

PHY 712 Electrodynamics
9-9:50 AM MWF Olin 105

Plan for Lecture 28:

Continue reading Chap. 14 –
Radiation by moving charges

- 1. Motion in a line**
- 2. Motion in a circle**
- 3. Spectral analysis of radiation**

03/028/2018 PHY 712 Spring 2018 – Lecture 28 1

Wake Forest College & Graduate School of Arts and Sciences

WFU Physics

People | Events and News | Undergraduate | Graduate | Research | Resources

Events

Colloquium: "Can shining laser light on our heads make us smarter? — Understanding femtosecond photomodulation by 1064-nm laser." Mar. 29, 2018 at 4 PM

Professor Jian Li, Professor of Biomechanics, Qingdao University Professor, Fellow, American Institute for Medical and Biological Engineering, University of Texas at Arlington George P. Williams, Jr. Lecture Hall, (Olin 105) ...

Public Lecture: "Charges and Electromagnetic Fields: From the Cathode Ray Tube to Light Emitting Devices." March 30, 2018, at 10 AM

Jinwei Xu Public Presentation in Olin 105

03/028/2018 PHY 712 Spring 2018 – Lecture 28 2

| | | | | | |
|----|-----------------|--------------|--------------------------------------|-----|------------|
| 23 | Fri: 03/16/2018 | Chap. 9 | Harmonic radiation | #15 | 03/21/2018 |
| 24 | Mon: 03/19/2018 | Chap. 9 & 10 | Interference and Scattering | #16 | 03/23/2018 |
| 25 | Wed: 03/21/2018 | Chap. 11 | Special relativity | #17 | 03/26/2018 |
| 26 | Fri: 03/23/2018 | Chap. 11 | Special relativity | #18 | 03/28/2018 |
| 27 | Mon: 03/26/2018 | Chap. 11 | Special relativity | | |
| 28 | Wed: 03/28/2018 | Chap. 14 | Radiation from accelerated particles | | |
| | Fri: 03/30/2018 | No class | Good Friday | | |
| 29 | Mon: 04/02/2018 | | | | |
| 30 | Wed: 04/04/2018 | | | | |
| 31 | Fri: 04/06/2018 | | | | |
| 32 | Mon: 04/09/2018 | | | | |
| 33 | Wed: 04/11/2018 | | | | |
| 34 | Fri: 04/13/2018 | | | | |
| 35 | Mon: 04/16/2018 | | | | |
| 36 | Wed: 04/18/2018 | | | | |
| 37 | Fri: 04/20/2018 | | | | |
| 38 | Mon: 04/23/2018 | | | | |
| 39 | Wed: 04/25/2018 | | | | |
| | Fri: 04/27/2018 | | Presentations I | | |
| | Mon: 04/30/2018 | | Presentations II | | |
| | Wed: 05/02/2018 | | Presentations III | | |

03/028/2018 PHY 712 Spring 2018 – Lecture 28 3

Radiation from a moving charged particle

Variables (notation):

$$\dot{\mathbf{R}}_q(t_r) \equiv \frac{d\mathbf{R}_q(t_r)}{dt_r} \equiv \mathbf{v}$$

$$\mathbf{R}(t_r) \equiv \mathbf{r} - \mathbf{R}_q(t_r) \equiv \mathbf{R}$$

03/028/2018 PHY 712 Spring 2018 – Lecture 28 4

Liénard-Wiechert fields (cgs Gaussian units):

$$\mathbf{E}(\mathbf{r}, t) = \frac{q}{(R - \mathbf{v} \cdot \mathbf{R})^3} \left[\left(\mathbf{R} - \frac{\mathbf{v}R}{c} \right) \left(1 - \frac{v^2}{c^2} \right) + \left(\mathbf{R} \times \left\{ \left(\mathbf{R} - \frac{\mathbf{v}R}{c} \right) \times \frac{\dot{\mathbf{v}}}{c^2} \right\} \right) \right]. \quad (19)$$

$$\mathbf{B}(\mathbf{r}, t) = \frac{q}{c} \left[\frac{-\mathbf{R} \times \mathbf{v}}{(R - \frac{\mathbf{v} \cdot \mathbf{R}}{c})^3} \left(1 - \frac{v^2}{c^2} + \frac{\dot{\mathbf{v}} \cdot \mathbf{R}}{c^2} \right) - \frac{\mathbf{R} \times \dot{\mathbf{v}}/c}{(R - \frac{\mathbf{v} \cdot \mathbf{R}}{c})^2} \right]. \quad (20)$$

