

PHY 712 Electrodynamics
9-9:50 AM MWF Olin 105

Plan for Lecture 26:

Continue reading Chap. 11 –
Theory of Special Relativity

A. Lorentz transformation relations
B. Electromagnetic field transformations
C. Connection to Liénard-Wiechert potentials
for constant velocity sources

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23	Fri: 03/16/2018	Chap. 9	Harmonic radiation	#15	03/21/2018
24	Mon: 03/19/2018	Chap. 9 & 10	Interference and Scattering	#16	03/23/2018
25	Wed: 03/21/2018	Chap. 11	Special relativity	#17	03/26/2018
26	Fri: 03/23/2018	Chap. 11	Special relativity	#18	03/28/2018
27	Mon: 03/26/2018				
28	Wed: 03/28/2018				
	Fri: 03/30/2018	No class	Good Friday		
29	Mon: 04/02/2018				
30	Wed: 04/04/2018				
31	Fri: 04/06/2018				
32	Mon: 04/09/2018				
33	Wed: 04/11/2018				
34	Fri: 04/13/2018				
35	Mon: 04/16/2018				
36	Wed: 04/18/2018				
37	Fri: 04/20/2018				
38	Mon: 04/23/2018				
39	Wed: 04/25/2018				
	Fri: 04/27/2018		Presentations I		
	Mon: 04/30/2018		Presentations II		
	Wed: 05/02/2018		Presentations III		

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Lorentz transformations

Convenient notation :

$$\beta_v \equiv \frac{v}{c}$$

$$\gamma_v \equiv \frac{1}{\sqrt{1 - \beta_v^2}}$$

Stationary frame

ct

x

y

z

Moving frame

$= \gamma(ct' + \beta x')$

$= \gamma(x' + \beta ct')$

$= y'$

$= z'$

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Lorentz transformations -- continued

For the moving frame with $\mathbf{v} = v\hat{\mathbf{x}}$:

$$\mathcal{L}_v = \begin{pmatrix} \gamma_v & \gamma_v\beta_v & 0 & 0 \\ \gamma_v\beta_v & \gamma_v & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \mathcal{L}_v^{-1} = \begin{pmatrix} \gamma_v & -\gamma_v\beta_v & 0 & 0 \\ -\gamma_v\beta_v & \gamma_v & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \mathcal{L}_v \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} \quad \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \mathcal{L}_v^{-1} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

Notice:

$$c^2t^2 - x^2 - y^2 - z^2 = c^2t'^2 - x'^2 - y'^2 - z'^2$$

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Special theory of relativity and Maxwell's equations

Continuity equation: $\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0$

Lorentz gauge condition: $\frac{1}{c} \frac{\partial \Phi}{\partial t} + \nabla \cdot \mathbf{A} = 0$

Potential equations: $\frac{1}{c^2} \frac{\partial^2 \Phi}{\partial t^2} - \nabla^2 \Phi = 4\pi\rho$

$$\frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla^2 \mathbf{A} = \frac{4\pi}{c} \mathbf{J}$$

Field relations: $\mathbf{E} = -\nabla\Phi - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t}$

$$\mathbf{B} = \nabla \times \mathbf{A}$$

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More 4-vectors:

Time and position: $\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \Rightarrow x^\alpha$

Charge and current: $\begin{pmatrix} c\rho \\ J_x \\ J_y \\ J_z \end{pmatrix} \Rightarrow J^\alpha$

Vector and scalar potentials: $\begin{pmatrix} \Phi \\ A_x \\ A_y \\ A_z \end{pmatrix} \Rightarrow A^\alpha$

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Lorentz transformations

$$\mathcal{L}_v = \begin{pmatrix} \gamma_v & \gamma_v \beta_v & 0 & 0 \\ \gamma_v \beta_v & \gamma_v & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Time and space : $x^\alpha = \mathcal{L}_v x'^\alpha \equiv \mathcal{L}_v^{\alpha\beta} x'^\beta$
 Charge and current : $J^\alpha = \mathcal{L}_v J'^\alpha \equiv \mathcal{L}_v^{\alpha\beta} J'^\beta$
 Vector and scalar potential : $A^\alpha = \mathcal{L}_v A'^\alpha \equiv \mathcal{L}_v^{\alpha\beta} A'^\beta$

