PHY 712 Electrodynamics
9-9:50 AM MWF Olin 105

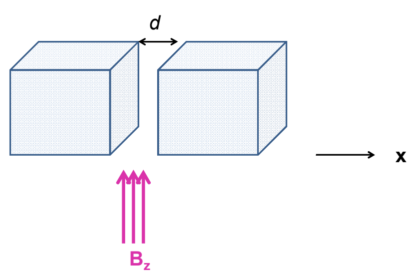
Plan for Lecture 37:

Special Topics in Electrodynamics:

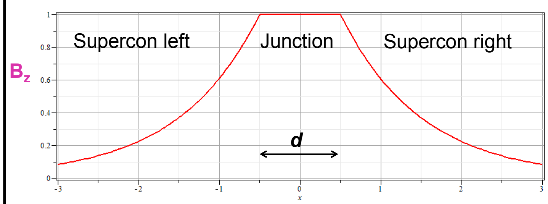
- **Tunneling between superconductors**
- **Optical properties of materials**
  - **Birefringence**
  - **Non-linear effects**

23	Fri: 03/16/2018	Chap. 9	Harmonic radiation	#15	03/21/2018
24	Mon: 03/19/2018	Chap. 9 & 10	Interference and Scattering	#16	03/23/2018
25	Wed: 03/21/2018	Chap. 11	Special relativity	#17	03/26/2018
26	Fri: 03/23/2018	Chap. 11	Special relativity	#18	03/28/2018
27	Mon: 03/26/2018	Chap. 11	Special relativity		
28	Wed: 03/28/2018	Chap. 14	Radiation from accelerated particles		
	Fri: 03/30/2018	No class	*Good Friday*		
29	Mon: 04/02/2018	Chap. 14	Synchrotron radiation	#19	04/06/2018
30	Wed: 04/04/2018	Chap. 14	Synchrotron radiation	#20	04/09/2018
31	Fri: 04/06/2018	Chap. 15	Radiation from collisions of charged particles		
32	Mon: 04/09/2018	Chap. 13	Cherenkov radiation		
33	Wed: 04/11/2018		Review		
34	Fri: 04/13/2018		Review		
35	Mon: 04/16/2018		Special topic: Superconductivity		
36	Wed: 04/18/2018		Special topic: Superconductivity		
37	Fri: 04/20/2018		Special topic: Topics in optics		
38	Mon: 04/23/2018				
39	Wed: 04/25/2018				
	Fri: 04/27/2018		Presentations I		
	Mon: 04/30/2018		Presentations II		
	Wed: 05/02/2018		Presentations III		

Mechanism for vortex detection --
Josephson junction -- tunneling current between two superconductors (Ref. Teplitz, ***Electromagnetism*** (1982))



Josephson junction -- continued

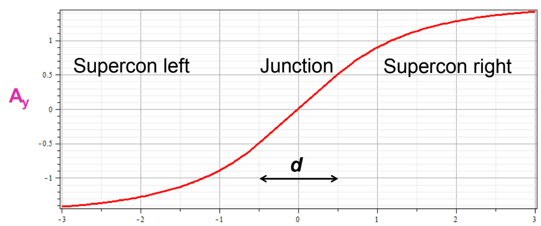


$\lambda_L \equiv$ London penetration length $= \sqrt{\dfrac{mc^2}{4\pi n e^2}}$

$$B_z(x) = \begin{cases} B_0 e^{(x+d/2)/\lambda_L} & x < -d/2 \\ B_0 & -d/2 < x < d/2 \\ B_0 e^{(-x+d/2)/\lambda_L} & x > d/2 \end{cases}$$

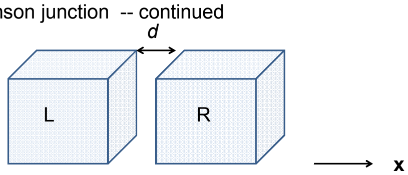
Josephson junction -- continued

$$\mathbf{B} = \nabla \times \mathbf{A}$$



$$A_y(x) = \begin{cases} B_0\left(\lambda_L e^{(x+d/2)/\lambda_L} - (\lambda_L + d/2)\right) & x < -d/2 \\ B_0 x & -d/2 < x < d/2 \\ B_0\left(-\lambda_L e^{(-x+d/2)/\lambda_L} + (\lambda_L + d/2)\right) & x > d/2 \end{cases}$$

Josephson junction -- continued



Quantum mechanical model of tunnelling current

Let $\Psi_L = \Psi_L{}^0 e^{i\phi_L}$ denote a wavefunction for a Cooper pair on left

