

**PHY 712 Electrodynamics**  
**9-9:50 AM MWF Olin 105**

**Plan for Lecture 6:**

**Continue reading Chapter 2**

- 1. Methods of images -- planes, spheres**
- 2. Solution of Poisson equation in for other geometries -- cylindrical**

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**PHY 712 Electrodynamics**

MWF 9-9:50 AM | OPL 105 | <http://www.wfu.edu/~natalie/s18phy712/>

Instructor: [Natalie Holzwarth](mailto:natalie@wfu.edu) Phone: 758-5510 Office: 300 OPL e-mail: [natalie@wfu.edu](mailto:natalie@wfu.edu)

**Course schedule for Spring 2018**

(Preliminary schedule -- subject to frequent adjustment.)

Lecture date	JDJ Reading	Topic	HW	Due date
Wed: 01/17/2018	No class	Snow		
1 Fri: 01/19/2018	Chap. 1 & Appen.	Introduction, units and Poisson equation	#1	01/26/2018
2 Mon: 01/22/2018	Chap. 1	Electrostatic energy calculations	#2	01/26/2018
3 Wed: 01/24/2018	Chap. 1	Poisson's equation and Green's theorem	#3	01/26/2018
4 Thu: 01/25/2018	Chap. 1 & 2	Poisson's equation in 2 and 3 dimensions		
5 Fri: 01/26/2018	Chap. 1 & 2	Brief introduction to numerical methods	#4	01/29/2018
6 Mon: 01/29/2018	Chap. 2	Method of image charges	#5	01/31/2018
7 Wed: 01/31/2018				
8 Fri: 02/02/2018				

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Survey of mathematical techniques for analyzing electrostatics – the Poisson equation

$$\nabla^2 \Phi(\mathbf{r}) = -\frac{\rho(\mathbf{r})}{\epsilon_0}$$

1. Direct solution of differential equation
2. Solution by means of an integral equation; Green's function techniques
3. Orthogonal function expansions
4. Numerical methods (finite differences and finite element methods)
5. Method of images

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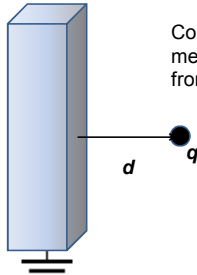
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Method of images  
 Clever trick for specialized geometries:  
 > Flat plane (surface)  
 > Sphere

Planar case:



Consider a grounded metal sheet, a distance  $d$  from a point charge  $q$ .

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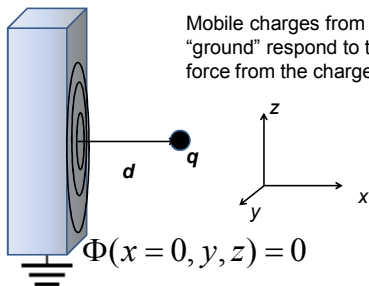
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A grounded metal sheet, a distance  $d$  from a point charge  $q$ .



Mobile charges from the "ground" respond to the force from the charge  $q$ .

$\Phi(x=0, y, z) = 0$

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A grounded metal sheet, a distance  $d$  from a point charge  $q$ .

$$\nabla^2 \Phi = -\frac{q}{\epsilon_0} \delta^3(\mathbf{r} - d\hat{\mathbf{x}})$$

$$\Phi(x=0, y, z) = 0$$

Trick for  $x \geq 0$ :

$$\Phi(x \geq 0, y, z) = \frac{1}{4\pi\epsilon_0} \left( \frac{q}{|\mathbf{r} - d\hat{\mathbf{x}}|} - \frac{q}{|\mathbf{r} + d\hat{\mathbf{x}}|} \right)$$

Surface charge density:

$$\sigma(y, z) = \epsilon_0 E(0, y, z) = -\epsilon_0 \frac{d\Phi(0, y, z)}{dx} = -\frac{q}{4\pi} \left( \frac{2d}{(d^2 + y^2 + z^2)^{3/2}} \right)$$

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A grounded metal sheet, a distance  $d$  from a point charge  $q$ .

Surface charge density :  $\sigma(y,z) = -\frac{q}{4\pi} \left( \frac{2d}{(d^2 + y^2 + z^2)^{3/2}} \right)$

Note :  $\iint dydz \sigma(y,z) = -\frac{q2d}{4\pi} 2\pi \int_0^\infty \frac{udu}{(d^2 + u^2)^{3/2}} = -q$

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A grounded metal sheet, a distance  $d$  from a point charge  $q$ .

