

PHY 712 Electrodynamics
9-9:50 AM MWF Olin 105

Plan for Lecture 14:

Continue reading Chapter 5

1. Recap of hyperfine interaction
2. Macroscopic magnetization density M
3. H field and its relation to B
4. Magnetic boundary values

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	Lecture date	JDJ Reading	Topic	HW	Due date
	Wed: 01/17/2018	No class	Snow		
1	Fri: 01/19/2018	Chap. 1 & Appen.	Introduction, units and Poisson equation	#1	01/26/2018
2	Mon: 01/22/2018	Chap. 1	Electrostatic energy calculations	#2	01/26/2018
3	Wed: 01/24/2018	Chap. 1	Poisson's equation and Green's theorem	#3	01/26/2018
4	Thu: 01/25/2018	Chap. 1 & 2	Poisson's equation in 2 and 3 dimensions		
5	Fri: 01/26/2018	Chap. 1 & 2	Brief introduction to numerical methods	#4	01/29/2018
6	Mon: 01/29/2018	Chap. 2	Method of image charges	#5	01/31/2018
7	Wed: 01/31/2018	Chap. 2 & 3	Cylindrical and spherical geometries	#6	02/02/2018
8	Fri: 02/02/2018	Chap. 3 & 4	Multipole analysis	#7	02/07/2018
9	Mon: 02/05/2018	Chap. 4	Dipoles and Dielectrics	#8	02/09/2018
10	Wed: 02/07/2018	Chap. 4	Dipoles and Dielectrics		
11	Fri: 02/09/2018	Chap. 1-4	Review		
12	Mon: 02/12/2018	Chap. 5	Magnetostatics	#9	02/16/2018
13	Wed: 02/14/2018	Chap. 5	Magnetic dipoles and hyperfine interaction	#10	02/19/2018
14	Fri: 02/16/2018	Chap. 5	Magnetic dipoles and dipolar fields	#11	02/21/2018
15	Mon: 02/19/2018				
16	Wed: 02/21/2018				
17	Fri: 02/23/2018				
18	Mon: 02/26/2018			Take Home Exam	

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Summary of hyperfine interaction form:

Interactions between magnetic dipoles

Sources of magnetic dipoles and other sources of magnetism in an atom:

- Intrinsic magnetic moment of a nucleus μ_N
- Intrinsic magnetic moment of an electron μ_e
- Magnetic field due to electron orbital current $\mathbf{J}_e(\mathbf{r})$

Interaction energy between a magnetic dipole \mathbf{m} and a magnetic field \mathbf{B} : $E_{int} = -\mathbf{m} \cdot \mathbf{B}$



In this case: $E_{int} = -\mu_e \cdot \mathbf{B}_{\mu_N} - \mu_N \cdot \mathbf{B}_{J_e}(0)$

$$\mathbf{B}_{\mu_N}(\mathbf{r}) = \frac{\mu_0}{4\pi} \left\{ \frac{3\hat{\mathbf{r}}(\boldsymbol{\mu}_N \cdot \hat{\mathbf{r}}) - \boldsymbol{\mu}_N}{r^3} + \frac{8\pi}{3} \boldsymbol{\mu}_N \delta^3(\mathbf{r}) \right\}$$

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Hyperfine interaction energy: -- continued

$$E_{int} = -\boldsymbol{\mu}_e \cdot \mathbf{B}_{\mu_N} - \mu_N \cdot \mathbf{B}_{\mathbf{J}_e}(0)$$

Evaluation of the magnetic field at the nucleus due to the electron current density:

The vector potential associated with an electron in a bound state of an atom as described by a quantum mechanical wavefunction $\psi_{nlm_l}(\mathbf{r})$ can be written:

$$\mathbf{A}_{\mathbf{J}_e}(\mathbf{r}) = -\frac{\mu_0}{4\pi} \frac{e\hbar}{m_e} m_l \int d^3r' \frac{\hat{\mathbf{z}} \times \mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} \frac{|\psi_{nlm_l}(\mathbf{r}')|^2}{r'^2 \sin^2 \theta'}$$

We want to evaluate the magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$ in the vicinity of the nucleus ($\mathbf{r} \rightarrow 0$).