Let $\Psi_R = \Psi_R{}^0 e^{i\phi_R}$ denote a wavefunction for a Cooper pair on right

$$-i\hbar \frac{\partial \Psi_L}{\partial t} = E_L \Psi_L + \varepsilon \Psi_R$$

$$-i\hbar \frac{\partial \Psi_R}{\partial t} = E_R \Psi_R + \varepsilon \Psi_L$$

Coupling parameter

Josephson junction -- continued

Solving for wavefunctions

$$\frac{1}{2}\frac{\partial|\Psi_L{}^0|^2}{\partial t} + i|\Psi_L{}^0|^2\frac{\partial\phi_L}{\partial t} = -\frac{i}{\hbar}\left(E_L|\Psi_L{}^0|^2 + \varepsilon\Psi_L{}^0\Psi_R{}^0 e^{i(\phi_R-\phi_L)}\right)$$

$$\frac{1}{2}\frac{\partial|\Psi_R{}^0|^2}{\partial t} + i|\Psi_R{}^0|^2\frac{\partial\phi_R}{\partial t} = -\frac{i}{\hbar}\left(E_R|\Psi_R{}^0|^2 + \varepsilon\Psi_L{}^0\Psi_R{}^0 e^{-i(\phi_R-\phi_L)}\right)$$

$$|\Psi_L{}^0|^2 \equiv n_L \qquad |\Psi_R{}^0|^2 \equiv n_R \qquad \phi_{LR} \equiv \phi_L - \phi_R$$

$$\frac{\partial n_L}{\partial t} = -\frac{\partial n_R}{\partial t} = -\frac{2\varepsilon}{\hbar}\sqrt{(n_L n_R)}\sin\phi_{LR}$$

$$\frac{\partial\phi_L}{\partial t} = -\frac{E_L}{\hbar} - \varepsilon\sqrt{\frac{n_R}{n_L}}\cos\phi_{LR}$$

$$\frac{\partial\phi_R}{\partial t} = -\frac{E_R}{\hbar} - \varepsilon\sqrt{\frac{n_L}{n_R}}\cos\phi_{LR}$$

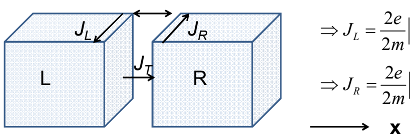
Note that

$$\frac{\partial\phi_{LR}}{\partial t} = -\frac{1}{\hbar}(E_L - E_R)$$

Josephson junction -- continued

Tunneling current: $J_T = 2e\frac{\partial n_L}{\partial t} = -\frac{4e\varepsilon}{\hbar}\sqrt{(n_L n_R)}\sin\phi_{LR}$

If $n_L = n_R$ and in absense of magnetic field, $\phi_{LR}(t) = \phi_{LR}(0) + \frac{E_R - E_L}{\hbar}t$



$$\Rightarrow J_L = \frac{2e}{2m}|\Psi_L{}^0|^2\left(\hbar\nabla\phi_L - \frac{2e}{c}\mathbf{A}\right)$$

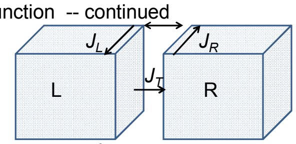
$$\Rightarrow J_R = \frac{2e}{2m}|\Psi_R{}^0|^2\left(\hbar\nabla\phi_R - \frac{2e}{c}\mathbf{A}\right)$$

x

Relationship between superconductor currents J_L and J_R and tunneling current. Within the superconductor, denote the generalized current operator acting on pair wavefunction $\Psi = \Psi^0 e^{i\phi}$

$$J = \frac{2e}{2}\left(\Psi^*(\hat{\mathbf{v}}\Psi) + \Psi(\hat{\mathbf{v}}\Psi)^*\right) \text{ with } \hat{\mathbf{v}} \equiv \frac{1}{2m}\left(-i\hbar\nabla - \frac{2e}{c}\mathbf{A}\right)$$

Josephson junction -- continued



x

Tunneling current: $J_T = 2e\frac{\partial n_L}{\partial t} = -\frac{4e\varepsilon}{\hbar}\sqrt{(n_L n_R)}\sin\phi_{LR}$

If $n_L = n_R = n$ and in absense of magnetic field, $\phi_{LR}(t) = \phi_{LR}(0) + \frac{E_R - E_L}{\hbar}t$

$\Rightarrow$ Constant Josephson tunneling current for $E_R - E_L = 0$

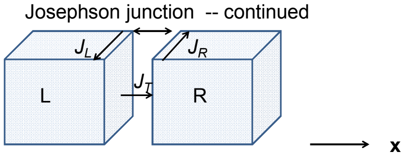
$$J_T = -\frac{4e\varepsilon}{\hbar}n\sin\phi_{LR}(0)$$

$\Rightarrow$ Oscillatory Josephson tunneling current for $E_R - E_L = 2eV$

$$J_T = -\frac{4e\varepsilon}{\hbar}n\sin\left(\phi_{LR}(0) + \frac{2eV}{\hbar}t\right)$$

Method for precise measurement of $e/\hbar$

Josephson junction -- continued



$$\Rightarrow J_L = \frac{2e}{2m}\left|\Psi_L{}^0\right|^2\left(\hbar\nabla\phi_L - \frac{2e}{c}\mathbf{A}\right) \equiv 2en_L\mathbf{v}_L$$

$$\Rightarrow J_R = \frac{2e}{2m}\left|\Psi_R{}^0\right|^2\left(\hbar\nabla\phi_R - \frac{2e}{c}\mathbf{A}\right) \equiv 2en_R\mathbf{v}_R$$

