

**PHY 712 Electrodynamics  
9-9:50 AM MWF Olin 105**

**Plan for Lecture 25:**

**Start reading Chap. 11**

**A. Equations in cgs (Gaussian) units**

**B. Special theory of relativity**

**C. Lorentz transformation relations**

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23	Fri: 03/16/2018	Chap. 9	Harmonic radiation	#15	03/21/2018
24	Mon: 03/19/2018	Chap. 9 & 10	Interference and Scattering	#16	03/23/2018
25	Wed: 03/21/2018	Chap. 11	Special relativity	#17	03/26/2018
26	Fri: 03/23/2018				
27	Mon: 03/26/2018				
28	Wed: 03/28/2018				
	Fri: 03/30/2018	No class	Good Friday		
29	Mon: 04/02/2018				
30	Wed: 04/04/2018				
31	Fri: 04/06/2018				
32	Mon: 04/09/2018				
33	Wed: 04/11/2018				
34	Fri: 04/13/2018				
35	Mon: 04/16/2018				
36	Wed: 04/18/2018				
37	Fri: 04/20/2018				
38	Mon: 04/23/2018				
39	Wed: 04/25/2018				
	Fri: 04/27/2018		Presentations I		
	Mon: 04/30/2018		Presentations II		
	Wed: 05/02/2018		Presentations III		

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**Events**

Colloquium: "Converting Agricultural Waste into Value-Added Products: The Case of the Citrus Peel" Mar. 21, 2018 at 4 PM  
Professor Wajler Grady, Emeritus Distinguished Professor of Mechanical Engineering, Baylor University & Emeritus Professor of Mechanical Engineering, The University of Georgia  
George H. Williams, Jr. Lecture Hall, Olin 101 Wed. Mar.

Colloquium: "Can shining laser light on our heads make us smarter?" - Understanding transcranial photobiomodulation by 1064-nm laser." Mar. 22, 2018 at 4 PM  
Professor Chang Liu, Professor of

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**Basic equations of electrodynamics**

CGS (Gaussian)	SI
$\nabla \cdot \mathbf{D} = 4\pi\rho$	$\nabla \cdot \mathbf{D} = \rho$
$\nabla \cdot \mathbf{B} = 0$	$\nabla \cdot \mathbf{B} = 0$
$\nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t}$	$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$
$\nabla \times \mathbf{H} = \frac{4\pi}{c} \mathbf{J} + \frac{1}{c} \frac{\partial \mathbf{D}}{\partial t}$	$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$
$\mathbf{F} = q(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B})$	$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$
$u = \frac{1}{8\pi} (\mathbf{E} \cdot \mathbf{D} + \mathbf{B} \cdot \mathbf{H})$	$u = \frac{1}{2} (\mathbf{E} \cdot \mathbf{D} + \mathbf{B} \cdot \mathbf{H})$
$\mathbf{S} = \frac{c}{4\pi} (\mathbf{E} \times \mathbf{H})$	$\mathbf{S} = (\mathbf{E} \times \mathbf{H})$

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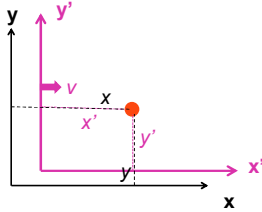
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**Notions of special relativity**

- The basic laws of physics are the same in all frames of reference (at rest or moving at constant velocity).
- The speed of light in vacuum  $c$  is the same in all frames of reference.



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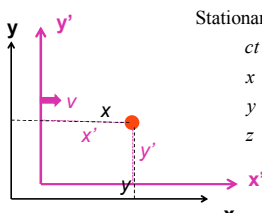
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**Lorentz transformations**

Convenient notation :

$$\beta \equiv \frac{v}{c}$$

$$\gamma \equiv \frac{1}{\sqrt{1 - \beta^2}}$$



Stationary frame	Moving frame
$ct$	$= \gamma(ct' + \beta x')$
$x$	$= \gamma(x' + \beta ct')$
$y$	$= y'$
$z$	$= z'$

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**Lorentz transformations -- continued**  $\beta \equiv \frac{v}{c}$   $\gamma \equiv \frac{1}{\sqrt{1-\beta^2}}$