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Hyperfine interaction energy: -- continued

$$\mathbf{B}_{\mathbf{J}_e}(0) = \nabla \times \mathbf{A}_{\mathbf{J}_e} \Big|_{\mathbf{r} \rightarrow 0} = -\frac{\mu_0}{4\pi} \frac{e\hbar}{m_e} m_l \int d^3r' \nabla \times \frac{\hat{\mathbf{z}} \times \mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} \frac{|\psi_{nlm_l}(\mathbf{r}')|^2}{r'^2 \sin^2 \theta'} \Big|_{\mathbf{r} \rightarrow 0}$$

$$\mathbf{B}_0(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{e\hbar}{m_e} m_l \int d^3r' \frac{(\mathbf{r}-\mathbf{r}') \times (\hat{\mathbf{z}} \times \mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} \frac{|\psi_{nlm_l}(\mathbf{r}')|^2}{r'^2 \sin^2 \theta'} \Big|_{\mathbf{r} \rightarrow 0}$$

$$\mathbf{B}_0(0) = -\frac{\mu_0}{4\pi} \frac{e\hbar}{m_e} m_l \int d^3r' \frac{\mathbf{r}' \times (\hat{\mathbf{z}} \times \mathbf{r}')}{r'^3} \frac{|\psi_{nlm_l}(\mathbf{r}')|^2}{r'^2 \sin^2 \theta'}$$

$$\hat{\mathbf{r}}' \times (\hat{\mathbf{z}} \times \hat{\mathbf{r}}') = \hat{\mathbf{z}}(1 - \cos^2 \theta') - \hat{\mathbf{x}} \cos \theta' \sin \theta' \cos \phi' - \hat{\mathbf{y}} \cos \theta' \sin \theta' \sin \phi'$$

$$\begin{aligned} \mathbf{B}_0(0) &= -\frac{\mu_0}{4\pi} \frac{e\hbar}{m_e} m_l \int d^3r' \frac{\hat{\mathbf{z}} r'^2 \sin^2 \theta'}{r'^3} \frac{|\psi_{nlm_l}(\mathbf{r}')|^2}{r'^2 \sin^2 \theta'} = -\frac{\mu_0}{4\pi} \frac{e\hbar}{m_e} m_l \hat{\mathbf{z}} \int d^3r' \frac{|\psi_{nlm_l}(\mathbf{r}')|^2}{r'^3} \\ &= -\frac{\mu_0}{4\pi} \frac{e\hbar}{m_e} m_l \hat{\mathbf{z}} \left\langle \frac{1}{r^3} \right\rangle \end{aligned}$$

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Hyperfine interaction energy: -- continued

$$E_{int} \equiv H_{HF} = -\boldsymbol{\mu}_e \cdot \mathbf{B}_{\mu_N} - \mu_N \cdot \mathbf{B}_{\mathbf{J}_e}(0)$$

Putting all of the terms together:

$$H_{HF} = -\frac{\mu_0}{4\pi} \left\langle \left(\frac{3(\boldsymbol{\mu}_N \cdot \hat{\mathbf{r}})(\boldsymbol{\mu}_e \cdot \hat{\mathbf{r}}) - \boldsymbol{\mu}_N \cdot \boldsymbol{\mu}_e}{r^3} + \frac{8\pi}{3} \boldsymbol{\mu}_N \cdot \boldsymbol{\mu}_e \delta^3(\mathbf{r}) \right) + \frac{e}{m_e} \left\langle \frac{\mathbf{L} \cdot \boldsymbol{\mu}_N}{r^3} \right\rangle \right\rangle$$

In this expression the brackets $\langle \rangle$ indicate evaluating the expectation value relative to the electronic state.