Surface charge density :

$$\sigma(y,z) = -\frac{q}{4\pi} \left( \frac{2d}{(d^2 + y^2 + z^2)^{3/2}} \right)$$

Force between charge and sheet :

$$\mathbf{F} = \frac{-q^2 \hat{\mathbf{x}}}{4\pi\epsilon_0 (2d)^2}$$

Image potential between charge and sheet at distance  $x$  :

$$V(x) = \frac{-q^2}{4\pi\epsilon_0 (4x)}$$

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A grounded metal sphere of radius  $a$ , in the presence of a point charge  $q$  at a distance  $d$  from its center.

Trick for  $r \geq a$  :

$$\Phi(r \geq a) = \frac{1}{4\pi\epsilon_0} \left( \frac{q}{|\mathbf{r} - \mathbf{d}|} - \frac{q}{|\mathbf{r} - \frac{a^2}{d}\hat{\mathbf{d}}|} \right)$$

Interpreted as  
Image charge of  $q' = -q \frac{a}{d}$   
Located along  $\hat{\mathbf{d}}$  at  $\hat{\mathbf{d}} \frac{a}{d}$

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A grounded metal sphere of radius  $a$ , in the presence of a point charge  $q$  at a distance  $d$  from its center.

Surface charge density:

$$\sigma(\hat{\mathbf{r}}) = -\epsilon_0 \left. \frac{\partial \Phi}{\partial r} \right|_{r=a} = -\frac{q}{4\pi a^2} \frac{a}{d} \frac{\left(1 - \frac{a^2}{d^2}\right)}{\left(1 + \frac{a^2}{d^2} - 2\frac{a}{d} \hat{\mathbf{r}} \cdot \hat{\mathbf{d}}\right)^{3/2}}$$

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A grounded metal sphere of radius  $a$ , in the presence of a point charge  $q$  at a distance  $d$  from its center.

Surface charge density:

$$\sigma(\hat{\mathbf{r}}) = -\epsilon_0 \left. \frac{\partial \Phi}{\partial r} \right|_{r=a} = -\frac{q}{4\pi a^2} \frac{a}{d} \frac{\left(1 - \frac{a^2}{d^2}\right)}{\left(1 + \frac{a^2}{d^2} - 2\frac{a}{d} \hat{\mathbf{r}} \cdot \hat{\mathbf{d}}\right)^{3/2}}$$

Force between  $q$  and sphere

$$|\mathbf{F}| = \frac{1}{4\pi\epsilon_0} \frac{q^2 (a/d)}{(d - a^2/d)^2} = \frac{q^2}{4\pi\epsilon_0} \frac{ad}{(d^2 - a^2)^2}$$

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Use of image charge formalism to construct Green's function

Example :

Suppose we have a Dirichlet boundary value problem on a sphere of radius  $a$  :

$$\nabla^2 \Phi = -\frac{\rho(\mathbf{r})}{\epsilon_0} \quad \Phi(r=a) = 0$$

$$\nabla^2 G(\mathbf{r}, \mathbf{r}') = -4\pi \delta^3(\mathbf{r} - \mathbf{r}')$$

$$\Rightarrow \text{For } r, r' > a : \quad G(\mathbf{r}, \mathbf{r}') = \frac{1}{|\mathbf{r} - \mathbf{r}'|} - \frac{1}{\frac{r'}{a} \left| \mathbf{r} - \frac{a^2}{r'^2} \mathbf{r}' \right|}$$

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Analysis of Poisson/Laplace equation in various regular geometries

1. Rectangular geometries
2. Cylindrical geometries
3. Spherical geometries

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Solution of the Poisson/Laplace equation in various geometries -- cylindrical geometry with no z-dependence (infinitely long wire, for example):

Corresponding orthogonal functions from solution of Laplace equation :  $\nabla^2\Phi = 0$



$$\frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial \Phi}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 \Phi}{\partial \phi^2} = 0$$

$$\Phi(\rho, \phi) = \Phi(\rho, \phi + m2\pi)$$

⇒ General solution of the Laplace equation in these coordinates :

$$\Phi(\rho, \phi) = A_0 + B_0 \ln(\rho) + \sum_{m=1}^{\infty} (A_m \rho^m + B_m \rho^{-m}) \sin(m\phi + \alpha_m)$$

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Solution of the Poisson/Laplace equation in various geometries -- cylindrical geometry with no z-dependence (infinitely long wire, for example):