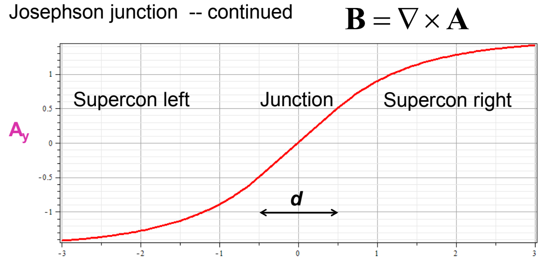
$$\nabla\phi_L = \frac{2m\mathbf{v}_L}{\hbar} + \frac{2e}{\hbar c}\mathbf{A} \qquad \nabla\phi_R = \frac{2m\mathbf{v}_R}{\hbar} + \frac{2e}{\hbar c}\mathbf{A}$$

Tunneling current: $J_T = 2e\frac{\partial n_L}{\partial t} = -\frac{4e\varepsilon}{\hbar}\sqrt{(n_L n_R)}\sin\phi_{LR}$

Need to evaluate ϕ_{LR} in presence of magnetic field

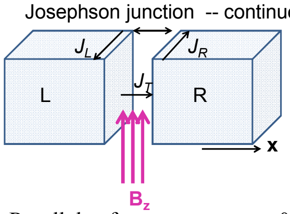
Josephson junction -- continued

$$\mathbf{B} = \nabla\times\mathbf{A}$$



$$A_y(x) = \begin{cases} B_0\left(\lambda_L e^{(x+d/2)/\lambda_L} - (\lambda_L + d/2)\right) & x < -d/2 \\ B_0 x & -d/2 < x < d/2 \\ B_0\left(-\lambda_L e^{(-x+d/2)/\lambda_L} + (\lambda_L + d/2)\right) & x > d/2 \end{cases}$$

Josephson junction -- continued



Tunneling current:

$$J_T = -\frac{4e\varepsilon}{\hbar}\sqrt{(n_L n_R)}\sin\phi_{LR}$$

$$\mathbf{v}_L = \left(\frac{\hbar}{2m}\nabla\phi_L - \frac{2e}{2mc}\mathbf{A}\right)$$

$$\mathbf{v}_R = \left(\frac{\hbar}{2m}\nabla\phi_R - \frac{2e}{2mc}\mathbf{A}\right)$$

Recall that for $x\to-\infty$ $\mathbf{v}_L \to 0$ and $\mathbf{A}\to -(\lambda_L + d/2)B_0\hat{\mathbf{y}}$

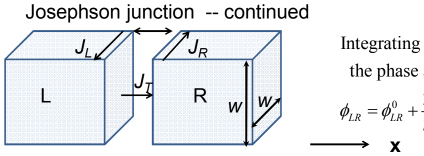
for $x\to\infty$ $\mathbf{v}_R\to 0$ and $\mathbf{A}\to(\lambda_L + d/2)B_0\hat{\mathbf{y}}$

Integrating the difference of the phase angles along y:

$$\phi_{LR} \equiv \phi_L\left(-\tfrac{d}{2},y\right) - \phi_L\left(-\tfrac{d}{2},0\right) - \phi_R\left(\tfrac{d}{2},y\right) + \phi_R\left(\tfrac{d}{2},0\right)$$

$$= \phi_{LR}^0 + \frac{2e}{\hbar c}B_0(2\lambda_L + d)y$$

Josephson junction -- continued



Integrating the difference of the phase angles along y:

$$\phi_{LR} = \phi^0_{LR} + \frac{2e}{\hbar c} B_0 (2\lambda_L + d) y$$

Tunneling current density: $J_T = \frac{4e\varepsilon}{\hbar} n_L \sin\phi_{LR} = J_{T0} \sin\left(\phi^0_{LR} + \frac{2e}{\hbar c} B_0(2\lambda_L + d) y\right)$

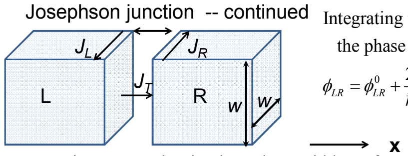
Integrating current density throughout width w of superconductors

$$I_T = w \int_{-w/2}^{w/2} J_T dy$$

$$= \frac{wJ_{T0}}{\frac{2e}{\hbar c} B_0 (2\lambda_L + d)} \left(\cos\left(\phi^0_{LR} - \frac{ew}{\hbar c} B_0 (2\lambda_L + d)\right) - \cos\left(\phi^0_{LR} + \frac{ew}{\hbar c} B_0 (2\lambda_L + d)\right)\right)$$

Define: $\Phi = B_0 w (2\lambda_L + d)$ and $\Phi^0 = \frac{2\pi\hbar c}{2e}$ $\Rightarrow \frac{ew}{\hbar c} B_0 (2\lambda_L + d) = \frac{\pi\Phi}{\Phi^0}$

Josephson junction -- continued



Integrating the difference of the phase angles along y:

$$\phi_{LR} = \phi^0_{LR} + \frac{2e}{\hbar c} B_0 (2\lambda_L + d) y$$

Integrating current density throughout width w of superconductors

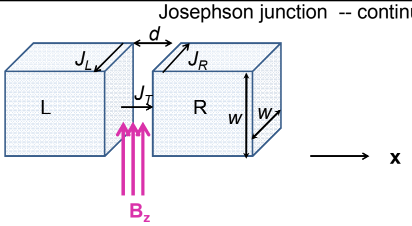
$$I_T = w \int_{-w/2}^{w/2} J_T dy =$$

$$= \frac{wJ_{T0}}{\frac{2e}{\hbar c} B_0 (2\lambda_L + d)} \left(\cos\left(\phi^0_{LR} - \frac{ew}{\hbar c} B_0 (2\lambda_L + d)\right) - \cos\left(\phi^0_{LR} + \frac{ew}{\hbar c} B_0 (2\lambda_L + d)\right)\right)$$