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Macroscopic dipolar effects --
Magnetic dipole moment

$$\mathbf{m} = \frac{1}{2} \int d^3r \mathbf{r} \times \mathbf{J}(\mathbf{r})$$

Note that the intrinsic spin of elementary particles is associated with a magnetic dipole moment, but we often do not have a detailed knowledge of $\mathbf{J}(\mathbf{r})$.

Vector potential for magnetic dipole moment

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{\mathbf{m} \times \mathbf{r}}{|\mathbf{r}|^3}$$

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Macroscopic magnetization

$$\mathbf{M}(\mathbf{r}) = \sum_i \mathbf{m}_i \delta^3(\mathbf{r} - \mathbf{r}_i)$$

Vector potential due to "free" current $\mathbf{J}_{\text{free}}(\mathbf{r})$ and macroscopic magnetization $\mathbf{M}(\mathbf{r})$. Note: the designation $\mathbf{J}_{\text{free}}(\mathbf{r})$ implies that this current does not also contribute to the magnetization density.

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int d^3r' \left(\frac{\mathbf{J}_{\text{free}}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} + \frac{\mathbf{M}(\mathbf{r}') \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \right)$$

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Vector potential contributions from macroscopic magnetization -- continued

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int d^3r' \left(\frac{\mathbf{J}_{\text{free}}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} + \frac{\mathbf{M}(\mathbf{r}') \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \right)$$

Note that :

$$\begin{aligned} \frac{\mathbf{M}(\mathbf{r}') \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} &= \mathbf{M}(\mathbf{r}') \times \nabla' \frac{1}{|\mathbf{r} - \mathbf{r}'|} \\ &= -\nabla' \times \left(\frac{\mathbf{M}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \right) + \frac{\nabla' \times \mathbf{M}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \end{aligned}$$

$$\Rightarrow \mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int d^3r' \frac{\mathbf{J}_{\text{free}}(\mathbf{r}') + \nabla' \times \mathbf{M}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}$$

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Vector potential contributions from macroscopic magnetization -- continued

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int d^3r' \frac{\mathbf{J}_{free}(\mathbf{r}') + \nabla' \times \mathbf{M}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}$$

Note that for the case that $\nabla \cdot \mathbf{A} = 0$:

$$\begin{aligned} \nabla \times \mathbf{B}(\mathbf{r}) &= \nabla \times (\nabla \times \mathbf{A}(\mathbf{r})) = -\nabla^2 \mathbf{A}(\mathbf{r}) \\ &= \frac{\mu_0}{4\pi} \int d^3r' (4\pi\delta^3(\mathbf{r} - \mathbf{r}')) (\mathbf{J}_{free}(\mathbf{r}') + \nabla' \times \mathbf{M}(\mathbf{r}')) \\ &= \mu_0 (\mathbf{J}_{free}(\mathbf{r}) + \nabla \times \mathbf{M}(\mathbf{r})) \\ \Rightarrow \nabla \times (\mathbf{B}(\mathbf{r}) - \mu_0 \mathbf{M}(\mathbf{r})) &= \mu_0 \mathbf{J}_{free}(\mathbf{r}) \end{aligned}$$

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Magnetic field contributions

$$\nabla \times (\mathbf{B}(\mathbf{r}) - \mu_0 \mathbf{M}(\mathbf{r})) = \mu_0 \mathbf{J}_{free}(\mathbf{r})$$

Define the magnetic flux density:

$$\begin{aligned} \mu_0 \mathbf{H}(\mathbf{r}) &\equiv \mathbf{B}(\mathbf{r}) - \mu_0 \mathbf{M}(\mathbf{r}) \\ \Rightarrow \nabla \times \mathbf{H}(\mathbf{r}) &= \mathbf{J}_{free}(\mathbf{r}) \end{aligned}$$

$$\nabla \times (\mathbf{B}(\mathbf{r}) - \mu_0 \mathbf{M}(\mathbf{r})) = \mu_0 \mathbf{J}_{free}(\mathbf{r})$$

