

**PHY 712 Electrodynamics  
9-9:50 AM MWF Olin 105**

**Plan for Lecture 29:**

**Continue reading Chap. 14 – Synchrotron radiation**

- 1. Radiation from electron synchrotron devices**
- 2. Radiation from astronomical objects in circular orbits**

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23	Fri: 03/16/2018	Chap. 9	Harmonic radiation	#15	03/21/2018
24	Mon: 03/19/2018	Chap. 9 & 10	Interference and Scattering	#16	03/23/2018
25	Wed: 03/21/2018	Chap. 11	Special relativity	#17	03/26/2018
26	Fri: 03/23/2018	Chap. 11	Special relativity	#18	03/28/2018
27	Mon: 03/26/2018	Chap. 11	Special relativity		
28	Wed: 03/28/2018	Chap. 14	Radiation from accelerated particles		
	Fri: 03/30/2018	No class	Good Friday		
29	Mon: 04/02/2018	Chap. 14	Synchrotron radiation	#19	04/06/2018
30	Wed: 04/04/2018				
31	Fri: 04/06/2018				
32	Mon: 04/09/2018				
33	Wed: 04/11/2018				
34	Fri: 04/13/2018				
35	Mon: 04/16/2018				
36	Wed: 04/18/2018				
37	Fri: 04/20/2018				
38	Mon: 04/23/2018				
39	Wed: 04/25/2018				
	Fri: 04/27/2018		Presentations I		
	Mon: 04/30/2018		Presentations II		
	Wed: 05/02/2018		Presentations III		

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**Radiation from charged particle in circular path**

Power distribution for circular acceleration

$$\frac{dP_r(t_r)}{d\Omega} = \frac{q^2}{4\pi c} \left| \hat{\mathbf{R}} \times [(\hat{\mathbf{R}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}] \right|^2 \Big|_{t_r = t - R/c}$$

$$= \frac{q^2}{4\pi c} \frac{|\dot{\boldsymbol{\beta}}|^2 (1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}})^2 - (\hat{\mathbf{R}} \cdot \dot{\boldsymbol{\beta}})^2 (1 - \beta^2)}{(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}})^5} \Big|_{t_r = t - R/c}$$

$$P_r(t_r) = \int d\Omega \frac{dP_r(t_r)}{d\Omega} = \frac{2}{3} \frac{q^2}{c^3} |\dot{\mathbf{v}}|^2 \gamma^4$$

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Spectral composition of electromagnetic radiation -- continued

When the dust clears, the spectral intensity depends on the following integral :

$$\frac{\partial^2 I}{\partial \omega \partial \Omega} = \frac{q^2 \omega^2}{4\pi^2 c} \left| \int_{-\infty}^{\infty} dt_r [\hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \boldsymbol{\beta}(t_r))] e^{i\omega(t_r - \hat{\mathbf{r}} \cdot \mathbf{R}_q(t_r)/c)} \right|^2$$

Recall that the spectral intensity is related to the time integrated power:

$$\int_{-\infty}^{\infty} dt \frac{dP(t)}{d\Omega} = \int_{-\infty}^{\infty} d\omega \frac{\partial^2 I}{\partial \omega \partial \Omega}$$

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Synchrotron radiation light source installations

Synchrotron radiation center in Madison, Wisconsin



$E_c = 0.5 \text{ GeV}$  and  $1 \text{ GeV}$ ;  $\lambda_c = 20 \text{ \AA}$  and  $10 \text{ \AA}$

<http://www.src.wisc.edu/>

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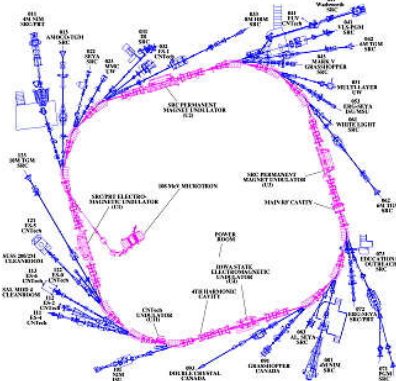
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SRC – “Aladdin” -- Madison Wisconsin



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Brookhaven National Laboratory – National Light Source