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4-vector relationships

$$\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \Leftrightarrow \begin{pmatrix} A^0 \\ A^1 \\ A^2 \\ A^3 \end{pmatrix} \Leftrightarrow (A^0, \mathbf{A}) : \text{upper index 4-vector } A^\alpha \text{ for } (\alpha = 0, 1, 2, 3)$$

Keeping track of signs -- lower index 4-vector $A_\alpha = (A^0, -\mathbf{A})$
 Derivative operators :
 $\partial^\alpha = \left(\frac{\partial}{c\partial t}, -\nabla \right) \quad \partial_\alpha = \left(\frac{\partial}{c\partial t}, \nabla \right)$

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Special theory of relativity and Maxwell's equations

Continuity equation : $\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0 \quad \partial_\alpha J^\alpha = 0$
 Lorentz gauge condition : $\frac{1}{c} \frac{\partial \Phi}{\partial t} + \nabla \cdot \mathbf{A} = 0 \quad \partial_\alpha A^\alpha = 0$
 Potential equations : $\frac{1}{c^2} \frac{\partial^2 \Phi}{\partial t^2} - \nabla^2 \Phi = 4\pi\rho$
 $\frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla^2 \mathbf{A} = \frac{4\pi}{c} \mathbf{J} \quad \partial_\alpha \partial^\alpha A^\beta = \frac{4\pi}{c} J^\beta$
 Field relations : $\mathbf{E} = -\nabla\Phi - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t}$
 $\mathbf{B} = \nabla \times \mathbf{A}$

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Electric and Magnetic field relationships

$$\mathbf{E} = -\nabla\Phi - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t}$$

$$E_x = -\frac{\partial\Phi}{\partial x} - \frac{\partial A_x}{c\partial t} = -(\partial^0 A^1 - \partial^1 A^0)$$

$$E_y = -\frac{\partial\Phi}{\partial y} - \frac{\partial A_y}{c\partial t} = -(\partial^0 A^2 - \partial^2 A^0)$$

$$E_z = -\frac{\partial\Phi}{\partial z} - \frac{\partial A_z}{c\partial t} = -(\partial^0 A^3 - \partial^3 A^0)$$

$\mathbf{B} = \nabla \times \mathbf{A}$

$$B_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = -(\partial^2 A^3 - \partial^3 A^2)$$

$$B_y = \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} = -(\partial^3 A^1 - \partial^1 A^3)$$

$$B_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = -(\partial^1 A^2 - \partial^2 A^1)$$

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Field strength tensor $F^{\alpha\beta} \equiv (\partial^\alpha A^\beta - \partial^\beta A^\alpha)$

$$F^{\alpha\beta} \equiv \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

Transformation of field strength tensor

$$F^{\alpha\beta} = \mathcal{L}^{\alpha\gamma} F^{\gamma\delta} \mathcal{L}_\delta^{\beta\epsilon}$$

$$\mathcal{L} = \begin{pmatrix} \gamma_v & \gamma_v \beta_v & 0 & 0 \\ \gamma_v \beta_v & \gamma_v & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$F^{\alpha\beta} = \begin{pmatrix} 0 & -E'_x & -\gamma_v(E'_y + \beta_v B'_z) & -\gamma_v(E'_z - \beta_v B'_y) \\ E'_x & 0 & -\gamma_v(B'_z + \beta_v E'_y) & \gamma_v(B'_y - \beta_v E'_z) \\ \gamma_v(E'_y + \beta_v B'_z) & \gamma_v(B'_z + \beta_v E'_y) & 0 & -B'_x \\ \gamma_v(E'_z - \beta_v B'_y) & -\gamma_v(B'_y - \beta_v E'_z) & B'_x & 0 \end{pmatrix}$$

