PHY 712 Electrodynamics
9-9:50 AM MWF Olin 105

Plan for Lecture 11:

Review of Electrostatics - Chapters 1-4

- **Homework problems and related**
- **Questions**

Course schedule for Spring 2018

(Preliminary schedule -- subject to frequent adjustment.)

	Lecture date	JDJ Reading	Topic	HW	Due date
	Wed: 01/17/2018	No class	Snow		
1	Fri: 01/19/2018	Chap. 1 & Appen.	Introduction, units and Poisson equation	#1	01/26/2018
2	Mon: 01/22/2018	Chap. 1	Electrostatic energy calculations	#2	01/26/2018
3	Wed: 01/24/2018	Chap. 1	Poisson's equation and Green's theorem	#3	01/26/2018
4	Thu: 01/25/2018	Chap. 1 & 2	Poisson's equation in 2 and 3 dimensions		
5	Fri: 01/26/2018	Chap. 1 & 2	Brief introduction to numerical methods	#4	01/29/2018
6	Mon: 01/29/2018	Chap. 2	Method of image charges	#5	01/31/2018
7	Wed: 01/31/2018	Chap. 2 & 3	Cylindrical and spherical geometries	#6	02/02/2018
8	Fri: 02/02/2018	Chap. 3 & 4	Multipole analysis	#7	02/07/2018
9	Mon: 02/05/2018	Chap. 4	Dipoles and Dielectrics	#8	02/09/2018
10	Wed: 02/07/2018	Chap. 4	Dipoles and Dielectrics		
11	Fri: 02/09/2018	Chap. 1-4	Review		
12	Mon: 02/12/2018				
13	Wed: 02/14/2018				
14	Fri: 02/16/2018				
15	Mon: 02/19/2018				

HW #4

1. Consider a two-dimensional charge distribution of the form:

$$\rho(x) = \rho_0 \sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{2\pi y}{a}\right),$$

where ρ_0 represents a density constant and a represents a length constant. In the problem, you are asked to determine the electrostatic potential $\Phi(x, y)$ for $0 \leq x \leq a$ and $0 \leq y \leq a$, which satisfies the Poisson equation for the charge density $\rho(x, y)$. and statisfies the boundary conditions $\Phi(0, y) = \Phi(a, y) = \Phi(x, 0) = \Phi(x, a) = 0$.

(a) Find the analytic form of the electrostatic potential $\Phi(x, y)$ for $0 \leq x \leq a$ and $0 \leq y \leq a$.

(b) Using the finite difference method for the two grids discussed in class, find $\Phi(x, y)$ on the grid points.

(c) Using the finite element method for the two grids discussed in class, find $\Phi(x, y)$ on the grid points.

(d) Compare the accuracy of the numerical solutions for this example.

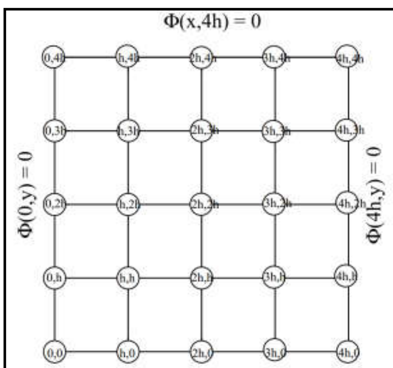
$$\nabla^2\Phi(x,y) = -\frac{\rho(x,y)}{\epsilon_0}$$

In this case: $\rho(x,y) = \rho_0 \sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi y}{a}\right)$

Solution which vanishes on boundaries: $x = 0, x = a, y = 0,$ and $y = a$

$$\Phi(x,y) = \frac{\rho_0}{\epsilon_0}\frac{a^2}{5\pi^2}\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi y}{a}\right)$$

Solution by finite difference method --



$$h = \frac{a}{4}$$

Algorithm:

$$\Phi(x,y) - \frac{1}{5}S_A - \frac{1}{20}S_B = \frac{3h^2}{10\varepsilon_0}\rho(x,y) + \frac{h^4}{40\varepsilon_0}\nabla^2\rho(x,y).$$

In our case:

$$\Phi(x,y) - \frac{1}{5}S_A - \frac{1}{20}S_B = \frac{3h^2}{10\varepsilon_0}\rho(x,y) + \frac{h^4}{40\varepsilon_0}\nabla^2\rho(x,y).$$

$$\Phi(x_m, y_n) = \Phi\left(\frac{ma}{4}, \frac{na}{4}\right)$$

$$RHS(x_m, y_n) = \frac{3h^2}{10\epsilon_0}\rho\left(\frac{ma}{4}, \frac{na}{4}\right) + \frac{h^4}{40\epsilon_0}\nabla^2\rho\left(\frac{ma}{4}, \frac{na}{4}\right)$$

$$= \frac{a^2\rho_0}{\epsilon_0}\left(\frac{3}{160} - \frac{5\pi^2}{10240}\right)\sin\left(\frac{\pi m}{4}\right)\sin\left(\frac{2\pi n}{4}\right)$$