For the moving frame with  $\mathbf{v} = v\hat{\mathbf{x}}$ :

$$\mathcal{L} = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \mathcal{L}^{-1} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \mathcal{L} \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} \quad \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \mathcal{L}^{-1} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

Notice:

$$c^2t'^2 - x'^2 - y'^2 - z'^2 = c^2t^2 - x^2 - y^2 - z^2$$

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**Examples of other 4-vectors applicable to the Lorentz transformation:**

For the moving frame with  $\mathbf{v} = v\hat{\mathbf{x}}$ :  $\beta \equiv \frac{v}{c}$   $\gamma \equiv \frac{1}{\sqrt{1-\beta^2}}$

$$\mathcal{L} = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \mathcal{L}^{-1} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} \omega/c \\ k_x \\ k_y \\ k_z \end{pmatrix} = \mathcal{L} \begin{pmatrix} \omega'/c \\ k'_x \\ k'_y \\ k'_z \end{pmatrix} \quad \begin{pmatrix} \omega'/c \\ k'_x \\ k'_y \\ k'_z \end{pmatrix} = \mathcal{L}^{-1} \begin{pmatrix} \omega/c \\ k_x \\ k_y \\ k_z \end{pmatrix} \quad \text{Note: } \omega t - \mathbf{k} \cdot \mathbf{r} = \omega' t' - \mathbf{k}' \cdot \mathbf{r}'$$

$$\begin{pmatrix} E \\ p_x c \\ p_y c \\ p_z c \end{pmatrix} = \mathcal{L} \begin{pmatrix} E' \\ p'_x c \\ p'_y c \\ p'_z c \end{pmatrix} \quad \begin{pmatrix} E' \\ p'_x c \\ p'_y c \\ p'_z c \end{pmatrix} = \mathcal{L}^{-1} \begin{pmatrix} E \\ p_x c \\ p_y c \\ p_z c \end{pmatrix} \quad \text{Note: } E^2 - p^2 c^2 = E'^2 - p'^2 c^2$$

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**The Doppler Effect**

For the moving frame with  $\mathbf{v} = v\hat{\mathbf{x}}$ :

$$\mathcal{L} = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \mathcal{L}^{-1} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} \omega/c \\ k_x \\ k_y \\ k_z \end{pmatrix} = \mathcal{L} \begin{pmatrix} \omega'/c \\ k'_x \\ k'_y \\ k'_z \end{pmatrix} \quad \begin{pmatrix} \omega'/c \\ k'_x \\ k'_y \\ k'_z \end{pmatrix} = \mathcal{L}^{-1} \begin{pmatrix} \omega/c \\ k_x \\ k_y \\ k_z \end{pmatrix} \quad \text{Note: } \omega t - \mathbf{k} \cdot \mathbf{r} = \omega' t' - \mathbf{k}' \cdot \mathbf{r}'$$

$$\omega'/c = \gamma(\omega/c - \beta k_x) \quad k'_x = \gamma(k_x - \beta \omega/c)$$

$$k'_y = k_y \quad k'_z = k_z$$

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### The Doppler Effect -- continued

$\omega' / c = \gamma(\omega / c - \beta k_x)$        $k'_x = \gamma(k_x - \beta \omega / c)$   
 $k'_y = k_y$        $k'_z = k_z$

More generally:

$\omega' / c = \gamma(\omega / c - \boldsymbol{\beta} \cdot \mathbf{k}) \equiv \gamma(\omega / c - \beta k \cos \theta)$   
 $\mathbf{k}' \cdot \hat{\boldsymbol{\beta}} = \gamma(\mathbf{k} \cdot \hat{\boldsymbol{\beta}} - \beta \omega / c) \equiv k' \cos \theta' = \gamma(k \cos \theta - \beta \omega / c)$   
 $\mathbf{k}' \times \hat{\boldsymbol{\beta}} = \mathbf{k} \times \hat{\boldsymbol{\beta}}$

For  $\theta = 0$ : ( $k = \omega / c$ )  
 $\omega' = \omega \gamma (1 - \beta)$        $\Rightarrow \omega' = \omega \sqrt{\frac{1 - \beta}{1 + \beta}}$

