

PHY 712 Electrodynamics
9-9:50 AM MWF Olin 105

Plan for Lecture 7:

Start reading Chapter 3

**Solution of Poisson/Laplace equation
in for special geometries –**


A. Cylindrical

B. Spherical

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Events

Colloquium: Jan. 31, 2018 at 4 PM
WFU Physics Colloquium TITLE: "Energy Policy and the EPA" SPEAKER: Professor Stan Melburg Director, Graduate Studies in Sustainability Wake Forest University TAG: Wed. Jan. 31, 2018, at 4:00 PM PLACE: ...

Colloquium: Feb. 7 at 4 PM
WFU Physics Colloquium TITLE: "Computational Engineering of Thermoelectric Materials" SPEAKER: Professor Kristian Berland Centre for Materials Science and Nanotechnology, University of Oslo TME: Wed. Feb. 7, 2018, at 4:00 PM ...


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Course schedule for Spring 2018
(Preliminary schedule -- subject to frequent adjustment.)

Lecture date	JDJ Reading	Topic	HW	Due date
Wed: 01/17/2018	No class	Snow		
1 Fri: 01/19/2018	Chap. 1 & Appen.	Introduction, units and Poisson equation	#1	01/26/2018
2 Mon: 01/22/2018	Chap. 1	Electrostatic energy calculations	#2	01/26/2018
3 Wed: 01/24/2018	Chap. 1	Poisson's equation and Green's theorem	#3	01/26/2018
4 Thu: 01/25/2018	Chap. 1 & 2	Poisson's equation in 2 and 3 dimensions		
5 Fri: 01/26/2018	Chap. 1 & 2	Brief introduction to numerical methods	#4	01/29/2018
6 Mon: 01/29/2018	Chap. 2	Method of image charges	#5	01/31/2018
7 Wed: 01/31/2018	Chap. 2 & 3	Cylindrical and spherical geometries	#6	02/02/2018
8 Fri: 02/02/2018				
9 Mon: 02/05/2018				
10 Wed: 02/07/2018				
11 Fri: 02/09/2018				

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Solution of the Poisson/Laplace equation in various geometries -- cylindrical geometry with no z-dependence (infinitely long wire, for example):



Corresponding orthogonal functions from solution of Laplace equation : $\nabla^2\Phi = 0$

$$\frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial \Phi}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 \Phi}{\partial \phi^2} = 0$$


$\Phi(\rho, \phi) = \Phi(\rho, \phi + m2\pi)$

⇒ General solution of the Laplace equation in these coordinates :

$$\Phi(\rho, \phi) = A_0 + B_0 \ln(\rho) + \sum_{m=1}^{\infty} (A_m \rho^m + B_m \rho^{-m}) \sin(m\phi + \alpha_m)$$

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Cylindrical coordinates with trivial z-dependence – some details: $\Phi(\rho, \phi) = f(\rho)g(\phi)$



Suppose $\frac{d^2 g(\phi)}{d\phi^2} = -m^2 g(\phi)$

$g(\phi) = \sin(m\phi + \alpha_m)$

$$\frac{1}{\rho} \frac{d}{d\rho} \left(\rho \frac{df_m(\rho)}{d\rho} \right) - \frac{m^2}{\rho^2} f_m(\rho) = 0$$


$f_0(\rho) = \begin{cases} 1 \\ \ln \rho \end{cases} \quad f_{m>0} = \rho^{\pm m}$

⇒ General solution of the Laplace equation in these coordinates:

$$\Phi(\rho, \phi) = A_0 + B_0 \ln(\rho) + \sum_{m=1}^{\infty} (A_m \rho^m + B_m \rho^{-m}) \sin(m\phi + \alpha_m)$$

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Solution of the Poisson/Laplace equation in various geometries -- cylindrical geometry with no z-dependence (infinitely long wire, for example):