$$\begin{pmatrix} 1 & -1/5 & -1/5 & -1/20 & 0 & 0 \\ -2/5 & 1 & -1/10 & -1/5 & 0 & 0 \\ -1/5 & -1/20 & 1 & -1/5 & -1/5 & -1/20 \\ -1/10 & -1/5 & -2/5 & 1 & -1/10 & -1/5 \\ 0 & 0 & -1/5 & -1/20 & 1 & -1/5 \\ 0 & 0 & -1/10 & -1/5 & -2/5 & 1 \end{pmatrix} \begin{pmatrix} \Phi(h,3h) \\ \Phi(2h,3h) \\ \Phi(h,2h) \\ \Phi(2h,2h) \\ \Phi(h,h) \\ \Phi(2h,h) \end{pmatrix} = \begin{pmatrix} RHS(h,3h) \\ RHS(2h,3h) \\ RHS(h,2h) \\ RHS(2h,2h) \\ RHS(h,h) \\ RHS(2h,h) \end{pmatrix}$$

For Maple:

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>with(LinearAlgebra)
>evalm(MatrixInverse(LHS).RHS)
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Solution by finite element method --

$$\Phi(x,y) = \sum_{ij} \psi_{ij}\phi_{ij}(x,y),$$

$$\iint dxdy\phi_{kl}(x,y)\left(-\nabla^2\Phi(x,y)\right) = \frac{1}{\epsilon_0}\iint dxdy\phi_{kl}(x,y)\rho(x,y)$$

$$\sum_{ij} M_{kl,ij}\psi_{ij} = G_{kl},$$

$$M_{kl,ij} \equiv \int dx \int dy \nabla\phi_{kl}(x,y)\cdot\nabla\phi_{ij}(x,y)$$

$$G_{kl} \equiv \int dx \int dy \phi_{kl}(x,y)\ \rho(x,y)/\epsilon_0.$$

$$\phi_{ij}(x,y) \equiv \mathcal{X}_i(x)\mathcal{Y}_j(y),$$

$$\mathcal{X}_i(x) \equiv \begin{cases} \left(1 - \frac{|x-x_i|}{h}\right) & \text{for } x_i - h \le x \le x_i + h \\ 0 & \text{otherwise} \end{cases}$$

$$G_{kl} = \frac{\rho_0}{\epsilon_0} X_k Y_l$$

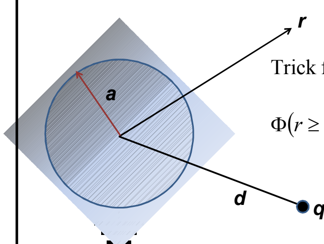
$$\text{where } X_k = \int_{(k-1)a/4}^{(k+1)a/4} dx \left(1 - \frac{|x - ka/4|}{a/4}\right)\sin\left(\frac{\pi x}{a}\right)$$

$$Y_l = \int_{(l-1)a/4}^{(l+1)a/4} dy \left(1 - \frac{|y - la/4|}{a/4}\right)\sin\left(\frac{2\pi y}{a}\right)$$

$$\begin{pmatrix} 8/3 & -1/3 & -1/3 & -1/3 & 0 & 0 \\ -2/3 & 8/3 & -2/3 & -1/3 & 0 & 0 \\ -1/3 & -1/3 & 8/3 & -1/3 & -1/3 & -1/3 \\ -2/3 & -1/3 & -2/3 & 8/3 & -2/3 & -1/3 \\ 0 & 0 & -1/3 & -1/3 & 8/3 & -1/3 \\ 0 & 0 & -2/3 & -1/3 & -2/3 & 8/3 \end{pmatrix} \begin{pmatrix} \Phi(h,3h) \\ \Phi(2h,3h) \\ \Phi(h,2h) \\ \Phi(2h,2h) \\ \Phi(h,h) \\ \Phi(2h,h) \end{pmatrix} = \begin{pmatrix} G(h,3h) \\ G(2h,3h) \\ G(h,2h) \\ G(2h,2h) \\ G(h,h) \\ G(2h,h) \end{pmatrix}$$

HW #5 Find total surface charge for *d>a* and for *d<a*

A grounded metal sphere of radius *a*, in the presence of a point charge ***q*** at a distance *d* from its center.



Trick for $r \geq a$:

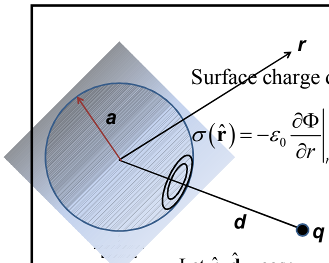
$$\Phi(r \geq a) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{|\mathbf{r}-\mathbf{d}|} - \frac{q}{\frac{d}{a}\left|\mathbf{r}-\mathbf{d}\frac{a^2}{d^2}\right|}\right)$$

Interpreted as

Image charge of $q' = -q\frac{a}{d}$

Located along $\hat{\mathbf{d}}$ at $\hat{\mathbf{d}}a\frac{a}{d}$

For *d>a*



Surface charge density:

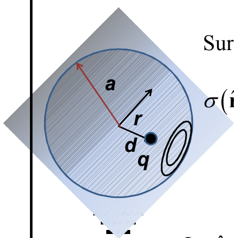
$$\sigma(\hat{\mathbf{r}}) = -\varepsilon_0 \left.\frac{\partial \Phi}{\partial r}\right|_{r=a} = -\frac{q}{4\pi a^2}\frac{a}{d}\frac{\left(1-\frac{a^2}{d^2}\right)}{\left(1+\frac{a^2}{d^2}-2\frac{a}{d}\hat{\mathbf{r}}\cdot\hat{\mathbf{d}}\right)^{3/2}}$$

Let $\hat{\mathbf{r}}\cdot\hat{\mathbf{d}} \equiv \cos\gamma$

$$Q = 2\pi a^2 \int_{-1}^{1} d\cos\gamma \left(-\frac{q}{4\pi a^2}\frac{a}{d}\frac{\left(1-\frac{a^2}{d^2}\right)}{\left(1+\frac{a^2}{d^2}-2\frac{a}{d}\cos\gamma\right)^{3/2}}\right)$$

$$= -\frac{qa}{d}$$

For *d*<*a*



Surface charge density:

$$\sigma(\hat{\mathbf{r}}) = \varepsilon_0 \frac{\partial \Phi}{\partial r}\bigg|_{r=a} = \frac{q}{4\pi a^2}\frac{a}{d}\frac{\left(1-\frac{a^2}{d^2}\right)}{\left(1+\frac{a^2}{d^2}-2\frac{a}{d}\hat{\mathbf{r}}\cdot\hat{\mathbf{d}}\right)^{3/2}}$$

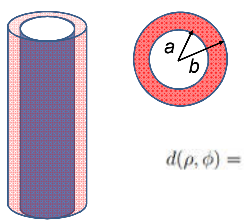
Let $\hat{\mathbf{r}}\cdot\hat{\mathbf{d}} \equiv \cos\gamma$

$$Q = 2\pi a^2 \int_{-1}^{1} d\cos\gamma \left(\frac{q}{4\pi a^2}\frac{a}{d}\frac{\left(1-\frac{a^2}{d^2}\right)}{\left(1+\frac{a^2}{d^2}-2\frac{a}{d}\cos\gamma\right)^{3/2}} \right)$$

$$= -q$$

HW#6

top view



$$d(\rho,\phi) = \begin{cases} 0 & \text{for} \quad 0 \le \rho < a \\ d_0 & \text{for} \quad a \le \rho \le b \\ 0 & \text{for} \quad \rho > b \end{cases}$$

Green's function appropriate for this geometry with boundary conditions at $\rho = 0$ and $\rho = \infty$:

$$\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}\right)G(\rho,\rho',\phi,\phi') =$$

$$-4\pi\frac{\delta(\rho-\rho')}{\rho}\delta(\phi-\phi')$$

$$G(\rho,\rho',\phi,\phi') = -\ln\left(\rho_>^{\,2}\right)+2\sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{\rho_<}{\rho_>}\right)^m \cos(m(\phi-\phi'))$$

$$\Phi(\rho,\varphi) = \frac{1}{4\pi\epsilon_0}\int_0^{2\pi} d\varphi' \int_0^{\infty} \rho' d\rho' G(\rho,\rho',\varphi,\varphi') d(\rho',\varphi')$$

$$= -\frac{1}{\epsilon_0}\int_0^{\infty} \rho' d\rho' \ln(\rho_>) d(\rho')$$

HW #7

PHY 712 -- Assignment #7

February 2, 2018

Complete reading Chapter 3 and start Chapter 4 in **Jackson** .

1. Consider the charge density of an electron bound to a proton in a hydrogen atom -- $\rho(r) = (1/\pi a_0^3)\, e^{-2r/a_0}$, where a_0 denotes the Bohr radius. Find the electrostatic potential $\Phi(r)$ associated with $\rho(r)$. Compare your result to HW#1.

Example for isolated charge density $\rho(\mathbf{r})$ with electrostatic potential vanishing for $r \to \infty$:

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int d^3r' \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$$

$$= \frac{1}{4\pi\varepsilon_0}\int d^3r'\,\rho(\mathbf{r}')\left(\sum_{lm}\frac{4\pi}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y_{lm}(\theta,\varphi)Y_{lm}^*(\theta',\varphi')\right)$$

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In our case, $\rho(r) = \frac{1}{\pi a_0^3}e^{-2r/a_0}$

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int d^3r'\,\rho(\mathbf{r}')\left(\sum_{lm}\frac{4\pi}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y_{lm}(\theta,\varphi)Y_{lm}^*(\theta',\varphi')\right)$$

Integral over solid angles

$$\int d\Omega' Y_{lm}^*(\theta',\phi') = \sqrt{4\pi}\,\delta_{lm,00}$$

Non-trivial radial integral

$$\Phi(\mathbf{r}) = \frac{1}{\varepsilon_0}\int_0^\infty r'^2\,dr'\rho(r')\frac{1}{r_>}$$

$$= \frac{1}{\varepsilon_0}\left(\frac{1}{r}\int_0^r r'^2\,dr'\,\rho(r') + \int_r^\infty r'\,dr'\,\rho(r')\right)$$

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