$E_c = 0.6 \text{ GeV}; \quad \lambda_c = 20 \text{ \AA}$

<http://www.bnl.gov/ps/>

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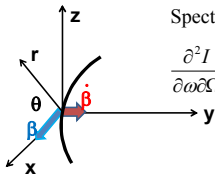
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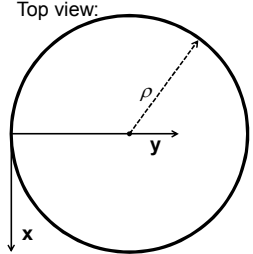
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Spectral intensity relationship :



$$\frac{\partial^2 I}{\partial \omega \partial \Omega} = \frac{q^2 \omega^2}{4\pi^2 c} \left| \int_{-\infty}^{\infty} dt_r [\hat{r} \times (\hat{r} \times \boldsymbol{\beta}(t_r))] e^{i\omega(t_r - \hat{r} \cdot \mathbf{R}_q(t_r)/c)} \right|^2$$

Top view:



$\mathbf{R}_q(t_r) = \rho \hat{x} \sin(vt_r / \rho) + \rho \hat{y} (1 - \cos(vt_r / \rho))$   
 $\boldsymbol{\beta}(t_r) = \beta (\hat{x} \cos(vt_r / \rho) + \hat{y} \sin(vt_r / \rho))$   
 For convenience, choose:  
 $\hat{r} = \hat{x} \cos \theta + \hat{z} \sin \theta$

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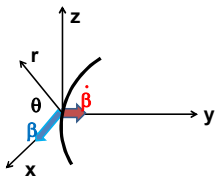
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$\mathbf{R}_q(t_r) = \rho \hat{x} \sin(vt_r / \rho) + \rho \hat{y} (1 - \cos(vt_r / \rho))$   
 $\boldsymbol{\beta}(t_r) = \beta (\hat{x} \cos(vt_r / \rho) + \hat{y} \sin(vt_r / \rho))$   
 For convenience, choose:  
 $\hat{r} = \hat{x} \cos \theta + \hat{z} \sin \theta$

Note that we have previous shown that in the radiation zone, the Poynting vector is in the  $\hat{r}$  direction; we can then choose to analyze two orthogonal polarization directions:

$\boldsymbol{\epsilon}_{\parallel} = \hat{y} \quad \boldsymbol{\epsilon}_{\perp} = -\hat{x} \sin \theta + \hat{z} \cos \theta$   
 $\hat{r} \times (\hat{r} \times \boldsymbol{\beta}) = \beta (-\boldsymbol{\epsilon}_{\parallel} \sin(vt_r / \rho) + \boldsymbol{\epsilon}_{\perp} \sin \theta \cos(vt_r / \rho))$

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$\epsilon_{\parallel} = \hat{y}$        $\epsilon_{\perp} = -\hat{x} \sin \theta + \hat{z} \cos \theta$   
 $\hat{r} \times (\hat{r} \times \beta) =$   
 $\beta (-\epsilon_{\parallel} \sin(vt_r / \rho) + \epsilon_{\perp} \sin \theta \cos(vt_r / \rho))$

$$\frac{d^2 I}{d\omega d\Omega} = \frac{q^2 \omega^2}{4\pi^2 c} \left| \int_{-\infty}^{\infty} \hat{r} \times (\hat{r} \times \beta) e^{i\omega(t - \hat{r} \cdot \mathbf{R}_q(t)/c)} dt \right|^2$$

$$\frac{d^2 I}{d\omega d\Omega} = \frac{q^2 \omega^2 \beta^2}{4\pi^2 c} \{ |C_{\parallel}(\omega)|^2 + |C_{\perp}(\omega)|^2 \}$$

$$C_{\parallel}(\omega) = \int_{-\infty}^{\infty} dt \sin(vt / \rho) e^{i\omega(t - \frac{\rho}{c} \cos \theta \sin(vt / \rho))}$$

$$C_{\perp}(\omega) = \int_{-\infty}^{\infty} dt \sin \theta \cos(vt / \rho) e^{i\omega(t - \frac{\rho}{c} \cos \theta \sin(vt / \rho))}$$