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Inverse transformation of field strength tensor

$$F^{\alpha\beta} = \mathcal{L}^{\alpha\gamma} F^{\gamma\delta} \mathcal{L}_\delta^{\beta\epsilon}$$

$$\mathcal{L}^{-1} = \begin{pmatrix} \gamma_v & -\gamma_v \beta_v & 0 & 0 \\ -\gamma_v \beta_v & \gamma_v & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$F^{\alpha\beta} = \begin{pmatrix} 0 & -E'_x & -\gamma_v(E'_y - \beta_v B'_z) & -\gamma_v(E'_z + \beta_v B'_y) \\ E'_x & 0 & -\gamma_v(B'_z - \beta_v E'_y) & \gamma_v(B'_y + \beta_v E'_z) \\ \gamma_v(E'_y - \beta_v B'_z) & \gamma_v(B'_z - \beta_v E'_y) & 0 & -B'_x \\ \gamma_v(E'_z + \beta_v B'_y) & -\gamma_v(B'_y + \beta_v E'_z) & B'_x & 0 \end{pmatrix}$$

Summary of results:

$$E'_x = E_x \quad B'_x = B_x$$

$$E'_y = \gamma_v(E_y - \beta_v B_z) \quad B'_y = \gamma_v(B_y + \beta_v E_z)$$

$$E'_z = \gamma_v(E_z + \beta_v B_y) \quad B'_z = \gamma_v(B_z - \beta_v E_y)$$

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Example:

Fields in moving frame :

$$\mathbf{E}' = \frac{q}{r'^3} (x' \hat{\mathbf{x}} + y' \hat{\mathbf{y}}) = \frac{q(-vt' \hat{\mathbf{x}} + b \hat{\mathbf{y}})}{((-vt')^2 + b^2)^{3/2}}$$

$$\mathbf{B}' = 0$$

Fields in stationary frame :

$$E_x = E'_x \qquad B_x = B'_x$$

$$E_y = \gamma_v (E'_y + \beta B'_z) \qquad B_y = \gamma_v (B'_y - \beta E'_z)$$

$$E_z = \gamma_v (E'_z - \beta B'_y) \qquad B_z = \gamma_v (B'_z + \beta E'_y)$$

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Example:

Fields in moving frame :

$$\mathbf{E}' = \frac{q}{r'^3} (x' \hat{\mathbf{x}} + y' \hat{\mathbf{y}}) = \frac{q(-vt' \hat{\mathbf{x}} + b \hat{\mathbf{y}})}{((-vt')^2 + b^2)^{3/2}}$$

$$\mathbf{B}' = 0$$

Fields in stationary frame :

$$E_x = E'_x = \frac{q(-vt')}{((-vt')^2 + b^2)^{3/2}}$$

$$E_y = \gamma_v (E'_y) = \frac{q(\gamma_v b)}{((-vt')^2 + b^2)^{3/2}}$$

$$B_z = \gamma_v (\beta_v E'_y) = \frac{q(\gamma_v \beta_v b)}{((-vt')^2 + b^2)^{3/2}}$$

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Example:

Fields in moving frame :

$$\mathbf{E}' = \frac{q}{r'^3} (x' \hat{\mathbf{x}} + y' \hat{\mathbf{y}}) = \frac{q(-vt' \hat{\mathbf{x}} + b \hat{\mathbf{y}})}{((-vt')^2 + b^2)^{3/2}}$$

$$\mathbf{B}' = 0$$

Fields in stationary frame :