Note that $\mathbf{B}(\mathbf{r}) \equiv$ the magnetic flux density

Define $\mathbf{H}(\mathbf{r}) \equiv$ the magnetic field

$$\begin{aligned} \mu_0 \mathbf{H}(\mathbf{r}) &\equiv \mathbf{B}(\mathbf{r}) - \mu_0 \mathbf{M}(\mathbf{r}) \\ \Rightarrow \nabla \times \mathbf{H}(\mathbf{r}) &= \mathbf{J}_{free}(\mathbf{r}) \end{aligned}$$

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Summary of equations of magnetostatics:

$$\begin{aligned} \nabla \times \mathbf{B}(\mathbf{r}) &= \mu_0 \mathbf{J}_{total}(\mathbf{r}) \\ \nabla \times \mathbf{H}(\mathbf{r}) &= \mathbf{J}_{free}(\mathbf{r}) \\ \mathbf{B}(\mathbf{r}) &= \mu_0 (\mathbf{H}(\mathbf{r}) + \mathbf{M}(\mathbf{r})) \\ \nabla \cdot \mathbf{B}(\mathbf{r}) &= 0 \end{aligned}$$

For the case that $\mathbf{J}_{free}(\mathbf{r}) = 0$:

$$\begin{aligned} \nabla \times \mathbf{H}(\mathbf{r}) &= 0 \\ \nabla \cdot \mathbf{B}(\mathbf{r}) &= 0 \end{aligned}$$

1

↑ \hat{n}

2

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For the case that $\mathbf{J}_{free}(\mathbf{r}) = 0$:

$\nabla \times \mathbf{H}(\mathbf{r}) = 0$
 $\nabla \cdot \mathbf{B}(\mathbf{r}) = 0$

At boundary :

1	↑ $\hat{\mathbf{n}}$
2	

$\mathbf{H}_1 \times \hat{\mathbf{n}} = \mathbf{H}_2 \times \hat{\mathbf{n}}$
 $\mathbf{B}_1 \cdot \hat{\mathbf{n}} = \mathbf{B}_2 \cdot \hat{\mathbf{n}}$

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Example magnetostatic boundary value problem

$$\mathbf{M}(\mathbf{r}) = \begin{cases} M_0 \hat{\mathbf{z}} & r \leq a \\ 0 & r > a \end{cases}$$

$\nabla \times \mathbf{H}(\mathbf{r}) = 0 \Rightarrow \mathbf{H}(\mathbf{r}) = -\nabla \Phi_H(\mathbf{r})$
 $\mathbf{B}(\mathbf{r}) = \mu_0(\mathbf{H}(\mathbf{r}) + \mathbf{M}(\mathbf{r}))$
 $\nabla \cdot \mathbf{B}(\mathbf{r}) = 0 = \mu_0 \nabla \cdot (\mathbf{H}(\mathbf{r}) + \mathbf{M}(\mathbf{r}))$
 $\Rightarrow \nabla^2 \Phi_H(\mathbf{r}) = \nabla \cdot \mathbf{M}(\mathbf{r})$

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Example magnetostatic boundary value problem -- continued

$$\mathbf{M}(\mathbf{r}) = \begin{cases} M_0 \hat{\mathbf{z}} & r \leq a \\ 0 & r > a \end{cases}$$

$\nabla^2 \Phi_H(\mathbf{r}) = \nabla \cdot \mathbf{M}(\mathbf{r})$
 $\Rightarrow \Phi_H(\mathbf{r}) = -\frac{1}{4\pi} \int d^3 r' \frac{\nabla' \cdot \mathbf{M}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}$
 $= -\frac{1}{4\pi} \int d^3 r' \left[\nabla' \cdot \left(\frac{\mathbf{M}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \right) - \mathbf{M}(\mathbf{r}') \cdot \nabla' \frac{1}{|\mathbf{r} - \mathbf{r}'|} \right]$
 $= -\frac{1}{4\pi} \nabla \cdot \int d^3 r' \frac{\mathbf{M}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}$

