equation for H-like atoms	#17	03/29/2017
27	Wed: 03/29/2017	Chap. 14	Angular momenta	#18	03/31/2017
28	Fri: 03/31/2017	Chap. 16	Time reversal symmetry	#19	04/05/2017
29	Mon: 04/03/2017	Chap. 16	Magnetic point groups		
30	Wed: 04/05/2017	Literature	Topology and group theory in Bloch states	#20	04/07/2017
31	Fri: 04/07/2017		Introduction to Lie groups	#21	04/10/2017
32	Mon: 04/10/2017		Introduction to Lie groups		
33	Wed: 04/12/2017		Introduction to Lie groups		
	Fri: 04/14/2017		*Good Friday Holiday -- no class*		
34	Mon: 04/17/2017		Introduction to Lie groups		
35	Wed: 04/19/2017		Introduction to Lie groups		
36	Fri: 04/21/2017		Introduction to Lie groups		
	Mon: 04/24/2017		Presentations I		
	Wed: 04/26/2017		Presentations II		



Some comments from Gilmore text:

Marius Sophus Lie (1842-1899) had the vision to use group theory to solve, analyze, simplify differential equations by exploring relationships between symmetry and group theory and related algebraic and geometric structures. His worked followed that of Evariste Galois (1811-1832).

Definition of a linear Lie group

1. A linear Lie group is a group
   - Each element of the group *T* forms a member of the group *T"* when "multiplied" by another member of the group *T"=T·T'*
   - One of the elements of the group is the identity *E*
   - For each element of the group *T,* there is a group member a group member T^{-1} such that $T \cdot T^{-1} = E$.
   - Associative property: *T·(T'·T")= (T·T')·T"*
2. Elements of group form a "topological space"
3. Elements also constitute an "analytic manifold"

➔Non countable number elements lying in a region "near" its identity

Definition: Linear Lie group of dimension *n*

A group G is a linear Lie group of dimension *n* if it satisfied the following four conditions:

1. G must have at least one faithful finite-dimensional representation Γ which defines the notion of distance.

For represent Γ having dimension m, the distance between two group elements T and T' can be defined:

$$d(T,T') \equiv \left\{ \sum_{j=1}^{m} \sum_{k=1}^{m} \left| \Gamma(T)_{jk} - \Gamma(T')_{jk} \right|^2 \right\}^{1/2}$$

Note that $d(T,T')$ has the following properties

(i) $d(T,T') = d(T',T)$

(ii) $d(T,T) = 0$

(iii) $d(T,T') > 0$ if $T \neq T'$

(iv) For elements $T, T',$ and T'',

$$d(T,T') \leq d(T,T'') + d(T',T'')$$

Definition: Linear Lie group of dimension *n* -- continued

2. Consider the distance between group elements *T* with respect to the identity *E* -- *d(T,E)*. It is possible to define a sphere M_δ that contains all elements *T'* such that $d(E,T') \leq \delta$.
   It follows that there must exist a $\delta > 0$ such that every T' of G lying in the sphere M_δ can be parameterized by n real parameters $x_1, x_2, \ldots x_n$ such each T' has a different set of parameters and for E the parameters are $x_1 = 0, x_2 = 0, \ldots x_n = 0$

Definition: Linear Lie group of dimension *n* -- continued

3. There must exist $\eta > 0$ such that for every parameter set $\{x_1, x_2, \ldots x_n\}$ corresponding to T' in the sphere M_δ:
$$\sum_{j=1}^{n} x_j^2 < \eta^2$$
4. There is a requirement that the corresponding representation is analytic
   For element T' within M_δ, $\Gamma(T'(x_1, x_2, .. x_n))$ must be an analytic (polynomial) function of $x_1, x_2, \ldots x_n$.