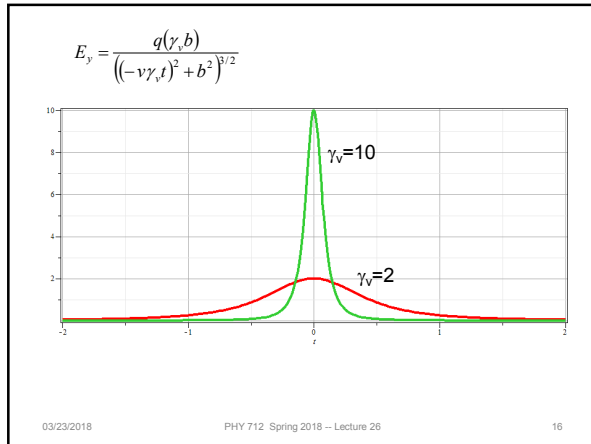
$$E_x = E'_x = \frac{q(-v\gamma_v t')}{((-v\gamma_v t')^2 + b^2)^{3/2}}$$

$$E_y = \gamma_v (E'_y) = \frac{q(\gamma_v b)}{((-v\gamma_v t')^2 + b^2)^{3/2}}$$

$$B_z = \gamma_v (\beta_v E'_y) = \frac{q(\gamma_v \beta_v b)}{((-v\gamma_v t')^2 + b^2)^{3/2}}$$

Expression in terms of consistent coordinates

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Examination of this system from the viewpoint of the Liénard-Wiechert potentials (temporarily keeping SI units)

$$\rho(\mathbf{r}, t) = q\delta^3(\mathbf{r} - \mathbf{R}_q(t)) \quad \mathbf{J}(\mathbf{r}, t) = q\dot{\mathbf{R}}_q(t)\delta^3(\mathbf{r} - \mathbf{R}_q(t)) \quad \dot{\mathbf{R}}_q(t) = \frac{d\mathbf{R}_q(t)}{dt}$$

$$\Phi(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \iint d^3r' \frac{\rho(\mathbf{r}', t')}{|\mathbf{r} - \mathbf{r}'|} \delta(t' - (t - |\mathbf{r} - \mathbf{r}'|/c))$$

$$\mathbf{A}(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0 c^2} \iint d^3r' \frac{\mathbf{J}(\mathbf{r}', t')}{|\mathbf{r} - \mathbf{r}'|} \delta(t' - (t - |\mathbf{r} - \mathbf{r}'|/c))$$

Evaluating integral over t' :

$$\int_{-\infty}^{\infty} dt' f(t') \delta(t' - (t - |\mathbf{r} - \mathbf{R}_q(t_r)|/c)) = \frac{f(t_r)}{1 - \frac{\mathbf{R}_q(t_r) \cdot (\mathbf{r} - \mathbf{R}_q(t_r))}{c|\mathbf{r} - \mathbf{R}_q(t_r)|}}$$

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Examination of this system from the viewpoint of the Liénard-Wiechert potentials – continued (SI units)

$$\Phi(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0} \frac{1}{R - \frac{\mathbf{v} \cdot \mathbf{R}}{c}}$$

$$\mathbf{A}(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0 c^2} \frac{\mathbf{v}}{R - \frac{\mathbf{v} \cdot \mathbf{R}}{c}}$$

where $\mathbf{R} = \mathbf{r} - \mathbf{R}_q(t_r)$ $\mathbf{v} = \frac{d\mathbf{R}_q(t_r)}{dt_r}$

$$\mathbf{E}(\mathbf{r}, t) = -\nabla\Phi(\mathbf{r}, t) - \frac{\partial\mathbf{A}(\mathbf{r}, t)}{\partial t}$$

$$\mathbf{B}(\mathbf{r}, t) = \nabla \times \mathbf{A}(\mathbf{r}, t)$$

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Examination of this system from the viewpoint of the Liénard-Wiechert potentials – continued (SI units)

$$\mathbf{E}(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0} \frac{1}{\left(R - \frac{\mathbf{v} \cdot \mathbf{R}}{c}\right)^3} \left[\left(\mathbf{R} - \frac{\mathbf{v}R}{c}\right) \left(1 - \frac{v^2}{c^2}\right) + \mathbf{R} \times \left\{ \left(\mathbf{R} - \frac{\mathbf{v}R}{c}\right) \times \frac{\dot{\mathbf{v}}}{c^2} \right\} \right]$$