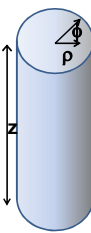
Green's function appropriate for this geometry with boundary conditions at $\rho = 0$ and $\rho = \infty$:

$$\left(\frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2}{\partial \phi^2} \right) G(\rho, \rho', \phi, \phi') = -4\pi \frac{\delta(\rho - \rho')}{\rho} \delta(\phi - \phi')$$

$$G(\rho, \rho', \phi, \phi') = -\ln(\rho_>^2) + 2 \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{\rho_<}{\rho_>} \right)^m \cos(m(\phi - \phi'))$$

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Solution of the Poisson/Laplace equation in various geometries -- cylindrical geometry with z-dependence



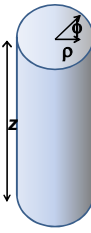
Laplace equation : $\nabla^2\Phi = 0$

$$\frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial \Phi}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 \Phi}{\partial \phi^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0$$

$$\Phi(\rho, \phi, z) = R(\rho)Q(\phi)Z(z)$$

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Cylindrical geometry continued:



Laplace equation : $\nabla^2\Phi = 0$

$$\Phi(\rho, \phi, z) = R(\rho)Q(\phi)Z(z)$$

One possibility :

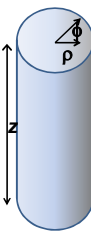
$$\frac{d^2 Z}{dz^2} - k^2 Z = 0 \quad \Rightarrow Z(z) = \sinh(kz), \cosh(kz), e^{\pm kz}$$

$$\frac{d^2 Q}{d\phi^2} + m^2 Q = 0 \quad \Rightarrow Q(\phi) = e^{\pm im\phi}$$

$$\frac{d^2 R}{d\rho^2} + \frac{1}{\rho} \frac{dR}{d\rho} + \left(k^2 - \frac{m^2}{\rho^2} \right) R = 0 \quad \Rightarrow J_m(k\rho), N_m(k\rho)$$

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Cylindrical geometry continued:



Laplace equation : $\nabla^2\Phi = 0$

$$\Phi(\rho, \phi, z) = R(\rho)Q(\phi)Z(z)$$

Another possibility :

$$\frac{d^2 Z}{dz^2} + k^2 Z = 0 \quad \Rightarrow Z(z) = \sin(kz), \cos(kz), e^{\pm ikz}$$

$$\frac{d^2 Q}{d\phi^2} + m^2 Q = 0 \quad \Rightarrow Q(\phi) = e^{\pm im\phi}$$

$$\frac{d^2 R}{d\rho^2} + \frac{1}{\rho} \frac{dR}{d\rho} + \left(-k^2 - \frac{m^2}{\rho^2} \right) R = 0 \quad \Rightarrow I_m(k\rho), K_m(k\rho)$$

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Solutions of Laplace equation inside cylindrical shape
 Example with non-trivial boundary value at $z=L$

$\Phi(\rho, \phi, z=L) = V(\rho, \phi)$
 $\Phi(\rho, \phi, z) = 0$ on all other boundaries

$$\Phi(\rho, \phi, z) = \sum_{n,m} A_{nm} J_m(k_{mn}\rho) \sinh(k_{mn}z) \sin(m\phi + \alpha_{mn})$$

where $J_m(k_{mn}a) = 0$

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Solutions of Laplace equation inside cylindrical shape
 Example with non-trivial boundary value at $z=L$

$\Phi(\rho, \phi, z=L) = V(\rho, \phi)$
 $\Phi(\rho, \phi, z) = 0$ on all other boundaries

$$\Phi(\rho, \phi, z) = \sum_{n,m} A_{nm} J_m(k_{mn}\rho) \sinh(k_{mn}z) \sin(m\phi + \alpha_{mn})$$

If $V(\rho, \phi)$ is an even function of ϕ so that $\alpha_{mn} = \pi/2$:

$$A_{mn} = \frac{\int_0^{2\pi} d\phi \cos(m\phi) \int_0^a \rho d\rho J_m(k_{mn}\rho) V(\rho, \phi)}{\sinh(k_{mn}L) \int_0^{2\pi} d\phi \cos^2(m\phi) \int_0^a \rho d\rho J_m^2(k_{mn}\rho)}$$

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Solutions of Laplace equation inside cylindrical shape
 Example with non-trivial boundary value at $\rho=a$