$$= w^2 J_{T0} \sin(\phi^0_{LR}) \frac{\sin(\pi\Phi/\Phi^0)}{\pi\Phi/\Phi^0}$$

where $\Phi = B_0 w (2\lambda_L + d)$ and $\Phi^0 = \frac{2\pi\hbar c}{2e}$

Josephson junction -- continued



Tunneling current density: $J_T = \frac{4e\varepsilon}{\hbar} n_L \sin\phi_{LR}$

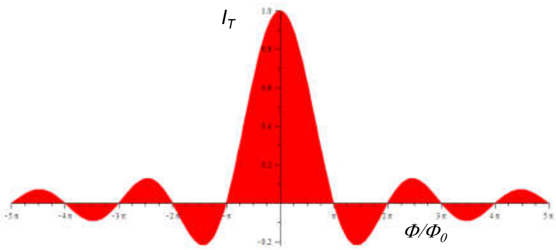
Integrating current density throughout width w of superconductors

$$I_T = w \int_{-w/2}^{w/2} J_T dy = w^2 J_{T0} \sin(\phi^0_{LR}) \frac{\sin(\pi\Phi/\Phi^0)}{\pi\Phi/\Phi^0}$$

where $\Phi = B_0 w (2\lambda_L + d)$ and $\Phi^0 = \frac{2\pi\hbar c}{2e}$

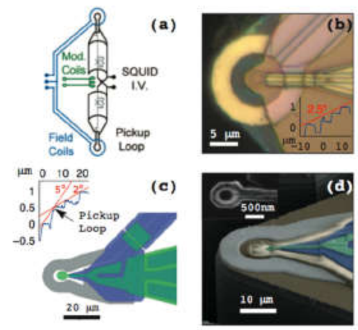
Josephson junction -- continued

SQUID =*superconducting quantum interference device*



Note: This very sensitive "SQUID" technology has been used in scanning probe techniques. See for example, J. R. Kirtley, Rep. Prog. Physics 73, 126501 (2010).

Scanning SQIUD microscopy
Ref. J. R. Kirtley, Rep. Prog. Phys. **73** 126501 (2010)



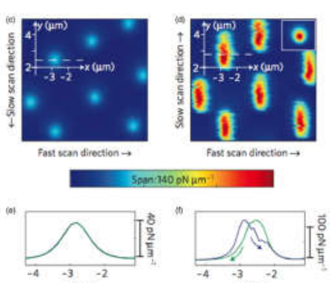


Figure 31. MFM imaging and manipulation of individual vortices in YBCO at $T = 22.3$ K. (*a*), (*b*) Schematic drawings of an MFM tip (triangle) that attracts a vortex (thick lines) in a sample with randomly distributed pinning sites (dots): at large heights (*a*) the force between tip and vortex is too weak to move the vortex; at small heights (*b*) the vortex moves right and then left as the tip rasters over it. (*c*) MFM image taken at $z = 420$ nm (maximum applied lateral force $F^{lat}_{max} = 6$ nN). (*d*) $z = 170$ nm ($F^{lat}_{max} = 12$ nN). Inset: scan taken at a comparable height at $T = 5.2$ K. (*e*) Line cut through the data in (*c*) along the dashed line, showing the signal from a stationary vortex (solid curve with peak to the left, blue online). Overlapping it is a line cut from the reverse scan (solid curve with peak to the right, green online). (*f*) Line cut through the data in (*d*) along the dashed line, showing a typical signal from a dragged vortex. Figure reprinted by permission from [312]. Copyright 2008 Macmillan Publishers Ltd.

NATURE | VOL 387 | 29 MAY 1997 pg. 481

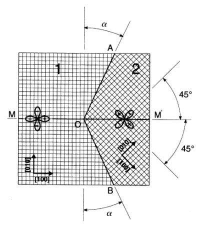


Figure 1 Schematic diagram illustrating the geometrical configuration of a tetracrystal (100) $SrTiO_3$ substrate, which effectively consists of two crystals rotated about the normal to the substrate plane by $\pi/4$ with respect to each other. For a given total misorientation angle $\theta = \theta_1 + \theta_2$, the pair tunnelling current is maximized[9] for a symmetric grain boundary $\theta_1 = \theta_2 = \pi/8$. We chose the angle $\alpha = \theta_1$ between the vertical ([010] in grain 1) and the grain boundaries (OA and OB) as 25°. Also shown are the polar plots of the assumed $d_{x^2-y^2}$ gap functions aligned with the crystallographic axes in the substrate.

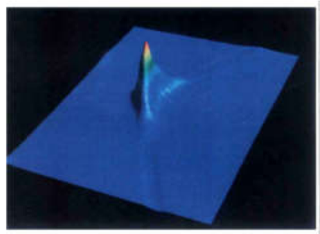


Figure 3 A three-dimensional presentation of the SSM image of the Josephson vortex trapped at the wedge tip in a Tl2201 blanket film deposited on a tetracrystal (100) $SrTiO_3$ substrate of the geometry shown in Fig. 1. The sample was cooled in a field of <1 mG and imaged at 4.2 K.