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We will analyze this expression for two different cases. The first case, is appropriate for man-made synchrotrons used as light sources. In this case, the light is produced by short bursts of electrons moving close to the speed of light ( $v \approx c(1 - 1/(2\gamma^2))$ ) passing a beam line port. In addition, because of the design of the radiation ports,  $\theta \approx 0$ , and the relevant integration times  $t$  are close to  $t \approx 0$ . This results in the form shown in Eq. 14.79 of your text. It is convenient to rewrite this form in terms of a critical frequency  $\omega_c \equiv \frac{3c\gamma^3}{2\rho}$ .

$$\frac{d^2 I}{d\omega d\Omega} = \frac{3q^2 \gamma^2}{4\pi^2 c} \left( \frac{\omega}{\omega_c} \right)^2 (1 + \gamma^2 \theta^2)^2 \left\{ \left[ K_{2/3} \left( \frac{\omega}{2\omega_c} (1 + \gamma^2 \theta^2)^{3/2} \right) \right]^2 + \frac{\gamma^2 \theta^2}{1 + \gamma^2 \theta^2} \left[ K_{1/3} \left( \frac{\omega}{2\omega_c} (1 + \gamma^2 \theta^2)^{3/2} \right) \right]^2 \right\}$$

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Some details:

Modified Bessel functions

$$K_{1/3}(\xi) = \sqrt{3} \int_0^{\infty} dx \cos\left[\frac{3}{2}\xi\left(x + \frac{1}{3}x^3\right)\right] \quad K_{2/3}(\xi) = \sqrt{3} \int_0^{\infty} dx x \sin\left[\frac{3}{2}\xi\left(x + \frac{1}{3}x^3\right)\right]$$

Exponential factor

$$\omega(t_r - \hat{r} \cdot \mathbf{R}_q(t_r)/c) = \omega\left(t_r - \frac{\rho}{c} \cos \theta \sin(vt_r / \rho)\right)$$

In the limit of  $t_r \approx 0$ ,  $\theta \approx 0$ ,  $v \approx c\left(1 - \frac{1}{2\gamma^2}\right)$

$$\omega(t_r - \hat{r} \cdot \mathbf{R}_q(t_r)/c) \approx \frac{\omega t_r}{2\gamma^2} (1 + \gamma^2 \theta^2) + \frac{\omega c^2 t_r^3}{6\rho^2} = \frac{3}{2} \xi \left(x + \frac{1}{3} x^3\right)$$

where  $\xi = \frac{\omega \rho}{3c\gamma^3} (1 + \gamma^2 \theta^2)^{3/2}$  and  $x = \frac{c\gamma t_r}{\rho(1 + \gamma^2 \theta^2)^{1/2}}$

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$$\frac{d^2 I}{d\omega d\Omega} = \frac{3q^2 \gamma^2}{4\pi^2 c} \left(\frac{\omega}{\omega_c}\right)^2 (1 + \gamma^2 \theta^2)^2 \left\{ \left[ K_{2/3} \left( \frac{\omega}{2\omega_c} (1 + \gamma^2 \theta^2)^{3/2} \right) \right]^2 + \frac{\gamma^2 \theta^2}{1 + \gamma^2 \theta^2} \left[ K_{1/3} \left( \frac{\omega}{2\omega_c} (1 + \gamma^2 \theta^2)^{3/2} \right) \right]^2 \right\}$$

By plotting the intensity as a function of  $\omega$ , we see that the intensity is largest near  $\omega \approx \omega_c$ . The plot below shows the intensity as a function of  $\omega/\omega_c$  for  $\gamma\theta=0, 0.5$  and  $1$ :

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More details

$$\frac{d^2 I}{d\omega d\Omega} = \frac{d^2 I_{\parallel}}{d\omega d\Omega} + \frac{d^2 I_{\perp}}{d\omega d\Omega}$$

$$\frac{d^2 I_{\parallel}}{d\omega d\Omega} = \frac{3q^2 \gamma^2}{4\pi^2 c} \left(\frac{\omega}{\omega_c}\right)^2 (1 + \gamma^2 \theta^2)^2 \left[ K_{2/3} \left( \frac{\omega}{2\omega_c} (1 + \gamma^2 \theta^2)^{3/2} \right) \right]^2$$

$$\frac{d^2 I_{\perp}}{d\omega d\Omega} = \frac{3q^2 \gamma^2}{4\pi^2 c} \left(\frac{\omega}{\omega_c}\right)^2 \frac{\gamma^2 \theta^2}{1 + \gamma^2 \theta^2} \left[ K_{1/3} \left( \frac{\omega}{2\omega_c} (1 + \gamma^2 \theta^2)^{3/2} \right) \right]^2$$