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Example magnetostatic boundary value problem -- continued

$$\mathbf{M}_0 \mathbf{M}(\mathbf{r}) = \begin{cases} M_0 \hat{\mathbf{z}} & r \leq a \\ 0 & r > a \end{cases} \quad \Phi_H(\mathbf{r}) = -\frac{1}{4\pi} \nabla \cdot \int d^3 r' \frac{\mathbf{M}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}$$

For this example:

$$\Phi_H(\mathbf{r}) = -\frac{M_0}{4\pi} \frac{\partial}{\partial z} \left(4\pi \int_0^a r'^2 dr' \frac{1}{r'} \right)$$

$$\text{For } r \leq a: \quad \Phi_H(\mathbf{r}) = -M_0 \frac{\partial}{\partial z} \left(\frac{a^2}{2} - \frac{r^2}{6} \right) = \frac{M_0 z}{3}$$

$$\text{For } r > a: \quad \Phi_H(\mathbf{r}) = -M_0 \frac{\partial}{\partial z} \left(\frac{a^3}{3r} \right) = \frac{M_0 a^3 z}{3r^3}$$

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Example magnetostatic boundary value problem -- continued

$$\mathbf{M}_0 \mathbf{M}(\mathbf{r}) = \begin{cases} M_0 \hat{\mathbf{z}} & r \leq a \\ 0 & r > a \end{cases}$$

$$\text{For } r \leq a: \quad \Phi_H(\mathbf{r}) = \frac{M_0 z}{3} \quad \mathbf{H}(\mathbf{r}) = -\nabla \Phi_H(\mathbf{r}) = -\frac{M_0}{3} \hat{\mathbf{z}}$$

$$\text{For } r > a: \quad \Phi_H(\mathbf{r}) = \frac{M_0 a^3 z}{3r^3} \quad \mathbf{H}(\mathbf{r}) = -\nabla \Phi_H(\mathbf{r}) = -\frac{M_0 a^3}{3} \left(\frac{\hat{\mathbf{z}}}{r^3} - \frac{3z\mathbf{r}}{r^5} \right)$$

$$\mathbf{B}(\mathbf{r}) = \mu_0 (\mathbf{H}(\mathbf{r}) + \mathbf{M}(\mathbf{r}))$$

$$\text{For } r \leq a: \quad \mathbf{H}(\mathbf{r}) = -\frac{M_0 \hat{\mathbf{z}}}{3} \quad \mathbf{B}(\mathbf{r}) = \mu_0 \frac{2M_0 \hat{\mathbf{z}}}{3}$$

$$\text{For } r > a: \quad \mathbf{H}(\mathbf{r}) = -\frac{M_0 a^3}{3} \left(\frac{\hat{\mathbf{z}}}{r^3} - \frac{3z\mathbf{r}}{r^5} \right)$$

$$\mathbf{B}(\mathbf{r}) = -\mu_0 \frac{M_0 a^3}{3} \left(\frac{\hat{\mathbf{z}}}{r^3} - \frac{3z\mathbf{r}}{r^5} \right)$$

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Check boundary values:

$$\text{For } r \leq a: \quad \mathbf{H}(\mathbf{r}) = -\frac{M_0 \hat{\mathbf{z}}}{3} \quad \mathbf{H}(a\hat{\mathbf{r}}) \times \hat{\mathbf{r}} = -\frac{M_0}{3} \hat{\mathbf{z}} \times \hat{\mathbf{r}}$$

$$\text{For } r > a: \quad \mathbf{H}(\mathbf{r}) = -\frac{M_0 a^3}{3} \left(\frac{\hat{\mathbf{z}}}{r^3} - \frac{3z\mathbf{r}}{r^5} \right)$$

$$\mathbf{H}(a\hat{\mathbf{r}}) \times \hat{\mathbf{r}} = -\frac{M_0 a^3}{3} \frac{\hat{\mathbf{z}} \times \hat{\mathbf{r}}}{a^3}$$

$$\text{For } r \leq a: \quad \mathbf{B}(\mathbf{r}) = \mu_0 \frac{2M_0 \hat{\mathbf{z}}}{3} \quad \mathbf{B}(a\hat{\mathbf{r}}) \cdot \hat{\mathbf{r}} = \mu_0 \frac{2M_0}{3} \hat{\mathbf{z}} \cdot \hat{\mathbf{r}}$$

$$\text{For } r > a: \quad \mathbf{B}(\mathbf{r}) = -\mu_0 \frac{M_0 a^3}{3} \left(\frac{\hat{\mathbf{z}}}{r^3} - \frac{3z\mathbf{r}}{r^5} \right)$$