In this case, the electric and magnetic fields are related according to

$$\mathbf{B}(\mathbf{r}, t) = \frac{\mathbf{R} \times \mathbf{E}(\mathbf{r}, t)}{R}. \quad (21)$$

$$\dot{\mathbf{R}}_q(t_r) \equiv \frac{d\mathbf{R}_q(t_r)}{dt_r} \equiv \mathbf{v} \quad \mathbf{R}(t_r) \equiv \mathbf{r} - \mathbf{R}_q(t_r) \equiv \mathbf{R} \quad \dot{\mathbf{v}} \equiv \frac{d^2\mathbf{R}_q(t_r)}{dt_r^2}$$

03/028/2018 PHY 712 Spring 2018 – Lecture 28 5

Electric field far from source:

$$\mathbf{E}(\mathbf{r}, t) = \frac{q}{\left(R - \frac{\mathbf{v} \cdot \mathbf{R}}{c} \right)^3} \left\{ \mathbf{R} \times \left[\left(\mathbf{R} - \frac{\mathbf{v}R}{c} \right) \times \frac{\dot{\mathbf{v}}}{c^2} \right] \right\}$$

$$\mathbf{B}(\mathbf{r}, t) = \frac{\mathbf{R} \times \mathbf{E}(\mathbf{r}, t)}{R}$$

Let $\hat{\mathbf{R}} \equiv \frac{\mathbf{R}}{R}$ $\boldsymbol{\beta} \equiv \frac{\mathbf{v}}{c}$ $\dot{\boldsymbol{\beta}} \equiv \frac{\dot{\mathbf{v}}}{c}$

$$\mathbf{E}(\mathbf{r}, t) = \frac{q}{cR(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}})^3} \left\{ \hat{\mathbf{R}} \times \left[(\hat{\mathbf{R}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}} \right] \right\}$$

$$\mathbf{B}(\mathbf{r}, t) = \hat{\mathbf{R}} \times \mathbf{E}(\mathbf{r}, t)$$

03/028/2018 PHY 712 Spring 2018 – Lecture 28 6

Poynting vector:

$$\mathbf{S}(\mathbf{r}, t) = \frac{c}{4\pi} (\mathbf{E} \times \mathbf{B})$$

$$\mathbf{E}(\mathbf{r}, t) = \frac{q}{cR(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}})^3} \left\{ \hat{\mathbf{R}} \times \left[(\hat{\mathbf{R}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}} \right] \right\}$$

$$\mathbf{B}(\mathbf{r}, t) = \hat{\mathbf{R}} \times \mathbf{E}(\mathbf{r}, t)$$

$$\mathbf{S}(\mathbf{r}, t) = \frac{c}{4\pi} \hat{\mathbf{R}} |\mathbf{E}(\mathbf{r}, t)|^2 = \frac{q^2}{4\pi c R^2} \hat{\mathbf{R}} \frac{|\hat{\mathbf{R}} \times [(\hat{\mathbf{R}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}]|^2}{(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}})^6}$$

Note: We have assumed that

$$\hat{\mathbf{R}} \cdot \mathbf{E}(\mathbf{r}, t) = 0$$

03/028/2018 PHY 712 Spring 2018 -- Lecture 28 7

Power radiated

$$\mathbf{S}(\mathbf{r}, t) = \frac{c}{4\pi} \hat{\mathbf{R}} |\mathbf{E}(\mathbf{r}, t)|^2 = \frac{q^2}{4\pi c R^2} \hat{\mathbf{R}} \frac{|\hat{\mathbf{R}} \times [(\hat{\mathbf{R}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}]|^2}{(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}})^6}$$

$$\frac{dP}{d\Omega} = \mathbf{S} \cdot \hat{\mathbf{R}} R^2 = \frac{q^2}{4\pi c} \frac{|\hat{\mathbf{R}} \times [(\hat{\mathbf{R}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}]|^2}{(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}})^6}$$