Some more details

4. There is a requirement that the corresponding representation is analytic
   For element T' within M_δ, $\Gamma(T'(x_1, x_2, .. x_n))$ must be an analytic (polynomial) function of $x_1, x_2, \ldots x_n$.
   Because of the mapping to the n parameters $x_1, x_2 \ldots x_n$ to each group element T', $\Gamma(T'(x_1, x_2 \ldots x_n)) = \Gamma(x_1, x_2 \ldots x_n)$.
   The analytic property of $\Gamma(x_1, x_2 \ldots x_n)$ also means that derivatives
$$\frac{\partial^\alpha \Gamma_{jk}(x_1, x_2 \ldots x_n)}{\partial^\alpha x_p} \text{ must exist for all } \alpha = 1, 2, \ldots$$
   Define n $m \times m$ matrices:
$$(\mathbf{a}_p)_{jk} \equiv \left.\frac{\partial \Gamma_{jk}(x_1, x_2 \ldots x_n)}{\partial x_p}\right|_{x_1=0, x_2=0, \ldots x_n=0}$$

Definition: Real Lie algebra

A real Lie algebra of dimension $n \geq 1$ is a real vector space of dimension n which includes a comutator $[M,N]$ as follows:

1. For all M,N in algebra, $[M,N]$ is also in algebra
2. For real numbers α and β, and members M,N,O, $[\alpha M + \beta N, O] = \alpha[M,O] + \beta[N,O]$
3. $[M,N] = -[N,M]$
4. $[M,[N,O]] + [N,[O,M]] + [O,[M,N]] = 0$

Generalizations for the notion of distance – choice of "metric". Here we have chosen the Hilbert-Schmidt metric:

$$d(T,T') \equiv \left\{ \sum_{j=1}^{m} \sum_{k=1}^{m} \left| \Gamma(T)_{jk} - \Gamma(T')_{jk} \right|^2 \right\}^{1/2}$$

Another choice of metric is based on the "regular" representation. For an n dimensional Lie group, a regular represention is based on $n \times n$ which satisfy the structure constant relations:

$$[X_i, X_j] = \sum_{r=1}^{n} c_{ij}^r X_r$$

A "Cartan Killing" inner product is defined:

$$(X_i, X_j)_{CK} = \sum_{r,s=1}^{n} c_{ir}^s c_{js}^r$$

Relationships of finite groups to continuous group – notions of "connectedness"

A maximal set of elements T of $\mathcal{G}$ that can be obtained from each other by continuously varying one or more matrix elements $\Gamma(T)_{jk}$ of the faithful finite dimensional representation Γ is said to form a "connected component" of $\mathcal{G}$.

Example: O(2) -- 2×2 orthogonal matrices

$$\mathbf{A}=\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22}\end{pmatrix} \quad \text{with } \mathbf{A}^{\dagger}\mathbf{A}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

Possibility #1: $\mathbf{A}_{\#1}(x_1)=\begin{pmatrix}\cos x_1 & \sin x_1\\ -\sin x_1 & \cos x_1\end{pmatrix}$

Possibility #2: $\mathbf{A}_{\#2}(x_1)=\begin{pmatrix}-\sin x_1 & \cos x_1\\ \cos x_1 & \sin x_1\end{pmatrix}$

$\Rightarrow \Gamma(\mathbf{A}_{\#1}(x_1))$ are connected

$\Rightarrow \Gamma(\mathbf{A}_{\#2}(x_1))$ are connected

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Theorem: The connected component of a linear Lie group $\mathcal{G}$ that contains the identity E is an invariant subgroup of $\mathcal{G}$.