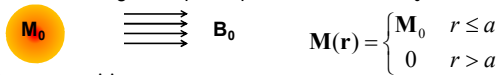
$$\mathbf{B}(a\hat{\mathbf{r}}) \cdot \hat{\mathbf{r}} = -\mu_0 \frac{M_0 a^3}{3} \hat{\mathbf{z}} \cdot \hat{\mathbf{r}} \left(\frac{1}{a^3} - \frac{3a^2}{a^5} \right)$$

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Variation; magnetic sphere plus external field \mathbf{B}_0



By superposition :

For $r \leq a$:

$$\mathbf{B}(\mathbf{r}) = \mathbf{B}_0 + \mu_0 \frac{2}{3} \mathbf{M}_0$$

$$\mathbf{H}(\mathbf{r}) = \frac{1}{\mu_0} \mathbf{B}_0 - \frac{1}{3} \mathbf{M}_0$$

$$\mathbf{B}(\mathbf{r}) + 2\mu_0 \mathbf{H}(\mathbf{r}) = 3\mathbf{B}_0$$

For an isotropic "paramagnetic" material, $\mathbf{B}(\mathbf{r}) = \mu \mathbf{H}(\mathbf{r})$

$$\mathbf{M}_0 = \frac{3}{\mu_0} \left(\frac{\mu - \mu_0}{\mu + 2\mu_0} \right) \mathbf{B}_0$$

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Summary of equations of magnetostatics :

$$\nabla \times \mathbf{B}(\mathbf{r}) = \mu_0 \mathbf{J}_{total}(\mathbf{r})$$

$$\nabla \times \mathbf{H}(\mathbf{r}) = \mathbf{J}_{free}(\mathbf{r})$$

$$\mathbf{B}(\mathbf{r}) = \mu_0 (\mathbf{H}(\mathbf{r}) + \mathbf{M}(\mathbf{r}))$$


$$\nabla \cdot \mathbf{B}(\mathbf{r}) = 0$$

For the case that $\mathbf{J}_{free}(\mathbf{r}) = 0$:

$$\nabla \times \mathbf{H}(\mathbf{r}) = 0$$

$$\nabla \cdot \mathbf{B}(\mathbf{r}) = 0$$

At boundary :



$$\mathbf{H}_1 \times \hat{\mathbf{n}} = \mathbf{H}_2 \times \hat{\mathbf{n}}$$

$$\mathbf{B}_1 \cdot \hat{\mathbf{n}} = \mathbf{B}_2 \cdot \hat{\mathbf{n}}$$

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Magnetism in materials

$$\mathbf{B}(\mathbf{r}) = \mu_0 (\mathbf{H}(\mathbf{r}) + \mathbf{M}(\mathbf{r}))$$

For materials with linear magnetism :

$$\mathbf{B} = \mu \mathbf{H}$$

$\mu > \mu_0 \Rightarrow$ paramagnetic material

$\mu < \mu_0 \Rightarrow$ diamagnetic material

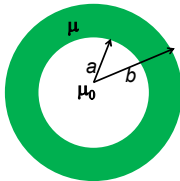
For ferromagnetic, antiferromagnetic materials

$$\mathbf{B} = f(\mathbf{H}) \quad (\text{with hysteresis})$$

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Example: permalloy, mumetal $\mu/\mu_0 \sim 10^4$

Spherical shell $a < r < b$:



μ

a b

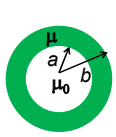
μ_0

\mathbf{B}_0

μ_0

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Example: permalloy, mumetal $\mu/\mu_0 \sim 10^4$ -- continued