In the non-relativistic limit: $\beta \ll 1$

$$\frac{dP}{d\Omega} = \frac{q^2}{4\pi c} \frac{|\hat{\mathbf{R}} \times [\dot{\mathbf{v}}]|^2}{4\pi c^3} = \frac{q^2}{4\pi c^3} |\dot{\mathbf{v}}|^2 \sin^2 \Theta$$

03/028/2018 PHY 712 Spring 2018 -- Lecture 28 8

Radiation from a moving charged particle

Variables (notation):

$$\dot{\mathbf{R}}_q(t_r) \equiv \frac{d\mathbf{R}_q(t_r)}{dt_r} \equiv \mathbf{v}$$

$$\mathbf{R}(t_r) \equiv \mathbf{r} - \mathbf{R}_q(t_r) \equiv \mathbf{R}$$

$$\frac{dP}{d\Omega} = \frac{q^2}{4\pi c^3} |\dot{\mathbf{v}}|^2 \sin^2 \Theta$$

03/028/2018 PHY 712 Spring 2018 -- Lecture 28 9

Radiation power in non-relativistic case -- continued

$$\frac{dP}{d\Omega} = \frac{q^2}{4\pi c^3} |\dot{\mathbf{v}}|^2 \sin^2 \Theta$$

$$P = \int d\Omega \frac{dP}{d\Omega} = \frac{2}{3} \frac{q^2}{c^3} |\dot{\mathbf{v}}|^2$$

03/028/2018 PHY 712 Spring 2018 -- Lecture 28 10

Radiation distribution in the relativistic case

$$\frac{dP}{d\Omega} = \mathbf{S} \cdot \hat{\mathbf{R}} R^2 = \frac{q^2}{4\pi c} \left. \frac{|\hat{\mathbf{R}} \times [(\hat{\mathbf{R}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}]|^2}{(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}})^3} \right|_{t_r = t - R/c}$$

This expression gives us the energy per unit field time t . We are often interested in the power per unit retarded time $t_r = t - R/c$:

$$\frac{dP(t)}{d\Omega} = \frac{dP_r(t_r)}{d\Omega} \frac{dt}{dt_r} \quad \frac{dt}{dt_r} = 1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}}$$

$$\frac{dP_r(t_r)}{d\Omega} = \frac{q^2}{4\pi c} \left. \frac{|\hat{\mathbf{R}} \times [(\hat{\mathbf{R}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}]|^2}{(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}})^5} \right|_{t_r = t - R/c}$$

03/028/2018 PHY 712 Spring 2018 -- Lecture 28 11

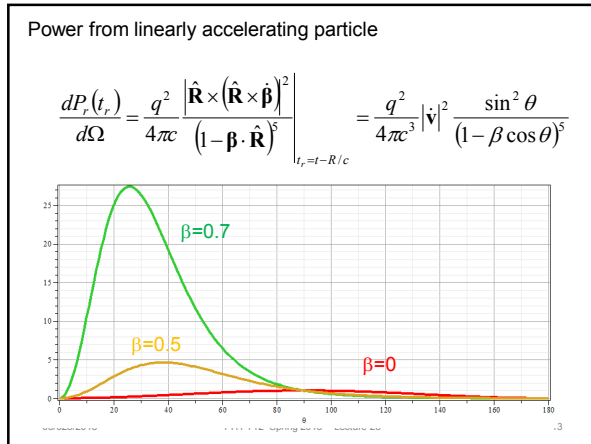
Radiation distribution in the relativistic case -- continued