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The second example of synchrotron radiation comes from a distant charged particle moving in a circular trajectory such that the spectrum represents a superposition of light generated over many complete circles. In this case, there is an interference effect which results in the spectrum consisting of discrete multiples of  $v/\rho$ . For this case we need to reconsider the analysis. There is a very convenient Bessel function identity of the form:

$$e^{-ia \sin \alpha} = \sum_{m=-\infty}^{\infty} J_m(a) e^{-im\alpha} \quad \text{Here } J_m(a) \text{ is a Bessel function of integer order } m.$$

In our case  $a = \frac{\omega\rho}{c} \cos \theta$  and  $\alpha = \frac{vt}{\rho}$ .

$$C_{\parallel}(\omega) = \int_{-\infty}^{\infty} dt \sin(vt/\rho) e^{i\omega(t - \frac{\rho}{c} \cos \theta \sin(vt/\rho))} = \frac{c}{-i\omega\rho} \frac{\partial}{\partial \cos \theta} \int_{-\infty}^{\infty} dt e^{i\omega(t - \frac{\rho}{c} \cos \theta \sin(vt/\rho))}$$

$$= \frac{c}{-i\omega\rho} \frac{\partial}{\partial \cos \theta} \sum_{m=-\infty}^{\infty} J_m \left( \frac{\omega\rho}{c} \cos \theta \right) 2\pi \delta(\omega - m \frac{v}{\rho}).$$

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Astronomical synchrotron radiation -- continued:

Note that:

$$\int_{-\infty}^{\infty} dt e^{i(\omega - m \frac{v}{\rho})t} = 2\pi \delta(\omega - m \frac{v}{\rho}).$$

$$\Rightarrow C_{\parallel}(\omega) = 2\pi i \sum_{m=-\infty}^{\infty} J'_m \left( \frac{\omega \rho}{c} \cos \theta \right) \delta(\omega - m \frac{v}{\rho}),$$

where  $J'_m(a) \equiv \frac{dJ_m(a)}{da}$

Similarly:

$$C_{\perp}(\omega) = \int_{-\infty}^{\infty} dt \sin \theta \cos(vt / \rho) e^{i\omega(t - \frac{\rho}{c} \cos \theta \sin(vt / \rho))}$$

$$= 2\pi \frac{\tan \theta}{v / c} \sum_{m=-\infty}^{\infty} J_m \left( \frac{\omega \rho}{c} \cos \theta \right) \delta(\omega - m \frac{v}{\rho}).$$

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Astronomical synchrotron radiation -- continued:

In both of the expressions, the sum over  $m$  includes both negative and positive values. However, only the positive values of  $\omega$  and therefore positive values of  $m$  are of interest. Using the identity:  $J_{-m}(a) = (-1)^m J_m(a)$ , the result becomes:

$$\frac{d^2 I}{d\omega d\Omega} = \frac{q^2 \omega^2 \beta^2}{c} \sum_{m=0}^{\infty} \delta(\omega - m \frac{v}{\rho}) \left\{ \left[ J'_m \left( \frac{\omega \rho}{c} \cos \theta \right) \right]^2 + \frac{\tan^2 \theta}{v^2 / c^2} \left[ J_m \left( \frac{\omega \rho}{c} \cos \theta \right) \right]^2 \right\}.$$

These results were derived by Julian Schwinger (Phys. Rev. **75**, 1912-1925 (1949)). The discrete case is similar to the result quoted in Problem 14.15 in Jackson's text. For information on man-made synchrotron sources, the following web page is useful:

[http://www.als.lbl.gov/als/synchrotron\\_sources.html](http://www.als.lbl.gov/als/synchrotron_sources.html).

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