$$\mathbf{B}(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0 c^2} \left[\frac{-\mathbf{R} \times \mathbf{v}}{\left(R - \frac{\mathbf{v} \cdot \mathbf{R}}{c}\right)^3} \left(1 - \frac{v^2}{c^2} + \frac{\dot{\mathbf{v}} \cdot \mathbf{R}}{c^2}\right) - \frac{\mathbf{R} \times \dot{\mathbf{v}} / c}{\left(R - \frac{\mathbf{v} \cdot \mathbf{R}}{c}\right)^2} \right]$$

$$\mathbf{B}(\mathbf{r}, t) = \frac{\mathbf{R} \times \mathbf{E}(\mathbf{r}, t)}{cR}$$

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Examination of this system from the viewpoint of the Liénard-Wiechert potentials – continued (Gaussian units)

$$\mathbf{E}(\mathbf{r}, t) = \frac{q}{\left(R - \frac{\mathbf{v} \cdot \mathbf{R}}{c}\right)^3} \left[\left(\mathbf{R} - \frac{\mathbf{v}R}{c}\right) \left(1 - \frac{v^2}{c^2}\right) + \mathbf{R} \times \left\{ \left(\mathbf{R} - \frac{\mathbf{v}R}{c}\right) \times \frac{\dot{\mathbf{v}}}{c^2} \right\} \right]$$

$$\mathbf{B}(\mathbf{r}, t) = \frac{q}{c} \left[\frac{-\mathbf{R} \times \mathbf{v}}{\left(R - \frac{\mathbf{v} \cdot \mathbf{R}}{c}\right)^3} \left(1 - \frac{v^2}{c^2} + \frac{\dot{\mathbf{v}} \cdot \mathbf{R}}{c^2}\right) - \frac{\mathbf{R} \times \dot{\mathbf{v}} / c}{\left(R - \frac{\mathbf{v} \cdot \mathbf{R}}{c}\right)^2} \right]$$

$$\mathbf{B}(\mathbf{r}, t) = \frac{\mathbf{R} \times \mathbf{E}(\mathbf{r}, t)}{R}$$

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Examination of this system from the viewpoint of the Liénard-Wiechert potentials – continued (Gaussian units)

$$\mathbf{E}(\mathbf{r}, t) = \frac{q}{\left(R - \frac{\mathbf{v} \cdot \mathbf{R}}{c}\right)^3} \left[\left(\mathbf{R} - \frac{\mathbf{v}R}{c}\right) \left(1 - \frac{v^2}{c^2}\right) \right]$$

For our example:
 $\mathbf{R}_r(t_r) = v t_r \hat{\mathbf{x}} \quad \mathbf{r} = b \hat{\mathbf{y}}$
 $\mathbf{R} = b \hat{\mathbf{y}} - v t_r \hat{\mathbf{x}} \quad R = \sqrt{v^2 t_r^2 + b^2}$
 $\mathbf{v} = v \hat{\mathbf{x}} \quad t_r = t - \frac{R}{c}$

$$\mathbf{B}(\mathbf{r}, t) = \frac{q}{c} \left[\frac{-\mathbf{R} \times \mathbf{v}}{\left(R - \frac{\mathbf{v} \cdot \mathbf{R}}{c}\right)^3} \left(1 - \frac{v^2}{c^2}\right) \right]$$

This should be equivalent to the result given in Jackson (11.152):

$$\mathbf{E}(x, y, z, t) = \mathbf{E}(0, b, 0, t) = q \frac{-v \gamma t \hat{\mathbf{x}} + \gamma b \hat{\mathbf{y}}}{\left(b^2 + (v \gamma t)^2\right)^{3/2}}$$

$$\mathbf{B}(x, y, z, t) = \mathbf{B}(0, b, 0, t) = q \frac{\gamma b b \hat{\mathbf{z}}}{\left(b^2 + (v \gamma t)^2\right)^{3/2}}$$

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