Green's function appropriate for this geometry with boundary conditions at  $\rho = 0$  and  $\rho = \infty$  :

$$\left( \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2}{\partial \phi^2} \right) G(\rho, \rho', \phi, \phi') = -4\pi \frac{\delta(\rho - \rho')}{\rho} \delta(\phi - \phi')$$

$$G(\rho, \rho', \phi, \phi') = -\ln(\rho_>^2) + 2 \sum_{m=1}^{\infty} \frac{1}{m} \left( \frac{\rho_<}{\rho_>} \right)^m \cos(m(\phi - \phi'))$$

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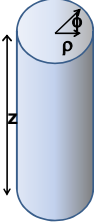
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Solution of the Poisson/Laplace equation in various geometries -- cylindrical geometry with z-dependence



Corresponding orthogonal functions from solution of Laplace equation :  $\nabla^2 \Phi = 0$

$$\frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial \Phi}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 \Phi}{\partial \phi^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0$$

$$\Phi(\rho, \phi, z) = \Phi(\rho, \phi + m2\pi, z)$$

$$\Phi(\rho, \phi, z) = R(\rho)Q(\phi)Z(z)$$

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Cylindrical geometry continued:

$$\frac{d^2 Z}{dz^2} - k^2 Z = 0 \quad \Rightarrow Z(z) = \sinh(kz), \cosh(kz), e^{\pm kz}$$

$$\frac{d^2 Q}{d\phi^2} + m^2 Q = 0 \quad \Rightarrow Q(\phi) = e^{\pm im\phi}$$

$$\frac{d^2 R}{d\rho^2} + \frac{1}{\rho} \frac{dR}{d\rho} + \left( k^2 - \frac{m^2}{\rho^2} \right) R = 0 \quad \Rightarrow J_m(k\rho), N_m(k\rho)$$

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Cylindrical geometry example:



$$\Phi(\rho, \phi, z = L) = V(\rho, \phi)$$

$$\Phi(\rho, \phi, z) = 0 \quad \text{on all other boundaries}$$

$$\Phi(\rho, \phi, z) = \sum_{n,m} A_{nm} J_m(k_{nm}\rho) \sinh(k_{nm}z) \sin(m\phi + \alpha_{nm})$$

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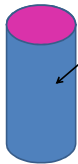
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Cylindrical geometry example:



$\Phi(\rho = a, \phi, z) = V(\phi, z)$   
 $\Phi(\rho, \phi, z) = 0$  on all other boundaries

$$\Phi(\rho, \phi, z) = \sum_{n,m} A_{nm} I_m \left( \frac{n\pi\rho}{L} \right) \sin \left( \frac{n\pi z}{L} \right) \sin(m\phi + \alpha_{nm})$$

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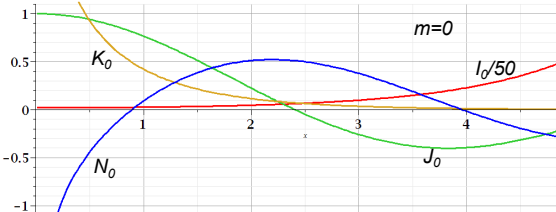
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Comments on cylindrical Bessel functions

$$\left( \frac{d^2}{du^2} + \frac{1}{u} \frac{d}{du} + \left( \pm 1 - \frac{m^2}{u^2} \right) \right) F_m^\pm(u) = 0$$

$F_m^+(u) = J_m(u), N_m(u), H_m(u) \equiv J_m(u) \pm iN_m(u)$   
 $F_m^-(u) = I_m(u), K_m(u)$



$m=0$

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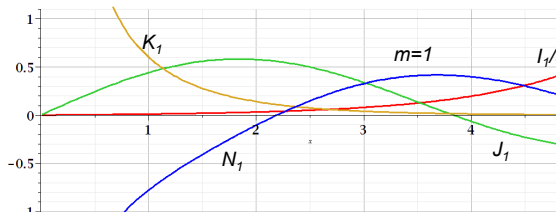
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Comments on cylindrical Bessel functions

$$\left( \frac{d^2}{du^2} + \frac{1}{u} \frac{d}{du} + \left( \pm 1 - \frac{m^2}{u^2} \right) \right) F_m^\pm(u) = 0$$

$F_m^+(u) = J_m(u), N_m(u), H_m(u) \equiv J_m(u) \pm iN_m(u)$   
 $F_m^-(u) = I_m(u), K_m(u)$



$m=1$

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