$\Phi(\rho=a, \phi, z) = V(\phi, z)$
 $\Phi(\rho, \phi, z) = 0$ on all other boundaries

$$\Phi(\rho, \phi, z) = \sum_{n,m} A_{nm} I_m\left(\frac{n\pi\rho}{L}\right) \sin\left(\frac{n\pi z}{L}\right) \sin(m\phi + \alpha_{nm})$$

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Comments on cylindrical Bessel functions

$$\left(\frac{d^2}{du^2} + \frac{1}{u} \frac{d}{du} + \left(\pm 1 - \frac{m^2}{u^2}\right)\right) F_m^\pm(u) = 0$$

$F_m^+(u) = J_m(u), N_m(u), H_m(u) \equiv J_m(u) \pm iN_m(u)$
 $F_m^-(u) = I_m(u), K_m(u)$

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Some useful identities involving cylindrical Bessel functions

$$\left(\frac{d^2}{du^2} + \frac{1}{u} \frac{d}{du} + \left(1 - \frac{m^2}{u^2}\right)\right) J_m(u) = 0 \quad \text{for integer } m$$

Properties of Bessel functions in terms of zeros: $x_{mn}; J_m(x_{mn}) = 0$

$$\int_0^a \rho d\rho J_m(x_{mn}\rho/a) J_m(x_{m'n'}\rho/a) = \frac{a^2}{2} (J_{m+1}(x_{mn}))^2 \delta_{nn'}$$

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Poisson and Laplace equation in spherical polar coordinates

$$\begin{aligned}
 x &= r \sin \theta \cos \varphi \\
 y &= r \sin \theta \sin \varphi \\
 z &= r \cos \theta
 \end{aligned}$$

<http://www.uic.edu/classes/eecs/eecs520/textbook/node32.html>
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Poisson and Laplace equation in spherical polar coordinates -- continued

Laplace equation for electrostatic potential $\Phi(r, \theta, \phi)$:

$$\frac{1}{r} \frac{\partial^2}{\partial r^2} (r\Phi) + \frac{1}{r^2} \left(\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right) \Phi = 0$$

$$\Phi(r, \theta, \phi) = \sum_{lm} R_{lm}(r) Y_{lm}(\theta, \phi)$$

Spherical harmonic functions :

$$\left(\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right) Y_{lm}(\theta, \phi) = -l(l+1) Y_{lm}(\theta, \phi)$$

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Properties of spherical harmonic functions

$$Y_{lm}(\theta, \phi) = (-1)^m Y_{l(-m)}^*(\theta, \phi) \quad (\text{standard Condon - Shortley convention})$$

$$\int d\Omega Y_{lm}(\theta, \phi) Y_{l'm'}^*(\theta, \phi) = \int \sin \theta d\theta d\phi Y_{lm}(\theta, \phi) Y_{l'm'}^*(\theta, \phi) = \delta_{ll'} \delta_{mm'}$$

Completeness :

$$\sum_{lm} Y_{lm}(\theta, \phi) Y_{lm}^*(\theta', \phi') = \delta(\hat{\mathbf{r}} - \hat{\mathbf{r}}') = \delta(\cos \theta - \cos \theta') \delta(\phi - \phi')$$

Relationship to Legendre polynomials :

$$Y_{l0}(\theta, \phi) = \sqrt{\frac{2l+1}{4\pi}} P_l(\cos \theta)$$

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Useful identity:

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \sum_{lm} \frac{4\pi}{2l+1} \frac{r_{<}^l}{r_{>}^{l+1}} Y_{lm}(\theta, \phi) Y_{lm}^*(\theta', \phi')$$

Example for isolated charge density $\rho(\mathbf{r})$ with electrostatic potential vanishing for $r \rightarrow \infty$:

$$\begin{aligned} \Phi(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0} \int d^3r' \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \\ &= \frac{1}{4\pi\epsilon_0} \int d^3r' \rho(\mathbf{r}') \left(\sum_{lm} \frac{4\pi}{2l+1} \frac{r_{<}^l}{r_{>}^{l+1}} Y_{lm}(\theta, \phi) Y_{lm}^*(\theta', \phi') \right) \end{aligned}$$