For  $\theta \neq 0$ : ( $k = \omega / c$ )  
 $\tan \theta' = \frac{\sin \theta}{\gamma(\cos \theta - \beta)}$

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### Electromagnetic Doppler Effect ( $\theta=0$ )

$\omega' = \omega \sqrt{\frac{1 - \beta}{1 + \beta}}$        $\beta \equiv \frac{v_{\text{source}} - v_{\text{detector}}}{c}$

### Sound Doppler Effect ( $\theta=0$ )

$\omega' = \omega \left( \frac{1 \pm v_{\text{detector}} / c_s}{1 \mp v_{\text{source}} / c_s} \right)$

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### Lorentz transformation of the velocity

Stationary frame	Moving frame
$ct$	$= \gamma(ct' + \beta x')$
$x$	$= \gamma(x' + \beta ct')$
$y$	$= y'$
$z$	$= z'$

For an infinitesimal increment:

Stationary frame	Moving frame
$cdt$	$= \gamma(cdt' + \beta dx')$
$dx$	$= \gamma(dx' + \beta cdt')$
$dy$	$= dy'$
$dz$	$= dz'$

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**Lorentz transformation of the velocity -- continued**

Stationary frame		Moving frame
$cdt$	$=$	$\gamma(cdt' + \beta dx')$
$dx$	$=$	$\gamma(dx' + \beta cdt')$
$dy$	$=$	$dy'$
$dz$	$=$	$dz'$

Define:  $u_x \equiv \frac{dx}{dt}$   $u_y \equiv \frac{dy}{dt}$   $u_z \equiv \frac{dz}{dt}$   
 $u'_x \equiv \frac{dx'}{dt'}$   $u'_y \equiv \frac{dy'}{dt'}$   $u'_z \equiv \frac{dz'}{dt'}$

$$\frac{dx}{dt} = \frac{\gamma(dx' + \beta cdt')}{\gamma(dt' + \beta dx'/c)} = \frac{u'_x + v}{1 + vu'_x/c^2} = u_x$$

$$\frac{dy}{dt} = \frac{dy'}{\gamma(dt' + \beta dx'/c)} = \frac{u'_y}{\gamma(1 + vu'_x/c^2)} = u_y$$

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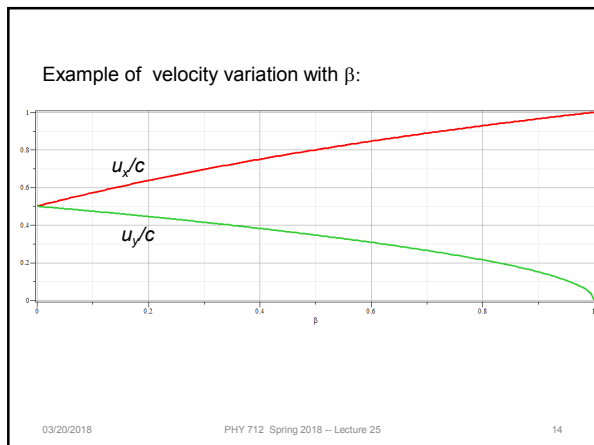
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**Extention to transformation of acceleration**

$$\mathbf{a}_{\parallel} = \frac{\left(1 - \frac{v^2}{c^2}\right)^{3/2}}{\left(1 + \frac{\mathbf{v} \cdot \mathbf{u}'}{c^2}\right)^3} \mathbf{a}'_{\parallel}$$

$$\mathbf{a}_{\perp} = \frac{\left(1 - \frac{v^2}{c^2}\right)}{\left(1 + \frac{\mathbf{v} \cdot \mathbf{u}'}{c^2}\right)^3} \left( \mathbf{a}'_{\perp} + \frac{\mathbf{v}}{c^2} \times (\mathbf{a}' \times \mathbf{u}') \right)$$

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**Velocity transformations continued:**

Consider:  $u_x = \frac{u'_x + v}{1 + vu'_x/c^2}$     $u_y = \frac{u'_y}{\gamma_v(1 + vu'_x/c^2)}$     $u_z = \frac{u'_z}{\gamma_v(1 + vu'_x/c^2)}$ .