Change of topics – review of reflection/refraction of electromagnetic plane waves.

Maxwell's equations

For linear isotropic media and no sources: $\mathbf{D} = \varepsilon \mathbf{E}; \quad \mathbf{B} = \mu \mathbf{H}$

Coulomb's law: $\nabla \cdot \mathbf{E} = 0$

Ampere-Maxwell's law: $\nabla \times \mathbf{B} - \mu\varepsilon \frac{\partial \mathbf{E}}{\partial t} = 0$

Faraday's law: $\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$

No magnetic monopoles: $\nabla \cdot \mathbf{B} = 0$

Analysis of Maxwell's equations without sources -- continued:

Summary of plane electromagnetic waves:

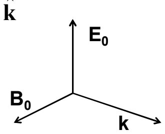
$$\mathbf{B}(\mathbf{r},t) = \Re\left(\frac{n\hat{\mathbf{k}} \times \mathbf{E}_0}{c} e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t} \right) \qquad \mathbf{E}(\mathbf{r},t) = \Re\left(\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t} \right)$$

$$|\mathbf{k}|^2 = \left(\frac{\omega}{v}\right)^2 = \left(\frac{n\omega}{c}\right)^2 \quad \text{where } n \equiv \sqrt{\frac{\mu\varepsilon}{\mu_0\varepsilon_0}} \quad \text{and } \hat{\mathbf{k}} \cdot \mathbf{E}_0 = 0$$

Poynting vector and energy density:

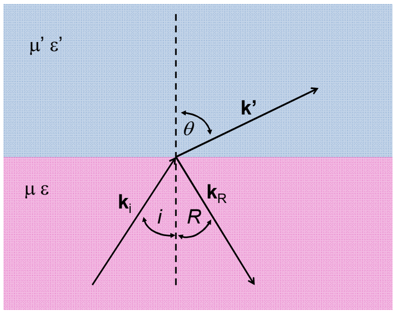
$$\langle \mathbf{S} \rangle_{avg} = \frac{n|\mathbf{E}_0|^2}{2\mu c} \hat{\mathbf{k}} = \frac{1}{2}\sqrt{\frac{\varepsilon}{\mu}} |\mathbf{E}_0|^2 \hat{\mathbf{k}}$$

$$\langle u \rangle_{avg} = \frac{1}{2} \varepsilon |\mathbf{E}_0|^2$$

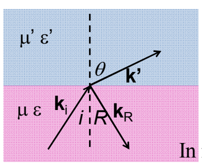


Reflection and refraction of plane electromagnetic waves at a plane interface between dielectrics (assumed to be lossless)

isotropic materials



Reflection and refraction -- continued



In medium $\mu'\varepsilon'$:

$$\mathbf{E}'(\mathbf{r},t) = \Re\left(\mathbf{E}'_0\, e^{i\frac{\omega}{c}\left(n'\hat{\mathbf{k}}'\cdot\mathbf{r}-ct\right)}\right)$$

$$\mathbf{B}'(\mathbf{r},t) = \frac{n'}{c}\hat{\mathbf{k}}'\times\mathbf{E}'(\mathbf{r},t) = \sqrt{\mu'\varepsilon'}\hat{\mathbf{k}}'\times\mathbf{E}(\mathbf{r},t)$$

In medium $\mu\varepsilon$:

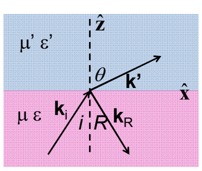
$$\mathbf{E}_i(\mathbf{r},t) = \Re\left(\mathbf{E}_{0i}\, e^{i\frac{\omega}{c}\left(n\hat{\mathbf{k}}_i\cdot\mathbf{r}-ct\right)}\right)$$

$$\mathbf{B}_i(\mathbf{r},t) = \frac{n}{c}\hat{\mathbf{k}}_i\times\mathbf{E}_i(\mathbf{r},t) = \sqrt{\mu\varepsilon}\hat{\mathbf{k}}_i\times\mathbf{E}_i(\mathbf{r},t)$$

$$\mathbf{E}_R(\mathbf{r},t) = \Re\left(\mathbf{E}_{0R}\, e^{i\frac{\omega}{c}\left(n\hat{\mathbf{k}}_R\cdot\mathbf{r}-ct\right)}\right)$$

$$\mathbf{B}_R(\mathbf{r},t) = \frac{n}{c}\hat{\mathbf{k}}_R\times\mathbf{E}_R(\mathbf{r},t) = \sqrt{\mu\varepsilon}\hat{\mathbf{k}}_R\times\mathbf{E}_R(\mathbf{r},t)$$

Reflection and refraction -- continued



Snell's law -- matching phase factors at boundary plane:

$$\mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + 0\hat{\mathbf{z}}$$

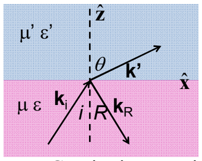
$$\hat{\mathbf{k}}'\cdot\mathbf{r} = x\sin\theta$$

$$\hat{\mathbf{k}}_i\cdot\mathbf{r} = x\sin i = \hat{\mathbf{k}}_R\cdot\mathbf{r} \quad \Rightarrow i = R$$

$$n'\hat{\mathbf{k}}'\cdot\mathbf{r} = n\hat{\mathbf{k}}_i\cdot\mathbf{r} \quad \Rightarrow n'x\sin\theta = nx\sin i$$