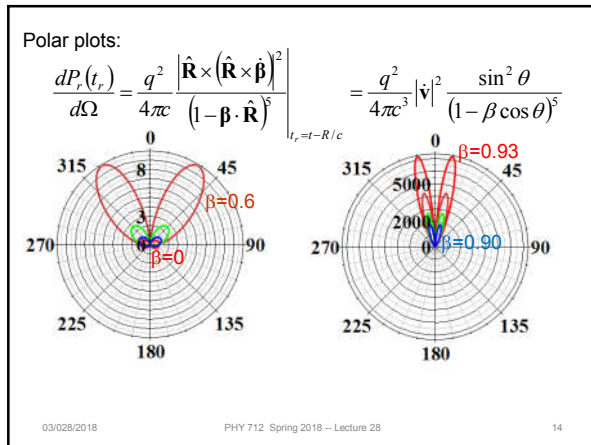
$$\frac{dP_r(t_r)}{d\Omega} = \frac{q^2}{4\pi c} \left. \frac{|\hat{\mathbf{R}} \times [(\hat{\mathbf{R}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}]|^2}{(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}})^5} \right|_{t_r = t - R/c}$$

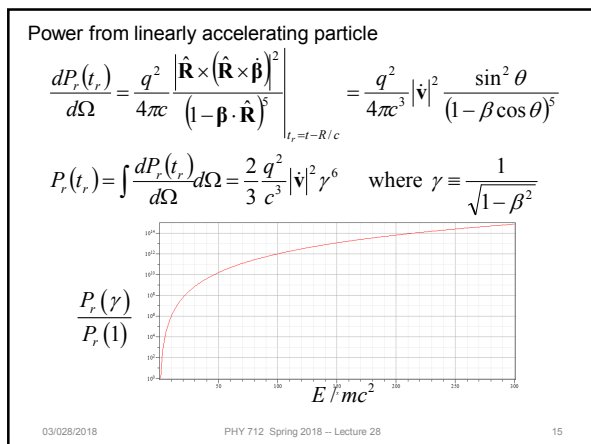
For linear acceleration: $\boldsymbol{\beta} \times \dot{\boldsymbol{\beta}} = 0$

$$\frac{dP_r(t_r)}{d\Omega} = \frac{q^2}{4\pi c} \left. \frac{|\hat{\mathbf{R}} \times (\dot{\boldsymbol{\beta}} \times \hat{\boldsymbol{\beta}})|^2}{(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}})^5} \right|_{t_r = t - R/c} = \frac{q^2}{4\pi c^3} |\dot{\mathbf{v}}|^2 \frac{\sin^2 \theta}{(1 - \beta \cos \theta)^5}$$

03/028/2018 PHY 712 Spring 2018 -- Lecture 28 12







Power distribution for linear acceleration -- continued

$$\frac{dP_r(t_r)}{d\Omega} = \frac{q^2}{4\pi c} \frac{|\hat{\mathbf{R}} \times (\hat{\mathbf{R}} \times \dot{\boldsymbol{\beta}})|^2}{(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}})^5} \bigg|_{t_r = t - R/c} = \frac{q^2}{4\pi c^3} |\dot{\mathbf{v}}|^2 \frac{\sin^2 \theta}{(1 - \beta \cos \theta)^5}$$

$$P_r(t_r) = \int d\Omega \frac{dP_r(t_r)}{d\Omega} = \frac{2}{3} \frac{q^2}{c^3} |\dot{\mathbf{v}}|^2 \gamma^6 \quad \text{where } \gamma \equiv \frac{1}{\sqrt{1 - \beta^2}}$$

03/028/2018 PHY 712 Spring 2018 -- Lecture 28 16

Power distribution for circular acceleration

$$\frac{dP_r(t_r)}{d\Omega} = \frac{q^2}{4\pi c} \frac{|\hat{\mathbf{R}} \times [(\hat{\mathbf{R}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}]|^2}{(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}})^5} \bigg|_{t_r = t - R/c}$$

$$= \frac{q^2}{4\pi c} \frac{|\dot{\boldsymbol{\beta}}|^2 (1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}})^2 - (\hat{\mathbf{R}} \cdot \dot{\boldsymbol{\beta}})^2 (1 - \beta^2)}{(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}})^5} \bigg|_{t_r = t - R/c}$$

$$P_r(t_r) = \int d\Omega \frac{dP_r(t_r)}{d\Omega} = \frac{2}{3} \frac{q^2}{c^3} |\dot{\mathbf{v}}|^2 \gamma^4$$