μ

a b

μ_0

\mathbf{B}_0

μ_0

For this case:

$\nabla \times \mathbf{H}(\mathbf{r}) = 0$

$\nabla \cdot \mathbf{B}(\mathbf{r}) = 0$

$\mathbf{B}(\mathbf{r}) = \mu \mathbf{H}(\mathbf{r})$

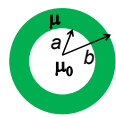
Continuity at boundaries:

$\mathbf{H} \times \hat{\mathbf{n}} = \text{continuous}$

$\mathbf{B} \cdot \hat{\mathbf{n}} = \text{continuous}$

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Example: permalloy, mumetal $\mu/\mu_0 \sim 10^4$ -- continued



μ

a b

μ_0

\mathbf{B}_0

μ_0

Let: $\mathbf{H}(\mathbf{r}) = -\nabla \Phi_H(\mathbf{r})$

$\nabla \cdot \mathbf{B}(\mathbf{r}) = 0 \Rightarrow \nabla^2 \Phi_H(\mathbf{r}) = 0$

For $0 \leq r \leq a$ $\Phi_H(\mathbf{r}) = \sum_l \delta_l r^l P_l(\cos \theta)$

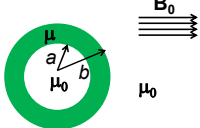
For $a \leq r \leq b$ $\Phi_H(\mathbf{r}) = \sum_l \left(\beta_l r^l + \frac{\gamma_l}{r^{l+1}} \right) P_l(\cos \theta)$

For $r \geq b$ $\Phi_H(\mathbf{r}) = -\frac{B_0}{\mu_0} r \cos \theta + \sum_l \frac{\alpha_l}{r^{l+1}} P_l(\cos \theta)$

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Example: permalloy, mumetal $\mu/\mu_0 \sim 10^4$ -- continued

Applying boundary conditions
(only $l = 1$ terms contribute):



At $r = a$ $\delta_1 = \frac{\mu}{\mu_0} \left(\beta_1 - 2 \frac{\gamma_1}{a^3} \right)$

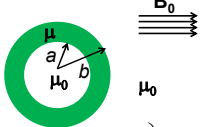
$$a \delta_1 = a \beta_1 + \frac{\gamma_1}{a^2}$$

At $r = b$ $\frac{\mu}{\mu_0} \left(\beta_1 - 2 \frac{\gamma_1}{b^3} \right) = -\frac{B_0}{\mu_0} - 2 \frac{\alpha_1}{b^3}$

$$b \beta_1 + \frac{\gamma_1}{b^2} = -b \frac{B_0}{\mu_0} + \frac{\alpha_1}{b^2}$$

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Example: permalloy, mumetal $\mu/\mu_0 \sim 10^4$ -- continued



When the dust clears:

$$\delta_1 = \left(\frac{-9\mu/\mu_0}{(2\mu/\mu_0 + 1)(\mu/\mu_0 + 2) - 2(a/b)^3(\mu/\mu_0 - 1)^2} \right) \frac{B_0}{\mu_0}$$

$$\approx \frac{1}{\mu/\mu_0} \left(\frac{-9/2}{(1 - (a/b)^3)} \frac{B_0}{\mu_0} \right)$$

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Energy associated with magnetic fields

Note: We previously used without proof --
the force on a magnetic dipole \mathbf{m} in an external \mathbf{B} field is:

$$\mathbf{F} = \nabla(\mathbf{m} \cdot \mathbf{B})$$

This implies that energy associated with aligning a
magnetic dipole \mathbf{m} in an external \mathbf{B} field is given by:

$$U = -\mathbf{m} \cdot \mathbf{B}$$

Macroscopic energies --

It can be shown that: $W_B = \frac{1}{2} \int d^3r \mathbf{B}(\mathbf{r}) \cdot \mathbf{H}(\mathbf{r})$

In analogy to: $W_E = \frac{1}{2} \int d^3r \mathbf{E}(\mathbf{r}) \cdot \mathbf{D}(\mathbf{r})$

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