Snell's law: $n'\sin\theta = n\sin i$

Reflection and refraction -- continued



Continuity equations at boundary with no sources:

$$\nabla \cdot \mathbf{D} = 0 \qquad \nabla \cdot \mathbf{B} = 0$$

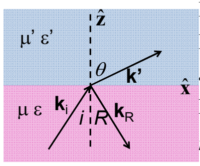
$$\nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = 0 \qquad \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$$

Matching field amplitudes at boundary plane:

$\mathbf{D}\cdot\hat{\mathbf{z}},\ \mathbf{B}\cdot\hat{\mathbf{z}}$ continuous

$\mathbf{H}\times\hat{\mathbf{z}}, \mathbf{E}\times\hat{\mathbf{z}}$ continuous

Reflection and refraction -- continued



Matching field amplitudes at boundary plane:

$\mathbf{D}\cdot\hat{\mathbf{z}}$ continuous:

$$\varepsilon\left(\mathbf{E}_{0i} + \mathbf{E}_{0R}\right)\cdot\hat{\mathbf{z}} = \varepsilon' \mathbf{E}'_0 \cdot \hat{\mathbf{z}}$$

$\mathbf{B}\cdot\hat{\mathbf{z}}$ continuous:

$$n\left(\hat{\mathbf{k}}_i \times \mathbf{E}_{0i} + \hat{\mathbf{k}}_R \times \mathbf{E}_{0R}\right)\cdot\hat{\mathbf{z}} = n'\hat{\mathbf{k}}' \times \mathbf{E}'_{0i} \cdot \hat{\mathbf{z}}$$

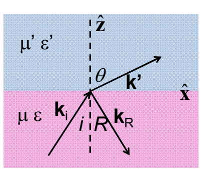
$\mathbf{E}\times\hat{\mathbf{z}}$ continuous:

$$\left(\mathbf{E}_{0i} + \mathbf{E}_{0R}\right)\times\hat{\mathbf{z}} = \mathbf{E}'_0 \times \hat{\mathbf{z}}$$

$\mathbf{H}\times\hat{\mathbf{z}}$ continuous:

$$\frac{n}{\mu}\left(\hat{\mathbf{k}}_i \times \mathbf{E}_{0i} + \hat{\mathbf{k}}_R \times \mathbf{E}_{0R}\right)\times\hat{\mathbf{z}} = \frac{n'}{\mu'}\hat{\mathbf{k}}' \times \mathbf{E}'_{0i} \times \hat{\mathbf{z}}$$

Reflection and refraction -- continued



s-polarization – **E** field "polarized" perpendicular to plane of incidence

$\mathbf{E}\times\hat{\mathbf{z}}$ continuous:

$$\left(\mathbf{E}_{0i} + \mathbf{E}_{0R}\right)\times\hat{\mathbf{z}} = \mathbf{E}'_0 \times \hat{\mathbf{z}}$$

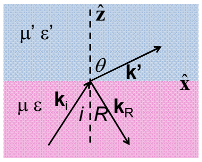
$\mathbf{H}\times\hat{\mathbf{z}}$ continuous:

$$\frac{n}{\mu}\left(\hat{\mathbf{k}}_i \times \mathbf{E}_{0i} + \hat{\mathbf{k}}_R \times \mathbf{E}_{0R}\right)\times\hat{\mathbf{z}} = \frac{n'}{\mu'}\hat{\mathbf{k}}' \times \mathbf{E}'_{0i} \times \hat{\mathbf{z}}$$

$$\frac{E_{0R}}{E_{0i}} = \frac{n\cos i - \frac{\mu}{\mu'} n'\cos\theta}{n\cos i + \frac{\mu}{\mu'} n'\cos\theta} \qquad \frac{E'_0}{E_{0i}} = \frac{2n\cos i}{n\cos i + \frac{\mu}{\mu'} n'\cos\theta}$$

Note that: $n'\cos\theta = \sqrt{n'^2 - n^2\sin^2 i}$

Reflection and refraction -- continued



p-polarization – **E** field "polarized" parallel to plane of incidence

$\mathbf{D}\cdot\hat{\mathbf{z}}$ continuous:

$$\varepsilon\left(\mathbf{E}_{0i}+\mathbf{E}_{0R}\right)\cdot\hat{\mathbf{z}}=\varepsilon'\mathbf{E}'_0\cdot\hat{\mathbf{z}}$$

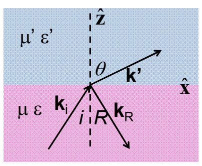
$\mathbf{H}\times\hat{\mathbf{z}}$ continuous:

$$\frac{n}{\mu}\left(\hat{\mathbf{k}}_i\times\mathbf{E}_{0i}+\hat{\mathbf{k}}_R\times\mathbf{E}_{0R}\right)\times\hat{\mathbf{z}}=\frac{n'}{\mu'}\hat{\mathbf{k}}'\times\mathbf{E}'_{0i}\times\hat{\mathbf{z}}$$

$$\frac{E_{0R}}{E_{0i}}=\frac{\frac{\mu}{\mu'}n'\cos i-n\cos\theta}{\frac{\mu}{\mu'}n'\cos i+n\cos\theta}\qquad\frac{E'_0}{E_{0i}}=\frac{2n\cos i}{\frac{\mu}{\mu'}n'\cos i+n\cos\theta}$$