03/028/2018 PHY 712 Spring 2018 -- Lecture 28 17

Power distribution for circular acceleration

$$\frac{dP_r(t_r)}{d\Omega} = \frac{q^2}{4\pi c} \frac{|\dot{\boldsymbol{\beta}}|^2 (1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}})^2 - (\hat{\mathbf{R}} \cdot \dot{\boldsymbol{\beta}})^2 (1 - \beta^2)}{(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}})^5} \bigg|_{t_r = t - R/c}$$

$$= \frac{q^2}{4\pi c^3} \frac{|\dot{\mathbf{v}}|^2}{(1 - \beta \cos(\theta))^3} \left(1 - \frac{\cos^2 \theta \sin^2 \phi}{\gamma^2 (1 - \beta \cos(\theta))^2} \right)$$

03/028/2018 PHY 712 Spring 2018 -- Lecture 28 18

Spectral composition of electromagnetic radiation
 Previously we determined the power distribution from a charged particle:

$$\frac{dP(t)}{d\Omega} = \mathbf{S} \cdot \hat{\mathbf{R}} R^2 = \frac{q^2}{4\pi c} \left. \frac{|\hat{\mathbf{R}} \times [(\hat{\mathbf{R}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}]|^2}{(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}})^3} \right|_{t_r = t - R/c}$$

$$\equiv |\mathbf{a}(t)|^2$$

where $\mathbf{a}(t) \equiv \sqrt{\frac{q^2}{4\pi c}} \left. \frac{|\hat{\mathbf{R}} \times [(\hat{\mathbf{R}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}]|}{(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}})^3} \right|_{t_r = t - R/c}$

Time integrated power per solid angle :

$$\frac{dW}{d\Omega} = \int_{-\infty}^{\infty} dt \frac{dP(t)}{d\Omega} = \int_{-\infty}^{\infty} dt |\mathbf{a}(t)|^2 = \int_{-\infty}^{\infty} d\omega |\tilde{\mathbf{a}}(\omega)|^2$$

03/028/2018 PHY 712 Spring 2018 -- Lecture 28 19

Spectral composition of electromagnetic radiation -- continued

Time integrated power per solid angle :

$$\frac{dW}{d\Omega} = \int_{-\infty}^{\infty} dt \frac{dP(t)}{d\Omega} = \int_{-\infty}^{\infty} dt |\mathbf{a}(t)|^2 = \int_{-\infty}^{\infty} d\omega |\tilde{\mathbf{a}}(\omega)|^2$$

Fourier amplitude :

$$\tilde{\mathbf{a}}(\omega) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt \mathbf{a}(t) e^{i\omega t} \quad \mathbf{a}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega \tilde{\mathbf{a}}(\omega) e^{-i\omega t}$$

Parseval's theorem
Marc-Antoine Parseval des Chênes 1755-1836
<http://www-history.mcs.st-andrews.ac.uk/Biographies/Parseval.html>

03/028/2018 PHY 712 Spring 2018 -- Lecture 28 20

Spectral composition of electromagnetic radiation -- continued

Consequences of Parseval's analysis :

$$\frac{dW}{d\Omega} = \int_{-\infty}^{\infty} dt \frac{dP(t)}{d\Omega} = \int_{-\infty}^{\infty} dt |\mathbf{a}(t)|^2 = \int_{-\infty}^{\infty} d\omega |\tilde{\mathbf{a}}(\omega)|^2$$

Note that : $\tilde{\mathbf{a}}(\omega) = \tilde{\mathbf{a}}^*(-\omega)$

$$\frac{dW}{d\Omega} = \int_{-\infty}^{\infty} d\omega |\tilde{\mathbf{a}}(\omega)|^2 = \int_0^{\infty} d\omega \left(|\tilde{\mathbf{a}}(\omega)|^2 + |\tilde{\mathbf{a}}(-\omega)|^2 \right) \equiv \int_0^{\infty} d\omega \frac{\partial^2 I}{\partial \Omega \partial \omega}$$

$$\frac{\partial^2 I}{\partial \Omega \partial \omega} \equiv 2 |\tilde{\mathbf{a}}(\omega)|^2$$