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Example -- continued

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int d^3r' \rho(\mathbf{r}') \left(\sum_{lm} \frac{4\pi}{2l+1} \frac{r_{<}^l}{r_{>}^{l+1}} Y_{lm}(\theta, \varphi) Y_{lm}^*(\theta', \varphi') \right)$$

Suppose: $\rho(\mathbf{r}') = \frac{Q}{a^3\pi^{3/2}} e^{-r'^2/a^2}$

$$\int d\Omega' Y_{lm}^*(\theta', \varphi') = \sqrt{4\pi} \delta_{l0} \delta_{m0}$$

$$\begin{aligned} \Phi(\mathbf{r}) &= \frac{4\pi}{4\pi\epsilon_0} \int_0^\infty r'^2 dr' \frac{r_{<}^0}{r_{>}^1} \frac{Q}{a^3\pi^{3/2}} e^{-r'^2/a^2} \\ &= \frac{Q}{4\pi\epsilon_0} \frac{\text{erf}(r/a)}{r} \end{aligned}$$

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Useful identity:

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \sum_{lm} \frac{4\pi}{2l+1} \frac{r_{<}^l}{r_{>}^{l+1}} Y_{lm}(\theta, \varphi) Y_{lm}^*(\theta', \varphi')$$

Elements of "proof":

$$\begin{aligned} \frac{1}{|\mathbf{r} - \mathbf{r}'|} &= \frac{1}{r_{>} \left(1 + \left(\frac{r_{<}}{r_{>}}\right)^2 - 2\left(\frac{r_{<}}{r_{>}}\right) \hat{\mathbf{r}} \cdot \hat{\mathbf{r}}' \right)^{1/2}} \\ &= \frac{1}{r_{>}} \left(1 + \left(\frac{r_{<}}{r_{>}}\right) \hat{\mathbf{r}} \cdot \hat{\mathbf{r}}' + \left(\frac{r_{<}}{r_{>}}\right)^2 \left(\frac{3}{2} (\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}')^2 - \frac{1}{2} \right) + \dots \right) \\ &= \frac{1}{r_{>}} \left(\sum_{l=0}^\infty \left(\frac{r_{<}}{r_{>}}\right)^l P_l(\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}') \right) \end{aligned}$$

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Useful identity:

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \sum_{lm} \frac{4\pi}{2l+1} \frac{r_{<}^l}{r_{>}^{l+1}} Y_{lm}(\theta, \varphi) Y_{lm}^*(\theta', \varphi')$$

Elements of "proof" -- continued:

Sum rule for spherical harmonics:

$$P_l(\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}') = \frac{4\pi}{2l+1} \sum_{m=-l}^l Y_{lm}(\hat{\mathbf{r}}) Y_{lm}^*(\hat{\mathbf{r}}')$$

Note that for $\hat{\mathbf{r}} = \hat{\mathbf{r}}'$, $P_l(\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}') = 1$

$$\Rightarrow \frac{4\pi}{2l+1} \sum_{m=-l}^l |Y_{lm}(\hat{\mathbf{r}})|^2 = 1$$

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Some spherical harmonic functions:

$$Y_{00}(\hat{\mathbf{r}}) = \frac{1}{\sqrt{4\pi}}$$

$$Y_{1(\pm 1)}(\hat{\mathbf{r}}) = \mp \sqrt{\frac{3}{8\pi}} \sin \theta e^{\pm i\phi}$$

$$Y_{10}(\hat{\mathbf{r}}) = \sqrt{\frac{3}{4\pi}} \cos \theta$$

$$Y_{2(\pm 2)}(\hat{\mathbf{r}}) = \sqrt{\frac{15}{32\pi}} \sin^2 \theta e^{\pm 2i\phi}$$

$$Y_{2(\pm 1)}(\hat{\mathbf{r}}) = \mp \sqrt{\frac{15}{8\pi}} \sin \theta \cos \theta e^{\pm i\phi}$$

$$Y_{20}(\hat{\mathbf{r}}) = \sqrt{\frac{5}{4\pi}} \left(\frac{3}{2} \cos^2 \theta - \frac{1}{2} \right)$$

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