Note that  $\gamma_u = \frac{1}{\sqrt{1 - (u/c)^2}} = \frac{1 + vu'_x/c^2}{\sqrt{1 - (u'/c)^2} \sqrt{1 - (v/c)^2}} = \gamma_v \gamma_u' (1 + vu'_x/c^2)$

$\Rightarrow \gamma_u c = \gamma_v (\gamma_u' c + \beta \gamma_u' u'_x)$   
 $\Rightarrow \gamma_u u_x = \gamma_v (\gamma_u' u'_x + \gamma_u' v) = \gamma_v (\gamma_u' u'_x + \beta \gamma_u' c)$   
 $\Rightarrow \gamma_u u_y = \gamma_u' u'_y$     $\gamma_u u_z = \gamma_u' u'_z$

Velocity 4-vector: 
$$\begin{pmatrix} \gamma_u c \\ \gamma_u u_x \\ \gamma_u u_y \\ \gamma_u u_z \end{pmatrix} = \mathcal{L}_u \begin{pmatrix} \gamma_u' c \\ \gamma_u' u'_x \\ \gamma_u' u'_y \\ \gamma_u' u'_z \end{pmatrix}$$

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**Some details:**

$\gamma_u = \gamma_v \gamma_u' (1 + vu'_x/c^2) \Rightarrow \left(1 - \frac{v^2}{c^2}\right) \left(1 - \frac{u'^2}{c^2}\right) = \left(1 - \frac{u^2}{c^2}\right) \left(1 + \frac{u_x v}{c^2}\right)^2$

where  $u_x = \frac{u'_x + v}{1 + vu'_x/c^2}$     $u_y = \frac{u'_y}{\gamma_v(1 + vu'_x/c^2)}$     $u_z = \frac{u'_z}{\gamma_v(1 + vu'_x/c^2)}$ .

$\left(\frac{u_x^2}{c^2} + \frac{u_y^2}{c^2} + \frac{u_z^2}{c^2}\right) \left(1 + \frac{u_x v}{c^2}\right)^2 = \left(\frac{u'_x + v}{c}\right)^2 + \left(\frac{u'^2}{c^2} + \frac{u'^2}{c^2}\right) \left(1 - \frac{v^2}{c^2}\right)$

$\frac{u^2}{c^2} \left(1 + \frac{u_x v}{c^2}\right)^2 = \frac{u'^2}{c^2} \left(1 - \frac{v^2}{c^2}\right) + \left(1 + \frac{u_x v}{c^2}\right)^2 - \left(1 - \frac{v^2}{c^2}\right)$

$\Rightarrow \left(1 - \frac{u^2}{c^2}\right) \left(1 + \frac{u_x v}{c^2}\right)^2 = \left(1 - \frac{u'^2}{c^2}\right) \left(1 - \frac{v^2}{c^2}\right)$

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**Significance of 4-velocity vector:**

$$\begin{pmatrix} \gamma_u c \\ \gamma_u u_x \\ \gamma_u u_y \\ \gamma_u u_z \end{pmatrix}$$

Introduce the “rest” mass  $m$  of particle characterized by velocity  $\mathbf{u}$ :

$$mc \begin{pmatrix} \gamma_u c \\ \gamma_u u_x \\ \gamma_u u_y \\ \gamma_u u_z \end{pmatrix} = \begin{pmatrix} \gamma_u mc^2 \\ \gamma_u m u_x c \\ \gamma_u m u_y c \\ \gamma_u m u_z c \end{pmatrix} = \begin{pmatrix} E \\ p_x c \\ p_y c \\ p_z c \end{pmatrix}$$

**Properties of energy-moment 4-vector:**

$$\begin{pmatrix} E \\ p_x c \\ p_y c \\ p_z c \end{pmatrix} = \mathcal{L} \begin{pmatrix} E' \\ p'_x c \\ p'_y c \\ p'_z c \end{pmatrix} \quad \begin{pmatrix} E' \\ p'_x c \\ p'_y c \\ p'_z c \end{pmatrix} = \mathcal{L}^{-1} \begin{pmatrix} E \\ p_x c \\ p_y c \\ p_z c \end{pmatrix} \quad \text{Note: } E^2 - p^2 c^2 = E'^2 - p'^2 c^2$$