03/028/2018 PHY 712 Spring 2018 -- Lecture 28 21

Spectral composition of electromagnetic radiation -- continued

For our case:
$$\mathbf{a}(t) \equiv \sqrt{\frac{q^2}{4\pi}} \frac{\hat{\mathbf{R}} \times [(\hat{\mathbf{R}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}]}{(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}})^3} \Big|_{t_r = t - R/c}$$

Fourier amplitude:

$$\begin{aligned} \tilde{\mathbf{a}}(\omega) &\equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt \mathbf{a}(t) e^{i\omega t} \\ &= \sqrt{\frac{q^2}{8\pi^2 c}} \int_{-\infty}^{\infty} dt \frac{\hat{\mathbf{R}} \times [(\hat{\mathbf{R}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}]}{(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}})^3} \Big|_{t_r = t - R/c} e^{i\omega t} \end{aligned}$$

03/028/2018 PHY 712 Spring 2018 -- Lecture 28 22

Spectral composition of electromagnetic radiation -- continued

Fourier amplitude:

$$\begin{aligned} \tilde{\mathbf{a}}(\omega) &\equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt \mathbf{a}(t) e^{i\omega t} \\ &= \sqrt{\frac{q^2}{8\pi^2 c}} \int_{-\infty}^{\infty} dt \frac{\hat{\mathbf{R}} \times [(\hat{\mathbf{R}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}]}{(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}})^3} \Big|_{t_r = t - R/c} e^{i\omega t} \\ &= \sqrt{\frac{q^2}{8\pi^2 c}} \int_{-\infty}^{\infty} dt_r \frac{d}{dt_r} \frac{\hat{\mathbf{R}} \times [(\hat{\mathbf{R}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}]}{(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}})^3} \Big|_{t_r = t - R/c} e^{i\omega(t_r + R(t_r)/c)} \\ &= \sqrt{\frac{q^2}{8\pi^2 c}} \int_{-\infty}^{\infty} dt_r \frac{\hat{\mathbf{R}} \times [(\hat{\mathbf{R}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}]}{(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}})^2} \Big|_{t_r = t - R/c} e^{i\omega(t_r + R(t_r)/c)} \end{aligned}$$

03/028/2018 PHY 712 Spring 2018 -- Lecture 28 23

Spectral composition of electromagnetic radiation -- continued

Exact expression:

$$\tilde{\mathbf{a}}(\omega) = \sqrt{\frac{q^2}{8\pi^2 c}} \int_{-\infty}^{\infty} dt_r \frac{\hat{\mathbf{R}} \times [(\hat{\mathbf{R}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}]}{(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}})^2} \Big|_{t_r = t - R/c} e^{i\omega(t_r + R(t_r)/c)}$$

Recall: $\dot{\mathbf{R}}_q(t_r) \equiv \frac{d\mathbf{R}_q(t_r)}{dt_r} \equiv \mathbf{v}$ $\mathbf{R}(t_r) \equiv \mathbf{r} - \mathbf{R}_q(t_r) \equiv \mathbf{R}$

For $r \gg R_q(t_r)$ $R(t_r) \approx r - \hat{\mathbf{r}} \cdot \mathbf{R}_q(t_r)$ where $\hat{\mathbf{r}} \equiv \frac{\mathbf{r}}{r}$

At the same level of approximation: $\hat{\mathbf{R}} \approx \hat{\mathbf{r}}$

03/028/2018 PHY 712 Spring 2018 -- Lecture 28 24

Spectral composition of electromagnetic radiation -- continued

Exact expression:

$$\tilde{\mathbf{a}}(\omega) = \sqrt{\frac{q^2}{8\pi^2 c}} \int_{-\infty}^{\infty} dt_r \frac{\hat{\mathbf{R}} \times [(\hat{\mathbf{R}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}]}{(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}})^2} \Bigg|_{t_r = t - R/c} e^{i\omega(t_r + R(t_r)/c)}$$

Approximate expression:

$$\tilde{\mathbf{a}}(\omega) = \sqrt{\frac{q^2}{8\pi^2 c}} e^{i\omega(r/c)} \int_{-\infty}^{\infty} dt_r \frac{\hat{\mathbf{r}} \times [(\hat{\mathbf{r}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}]}{(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{r}})^2} \Bigg|_{t_r = t - R/c} e^{i\omega(t_r - \hat{\mathbf{r}} \cdot \mathbf{R}_q(t_r)/c)}$$