Note that: $n'\cos\theta=\sqrt{n'^2-n^2\sin^2 i}$

Reflection and refraction -- continued



Reflectance, transmittance:

$$\mathrm{R}=\frac{\mathbf{S}_R\cdot\hat{\mathbf{z}}}{\mathbf{S}_i\cdot\hat{\mathbf{z}}}=\left|\frac{E_{0R}}{E_{0i}}\right|^2\qquad\mathrm{T}=\frac{\mathbf{S}'\cdot\hat{\mathbf{z}}}{\mathbf{S}_i\cdot\hat{\mathbf{z}}}=\left|\frac{E'_0}{E_{0i}}\right|^2\frac{n'}{n}\frac{\mu}{\mu'}\frac{\cos\theta}{\cos i}$$

Note that $\mathrm{R}+\mathrm{T}=1$

For s-polarization

$$\frac{E_{0R}}{E_{0i}}=\frac{n\cos i-\frac{\mu}{\mu'}n'\cos\theta}{n\cos i+\frac{\mu}{\mu'}n'\cos\theta}\qquad\frac{E'_0}{E_{0i}}=\frac{2n\cos i}{n\cos i+\frac{\mu}{\mu'}n'\cos\theta}$$

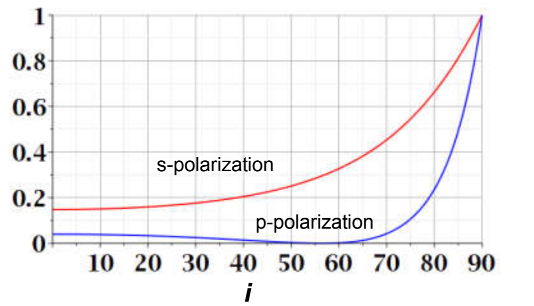
Note that: $n'\cos\theta=\sqrt{n'^2-n^2\sin^2 i}$

For p-polarization

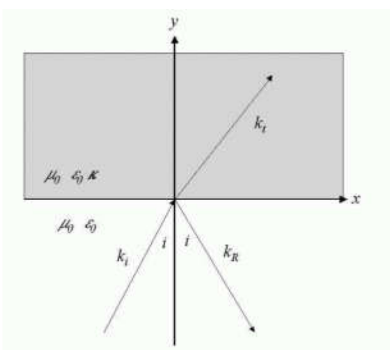
$$\frac{E_{0R}}{E_{0i}}=\frac{\frac{\mu}{\mu'}n'\cos i-n\cos\theta}{\frac{\mu}{\mu'}n'\cos i+n\cos\theta}\qquad\frac{E'_0}{E_{0i}}=\frac{2n\cos i}{\frac{\mu}{\mu'}n'\cos i+n\cos\theta}$$

Note that: $n'\cos\theta=\sqrt{n'^2-n^2\sin^2 i}$

Example for $n = 1$ and $n' = 1.5$



Extension of analysis to anisotropic media --



Consider the problem of determining the reflectance from an anisotropic medium with isotropic permeability μ_0 and anisotropic permittivity $\varepsilon_0 \boldsymbol{\kappa}$ where:

$$\boldsymbol{\kappa} \equiv \begin{pmatrix} \kappa_{xx} & 0 & 0 \\ 0 & \kappa_{yy} & 0 \\ 0 & 0 & \kappa_{zz} \end{pmatrix}$$

By assumption, the wave vector in the medium is confined to the *x-y* plane and will be denoted by

$$\mathbf{k}_t \equiv \frac{\omega}{c}(n_x \hat{\mathbf{x}} + n_y \hat{\mathbf{y}}), \text{where } n_x \text{ and } n_y \text{ are to be determined.}$$

The electric field inside the medium is given by:

$$\mathbf{E} = (E_x \hat{\mathbf{x}} + E_y \hat{\mathbf{y}} + E_z \hat{\mathbf{z}}) e^{i\frac{\omega}{c}(n_x x + n_y y) - i\omega t}.$$

Digression --

Note that the most general form of the dielectric matrix is:

$$\overline{\boldsymbol{\kappa}} \equiv \begin{pmatrix} \overline{\kappa}_{xx} & \overline{\kappa}_{xy} & \overline{\kappa}_{xz} \\ \overline{\kappa}_{yx} & \overline{\kappa}_{yy} & \overline{\kappa}_{yz} \\ \overline{\kappa}_{zx} & \overline{\kappa}_{zy} & \overline{\kappa}_{zz} \end{pmatrix}$$

Expecting this matrix to be symmetric/Hermitian, it can be diagonalized by a similarity transformation

$$\mathbf{S}\overline{\boldsymbol{\kappa}}\mathbf{S}^{-1} = \boldsymbol{\kappa} = \begin{pmatrix} \kappa_{xx} & 0 & 0 \\ 0 & \kappa_{yy} & 0 \\ 0 & 0 & \kappa_{zz} \end{pmatrix}$$