Example from O(2):

Possibility #1: $\mathbf{A}_{\#1}(x_1)=\begin{pmatrix}\cos x_1 & \sin x_1\\ -\sin x_1 & \cos x_1\end{pmatrix}$

$$\mathbf{A}_{\#1}(x_1=0)=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

$$\mathbf{A}_{\#1}(x_1)\mathbf{A}_{\#1}(x_2)=\begin{pmatrix}\cos x_1 & \sin x_1\\ -\sin x_1 & \cos x_1\end{pmatrix}\begin{pmatrix}\cos x_2 & \sin x_2\\ -\sin x_2 & \cos x_2\end{pmatrix}$$

$$=\begin{pmatrix}\cos(x_1+x_2) & \sin(x_1+x_2)\\ -\sin(x_1+x_2) & \cos(x_1+x_2)\end{pmatrix}$$

→ subgroup

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$\Rightarrow \mathbf{A}_{\#1}(x_1)=\begin{pmatrix}\cos x_1 & \sin x_1\\ -\sin x_1 & \cos x_1\end{pmatrix}$ is a subgroup $\mathcal{G}'$ of $\mathcal{G}$

To show that $\mathcal{G}'$ is invariant, must show that

$\mathbf{X}\mathbf{A}\mathbf{X}^{-1}\in\mathcal{G}'$ for all $\mathbf{X}\in\mathcal{G}$

A linear Lie group is said to be “connected” if it possesses only one connected component.

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Definition

A linear Lie group of dimension n with a finite number of connected components is compact if the parameters $y_1, y_2, \ldots y_n$ range over closed finite intervals such as $a_j \leq y_j \leq b_j$

Example from O(2):

Possibility #1: $\mathbf{A}_{\#1}(x_1) = \begin{pmatrix} \cos x_1 & \sin x_1 \\ -\sin x_1 & \cos x_1 \end{pmatrix}$

In this case $-\pi \leq x_1 \leq \pi \quad \Rightarrow$ compact

With the notion of a compact linear Lie groups, we can relate their properties, such as the great orthogonality theorem, to those of finite groups.

Consider a complex function f of a group element T:

Left invariant: $\sum_{T \in \mathcal{G}} f(T'T) = \sum_{T'' \in \mathcal{G}} f(T'') \qquad \Rightarrow \int_{\mathcal{G}} f(T)\, d_l T$

Right invariant: $\sum_{T \in \mathcal{G}} f(TT') = \sum_{T''' \in \mathcal{G}} f(T''') \qquad \Rightarrow \int_{\mathcal{G}} f(T)\, d_r T$

$$\int_{\mathcal{G}} f(T)\, d_l T = \int_{a_1}^{b_1} dy_1 \int_{a_2}^{b_2} dy_2 \ldots \int_{a_n}^{b_n} dy_n f(T(y_1.y_2 \ldots y_n))\sigma_l(y_1, y_2 \ldots y_n)$$

$$\int_{\mathcal{G}} f(T)\, d_r T = \int_{a_1}^{b_1} dy_1 \int_{a_2}^{b_2} dy_2 \ldots \int_{a_n}^{b_n} dy_n f(T(y_1.y_2 \ldots y_n))\sigma_r(y_1, y_2 \ldots y_n)$$

If $\sigma_l(y_1, y_2 \ldots y_n) = \sigma_r(y_1, y_2 \ldots y_n)$, the $\mathcal{G}$ is called unimodular.

For $\mathcal{G}$ compact and unimodular and $f(T)$ continuous, the integral exists and is finite.

Example: For $O(2)$

Possibility #1: $\mathbf{A}_{\#1}(x_1) = \begin{pmatrix} \cos x_1 & \sin x_1 \\ -\sin x_1 & \cos x_1 \end{pmatrix}$

In this case $-\pi \leq x_1 \leq \pi \quad \Rightarrow$ compact

$\int dT = \frac{1}{2\pi} \int_{-\pi}^{\pi} dx_1$

Other examples –

SU(2):

$$\mathbf{u}=\begin{pmatrix} \cos y_1 e^{iy_2} & \sin y_1 e^{iy_3} \\ -\sin y_1 e^{-iy_3} & \cos y_1 e^{-iy_2} \end{pmatrix}$$

$$\int dT = \frac{1}{4\pi^2}\int_0^{\pi/2} dy_1 \int_0^{2\pi} dy_2 \int_0^{2\pi} dy_3$$