Resulting spectral intensity expression:

$$\frac{\partial^2 I}{\partial \omega \partial \Omega} = \frac{q^2}{4\pi^2 c} \left| \int_{-\infty}^{\infty} dt_r \frac{\hat{\mathbf{r}} \times [(\hat{\mathbf{r}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}]}{(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{r}})^2} \Bigg|_{t_r = t - R/c} e^{i\omega(t_r - \hat{\mathbf{r}} \cdot \mathbf{R}_q(t_r)/c)} \right|^2$$

03/028/2018 PHY 712 Spring 2018 -- Lecture 28 25

Example - radiation from a collinear acceleration burst

$$\frac{\partial^2 I}{\partial \omega \partial \Omega} = \frac{q^2}{4\pi^2 c} \left| \int_{-\infty}^{\infty} dt_r \frac{\hat{\mathbf{r}} \times [(\hat{\mathbf{r}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}]}{(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{r}})^2} \Bigg|_{t_r = t - R/c} e^{i\omega(t_r - \hat{\mathbf{r}} \cdot \mathbf{R}_q(t_r)/c)} \right|^2$$

Suppose that $\dot{\boldsymbol{\beta}} = \begin{cases} \frac{\hat{\boldsymbol{\beta}} \Delta v}{c\tau} & 0 < t_r < \tau \\ 0 & \text{otherwise} \end{cases}$

$$\frac{\partial^2 I}{\partial \omega \partial \Omega} = \frac{q^2}{4\pi^2 c^3} \left| \frac{\hat{\mathbf{r}} \times [\hat{\mathbf{r}} \times \hat{\boldsymbol{\beta}}] \Delta v}{(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{r}})^2 \tau} \int_0^\tau dt_r e^{i\omega(t_r - \hat{\mathbf{r}} \cdot \boldsymbol{\beta} t_r)} \right|^2 \quad \text{Let } \boldsymbol{\beta} \cdot \hat{\mathbf{r}} = \beta \cos \theta$$

$$\frac{\partial^2 I}{\partial \omega \partial \Omega} = \frac{q^2}{4\pi^2 c^3} \left(\frac{\Delta v \sin \theta}{(1 - \beta \cos \theta)^2} \frac{\sin(\omega\tau(1 - \beta \cos \theta)/2)}{(\omega\tau(1 - \beta \cos \theta)/2)} \right)^2$$

03/028/2018 PHY 712 Spring 2018 -- Lecture 28 26

Example:

Suppose that $\dot{\boldsymbol{\beta}} = \begin{cases} \frac{\hat{\boldsymbol{\beta}} \Delta v}{c\tau} & 0 < t_r < \tau \\ 0 & \text{otherwise} \end{cases}$

$$\frac{\partial^2 I}{\partial \omega \partial \Omega} = \frac{q^2}{4\pi^2 c^3} \left(\frac{\Delta v \sin \theta}{(1 - \beta \cos \theta)^2} \frac{\sin(\omega\tau(1 - \beta \cos \theta)/2)}{(\omega\tau(1 - \beta \cos \theta)/2)} \right)^2$$

03/028/2018 PHY 712 Spring 2018 -- Lecture 28 27

Spectral composition of electromagnetic radiation -- continued

Alternative expression --

It can be shown that:

$$\frac{\hat{\mathbf{r}} \times [(\hat{\mathbf{r}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}]}{(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{r}})^2} = \frac{d}{dt_r} \left(\frac{\hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \boldsymbol{\beta})}{(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{r}})} \right)$$

Integration by parts and assumptions about the integration limit behavior shows that the spectral intensity depends on the following integral:

$$\frac{\partial^2 I}{\partial \omega \partial \Omega} = \frac{q^2 \omega^2}{4\pi^2 c} \left| \int_{-\infty}^{\infty} dt_r [\hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \boldsymbol{\beta}(t_r))] e^{i\omega(t_r - \hat{\mathbf{r}} \cdot \mathbf{R}_q(t_r)/c)} \right|^2$$

03/028/2018

PHY 712 Spring 2018 -- Lecture 28

28