Inside the anisotropic medium, Maxwell's equations for time harmonic fields are:

$$\nabla \cdot \mathbf{H} = 0 \qquad \nabla \cdot \boldsymbol{\kappa} \cdot \mathbf{E} = 0$$

$$\nabla \times \mathbf{E} - i\omega\mu_0 \mathbf{H} = 0 \qquad \nabla \times \mathbf{H} + i\omega\epsilon_0 \boldsymbol{\kappa} \cdot \mathbf{E} = 0$$

$$\Rightarrow \nabla \times (\nabla \times \mathbf{E}) - i\omega\mu_0 \nabla \times \mathbf{H} = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} - \frac{\omega^2}{c^2} \boldsymbol{\kappa} \cdot \mathbf{E} = 0$$

Suppose the electric field takes the general form:

$$\mathbf{E} = \left(E_x \hat{\mathbf{x}} + E_y \hat{\mathbf{y}} + E_z \hat{\mathbf{z}}\right) e^{i\left(\frac{\omega}{c}\left(n_x x + n_y y - ct\right)\right)}$$

$$\begin{pmatrix} \kappa_{xx} - n_y^2 & n_x n_y & 0 \\ n_x n_y & \kappa_{yy} - n_x^2 & 0 \\ 0 & 0 & \kappa_{zz} - (n_x^2 + n_y^2) \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = 0.$$

Inside the anisotropic medium, Maxwell's equations for time harmonic fields – continued --

$$\mathbf{E} = \left(E_x \hat{\mathbf{x}} + E_y \hat{\mathbf{y}} + E_z \hat{\mathbf{z}}\right) \mathrm{e}^{i\frac{\omega}{c}\left(n_x x + n_y y\right) - i\omega t}$$

$$\begin{pmatrix} \kappa_{xx} - n_y^2 & n_x n_y & 0 \\ n_x n_y & \kappa_{yy} - n_x^2 & 0 \\ 0 & 0 & \kappa_{zz} - (n_x^2 + n_y^2) \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = 0.$$

$$\mathbf{H} = \frac{1}{i\mu_0\omega} \nabla \times \mathbf{E}$$

$$\mathbf{H} = \frac{1}{\mu_0 c} \left\{ E_z (n_y \hat{\mathbf{x}} - n_x \hat{\mathbf{y}}) + (E_y n_x - E_x n_y) \hat{\mathbf{z}} \right\} \mathrm{e}^{i\frac{\omega}{c}(n_x x + n_y y) - i\omega t}.$$

The fields for the incident and reflected waves are the same as for the isotropic case.

$$\mathbf{k}_i = \frac{\omega}{c}(\sin i\hat{\mathbf{x}} + \cos i\hat{\mathbf{y}}),$$

$$\mathbf{k}_R = \frac{\omega}{c}(\sin i\hat{\mathbf{x}} - \cos i\hat{\mathbf{y}}).$$

Note that, consistent with Snell's law: $n_x = \sin i$
Continuity conditions at the *y=0* plane must be applied for the following fields:

$$\mathbf{H}(x,0,z,t),\ E_x(x,0,z,t),\ E_z(x,0,z,t),\ \text{and}\ D_y(x,0,z,t).$$

There will be two different solutions, depending of the polarization of the incident field.

Solution for s-polarization

$$E_x = E_y = 0 \qquad \Rightarrow n_y^2 = \kappa_{zz} - n_x^2 = \kappa_{zz} - \sin^2 i$$

$$\mathbf{E} = E_z\hat{\mathbf{z}}e^{i\frac{\omega}{c}(n_x x + n_y y) - i\omega t} \qquad \mathbf{H} = \frac{1}{\mu_0 c}\left\{E_z(n_y\hat{\mathbf{x}} - n_x\hat{\mathbf{y}})\right\}e^{i\frac{\omega}{c}(n_x x + n_y y) - i\omega t}$$

E_z must be determined from the continuity conditions:

$$E_0 + E_0'' = E_z \qquad (E_0 - E_0'')\cos i = E_z n_y \qquad (E_0 + E_0'')\sin i = E_z n_x$$

$$\frac{E_0''}{E_0} = \frac{\cos i - n_y}{\cos i + n_y}.$$

Solution for p-polarization

$$E_z = 0 \qquad \Rightarrow n_y^2 = \frac{\kappa_{xx}}{\kappa_{yy}}(\kappa_{yy} - n_x^2) = \frac{\kappa_{xx}}{\kappa_{yy}}(\kappa_{yy} - \sin^2 i)$$

$$\mathbf{E} = E_x\left(\hat{\mathbf{x}} - \frac{\kappa_{xx}n_x}{\kappa_{yy}n_y}\hat{\mathbf{y}}\right)e^{i\frac{\omega}{c}(n_x x + n_y y) - i\omega t}.$$

$$\mathbf{H} = -\frac{E_x}{\mu_0 c}\frac{\kappa_{xx}}{n_y}\hat{\mathbf{z}}e^{i\frac{\omega}{c}(n_x x + n_y y) - i\omega t}.$$

E_x must be determined from the continuity conditions:

$$(E_0 - E_0'')\cos i = E_x \qquad (E_0 + E_0'') = \frac{\kappa_{xx}}{n_y}E_x \qquad (E_0 + E_0'')\sin i = \frac{\kappa_{xx}n_x}{n_y}E_x.$$

$$\frac{E_0''}{E_0} = \frac{\kappa_{xx}\cos i - n_y}{\kappa_{xx}\cos i + n_y}.$$