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**Properties of Energy-momentum 4-vector -- continued**

$$\begin{pmatrix} E \\ p_x c \\ p_y c \\ p_z c \end{pmatrix} = \begin{pmatrix} \gamma_u m c^2 \\ \gamma_u m u_x c \\ \gamma_u m u_y c \\ \gamma_u m u_z c \end{pmatrix}$$

Note:  $E^2 - p^2 c^2 = \frac{(m c^2)^2}{1 - \beta_u^2} \left( 1 - \left(\frac{u_x}{c}\right)^2 - \left(\frac{u_y}{c}\right)^2 - \left(\frac{u_z}{c}\right)^2 \right) = (m c^2)^2 = E^2 - p^2 c^2$

Notion of "rest energy": For  $\mathbf{p} \equiv 0$ ,  $E = m c^2$

Define kinetic energy:  $E_K \equiv E - m c^2 = \sqrt{p^2 c^2 + m^2 c^4} - m c^2$

Non-relativistic limit: If  $\frac{p}{m c} \ll 1$ ,  $E_K = m c^2 \left( \sqrt{1 + \left(\frac{p}{m c}\right)^2} - 1 \right) \approx \frac{p^2}{2m}$

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**Summary of relativistic energy relationships**

$$\begin{pmatrix} E \\ p_x c \\ p_y c \\ p_z c \end{pmatrix} = \begin{pmatrix} \gamma_u m c^2 \\ \gamma_u m u_x c \\ \gamma_u m u_y c \\ \gamma_u m u_z c \end{pmatrix}$$

$E = \sqrt{p^2 c^2 + m^2 c^4} = \gamma_u m c^2$

Check:  $\sqrt{p^2 c^2 + m^2 c^4} = m c^2 \sqrt{\gamma_u^2 \beta_u^2 + 1} = \gamma_u m c^2$

Example: for an electron  $m c^2 = 0.5 \text{ MeV}$   
 for  $E = 200 \text{ GeV}$   
 $\gamma_u = \frac{E}{m c^2} = 4 \times 10^5$   
 $\beta_u = \sqrt{1 - \frac{1}{\gamma_u^2}} \approx 1 - \frac{1}{2\gamma_u^2} \approx 1 - 3 \times 10^{-12}$

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**Special theory of relativity and Maxwell's equations**

Continuity equation :  $\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0$

Lorentz gauge condition :  $\frac{1}{c} \frac{\partial \Phi}{\partial t} + \nabla \cdot \mathbf{A} = 0$

Potential equations :  $\frac{1}{c^2} \frac{\partial^2 \Phi}{\partial t^2} + \nabla^2 \Phi = 4\pi \rho$   
 $\frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} + \nabla^2 \mathbf{A} = \frac{4\pi}{c} \mathbf{J}$

Field relations :  $\mathbf{E} = -\nabla \Phi - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t}$   
 $\mathbf{B} = \nabla \times \mathbf{A}$

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More 4-vectors:

Time and position :  $\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \Rightarrow x^\alpha$

Charge and current :  $\begin{pmatrix} c\rho \\ J_x \\ J_y \\ J_z \end{pmatrix} \Rightarrow J^\alpha$

Vector and scalar potentials :  $\begin{pmatrix} \Phi \\ A_x \\ A_y \\ A_z \end{pmatrix} \Rightarrow A^\alpha$

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Lorentz transformations

$$\mathcal{L}_v = \begin{pmatrix} \gamma_v & \gamma_v \beta_v & 0 & 0 \\ \gamma_v \beta_v & \gamma_v & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Time and space :  $x^\alpha = \mathcal{L}_v x'^\alpha \equiv \mathcal{L}_v^{\alpha\beta} x'^\beta$

Charge and current :  $J^\alpha = \mathcal{L}_v J'^\alpha \equiv \mathcal{L}_v^{\alpha\beta} J'^\beta$

Vector and scalar potential :  $A^\alpha = \mathcal{L}_v A'^\alpha \equiv \mathcal{L}_v^{\alpha\beta} A'^\beta$

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