PHY 712 Electrodynamics
9-9:50 AM MWF Olin 105

Plan for Lecture 9:

Continue reading Chapter 4

Dipolar fields and dielectrics

A. Electric field due to a dipole

B. Electric polarization P

C. Electric displacement D and dielectric functions

Course schedule for Spring 2018

(Preliminary schedule -- subject to frequent adjustment.)

	Lecture date	JDJ Reading	Topic	HW	Due date
	Wed: 01/17/2018	No class	Snow		
1	Fri: 01/19/2018	Chap. 1 & Appen.	Introduction, units and Poisson equation	#1	01/26/2018
2	Mon: 01/22/2018	Chap. 1	Electrostatic energy calculations	#2	01/26/2018
3	Wed: 01/24/2018	Chap. 1	Poisson's equation and Green's theorem	#3	01/26/2018
4	Thu: 01/25/2018	Chap. 1 & 2	Poisson's equation in 2 and 3 dimensions		
5	Fri: 01/26/2018	Chap. 1 & 2	Brief introduction to numerical methods	#4	01/29/2018
6	Mon: 01/29/2018	Chap. 2	Method of image charges	#5	01/31/2018
7	Wed: 01/31/2018	Chap. 2 & 3	Cylindrical and spherical geometries	#6	02/02/2018
8	Fri: 02/02/2018	Chap. 3 & 4	Multipole analysis	#7	02/07/2018
9	Mon: 02/05/2018	Chap. 4	Dipoles and Dielectrics	#8	02/09/2018
10	Wed: 02/07/2018				
11	Fri: 02/09/2018				
12	Mon: 02/12/2018				
13	Wed: 02/14/2018				
14	Fri: 02/16/2018				
15	Mon: 02/19/2018				

Review: General results for a multipole analysis of the electrostatic potential due to an isolated charge distribution:

General form of electrostatic potential with boundary value $\Phi(r \to \infty) = 0$ for confined charge density $\rho(\mathbf{r})$:

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int d^3r' \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$$

$$= \frac{1}{4\pi\varepsilon_0}\int d^3r' \rho(\mathbf{r}')\left(\sum_{lm}\frac{4\pi}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y_{lm}(\theta,\varphi)Y_{lm}^*(\theta',\varphi')\right)$$

Suppose that $\rho(\mathbf{r}) = \sum_{lm}\rho_{lm}(r)Y_{lm}(\theta,\varphi)$

$$\Rightarrow \Phi(\mathbf{r}) = \frac{1}{\varepsilon_0}\sum_{lm}\frac{1}{2l+1}Y_{lm}(\theta,\varphi)\left(\frac{1}{r^{l+1}}\int_0^r r'^{2+l}dr'\rho_{lm}(r') + r^l\int_r^\infty r'^{1-l}dr'\rho_{lm}(r')\right)$$

For $r \to \infty$: $\Phi(\mathbf{r}) = \frac{1}{\varepsilon_0}\sum_{lm}\frac{1}{2l+1}Y_{lm}(\theta,\varphi)\frac{1}{r^{l+1}}\underbrace{\int_0^\infty r'^{2+l}dr'\rho_{lm}(r')}_{q_{lm}}$

Notion of multipole moment:

In the spherical harmonic representation - -

define the moment q_{lm} of the (confined) charge distribution $\rho(\mathbf{r})$:

$$q_{lm} \equiv \int d^3r' \, r'^l \, Y^*_{lm}(\theta', \phi') \rho(\mathbf{r}')$$

In the Cartesian representation - -

define the monopole moment q :

$$q \equiv \int d^3r' \, \rho(\mathbf{r}')$$

define the dipole moment $\mathbf{p}$:

$$\mathbf{p} \equiv \int d^3r' \, \mathbf{r}' \rho(\mathbf{r}')$$

define the quadrupole moment components Q_{ij} $(i,j \rightarrow x,y,z)$:

$$Q_{ij} \equiv \int d^3r' \left(3r'_i \, r'_j - r'^2 \, \delta_{ij}\right) \rho(\mathbf{r}')$$

General form of electrostatic potential in terms of multipole moments:

For r outside the extent of $\rho(\mathbf{r})$:

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \sum_{lm} \frac{4\pi}{2l+1} \frac{Y_{lm}(\theta,\varphi)}{r^{l+1}} \left(\int d^3r' \, r'^l \, Y_{lm}{}^*(\theta', \varphi') \rho(\mathbf{r}') \right)$$

$$= \frac{1}{4\pi\varepsilon_0} \sum_{lm} \frac{4\pi \, q_{lm}}{2l+1} \frac{Y_{lm}(\theta,\varphi)}{r^{l+1}}$$

In terms of Cartesian expansion :

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \left(\frac{q}{r} + \frac{\mathbf{p}\cdot\mathbf{r}}{r^3} + \frac{1}{2} \sum_{i,j} Q_{ij} \frac{r_i r_j}{r^5} \cdots \right)$$

Focus on dipolar contributions:

For r outside the extent of $\rho(\mathbf{r})$:

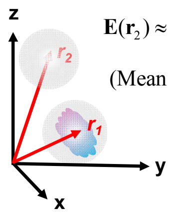
Electrostatic potential :

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \left(\frac{\mathbf{p}\cdot\mathbf{r}}{r^3} \right)$$

Electrostatic field :

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \left(\frac{3\mathbf{r}\,(\mathbf{p}\cdot\mathbf{r}) - r^2\mathbf{p}}{r^5} - \frac{4\pi}{3} \mathbf{p} \, \delta^3(\mathbf{r}) \right)$$

Summary of key argument:



$$\mathbf{E}(\mathbf{r}_2) \approx \frac{3}{4\pi R^3} \int_{r \le R} d^3r \; \mathbf{E}(\mathbf{r}_2 + \mathbf{r}) = \mathbf{E}(\mathbf{r}_2)$$

(Mean value theorem for Laplace equation)

$$\mathbf{E}(\mathbf{r}_1) \approx \frac{3}{4\pi R^3} \int_{r \le R} d^3r \; \mathbf{E}(\mathbf{r}_1 + \mathbf{r})$$

$$\approx \frac{3}{4\pi R^3}\left(-\frac{\mathbf{p}}{3\epsilon_0}\right) \approx -\frac{\mathbf{p}}{3\epsilon_0}\delta(\mathbf{r}-\mathbf{r}_1)$$

Summary:

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\left(\frac{3\mathbf{r}\;(\mathbf{p}\cdot\mathbf{r}) - r^2\mathbf{p}}{r^5} - \frac{4\pi}{3}\mathbf{p}\delta^3(\mathbf{r})\right)$$

Coarse grain representation of macroscopic distribution of dipoles:

Electric polarization $\mathbf{P}(\mathbf{r})$:

$$\mathbf{P}(\mathbf{r}) \equiv \sum_i \mathbf{p}_i \delta^3(\mathbf{r}-\mathbf{r}_i)$$

Mono electric charge density $\rho_{\text{mono}}(\mathbf{r})$:

$$\rho_{\text{mono}}(\mathbf{r}) \equiv \sum_i q_i \delta^3(\mathbf{r}-\mathbf{r}_i)$$

Electrostatic potential for a single monopole charge q and a single dipole $\mathbf{p}$:

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r} + \frac{\mathbf{p}\cdot\mathbf{r}}{r^3}\right)$$

Coarse grain representation of macroscopic distribution of dipoles -- continued:

Electrostatic potential for a single monopole charge q and a single dipole $\mathbf{p}$:

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r} + \frac{\mathbf{p}\cdot\mathbf{r}}{r^3}\right)$$

Electrostatic potential for collections of monopole charges q_i and dipoles $\mathbf{p}_i$:

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\left(\int d^3r' \frac{\rho_{mono}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} + \int d^3r' \frac{\mathbf{P}(\mathbf{r}')\cdot(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\right)$$

Note: $\int d^3r' \frac{\mathbf{P}(\mathbf{r}')\cdot(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} = \int d^3r' \, \mathbf{P}(\mathbf{r}')\cdot\nabla' \frac{1}{|\mathbf{r}-\mathbf{r}'|} = -\int d^3r' \frac{\nabla'\cdot\mathbf{P}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$

Coarse grain representation of macroscopic distribution of dipoles -- continued:

Electrostatic potential for collections of monopole charges q_i and dipoles $\mathbf{p}_i$:

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\left(\int d^3r' \frac{\rho_{mono}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} - \int d^3r' \frac{\nabla'\cdot\mathbf{P}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\right)$$

$$-\nabla^2\Phi(\mathbf{r}) = \nabla\cdot\mathbf{E}(\mathbf{r}) = \frac{1}{\varepsilon_0}\left(\rho_{mono}(\mathbf{r}) - \nabla\cdot\mathbf{P}(\mathbf{r})\right)$$

$$\Rightarrow \nabla\cdot\left(\varepsilon_0\mathbf{E}(\mathbf{r}) + \mathbf{P}(\mathbf{r})\right) = \rho_{mono}(\mathbf{r})$$

Define Displacement field : $\mathbf{D}(\mathbf{r}) \equiv \varepsilon_0\mathbf{E}(\mathbf{r}) + \mathbf{P}(\mathbf{r})$

Macroscopic form of Gauss's law : $\nabla\cdot\mathbf{D}(\mathbf{r}) = \rho_{mono}(\mathbf{r})$

Coarse grain representation of macroscopic distribution of dipoles -- continued:

Many materials are polarizable and produce a polarization field in the presence of an electric field with a proportionality constant χ_e :

$$\mathbf{P}(\mathbf{r}) = \varepsilon_0\chi_e\mathbf{E}(\mathbf{r})$$

$$\mathbf{D}(\mathbf{r}) \equiv \varepsilon_0\mathbf{E}(\mathbf{r}) + \mathbf{P}(\mathbf{r}) = \varepsilon_0(1+\chi_e)\mathbf{E}(\mathbf{r}) \equiv \varepsilon\mathbf{E}(\mathbf{r})$$

ε represents the dielectric function of the material

Boundary value problems in dielectric materials

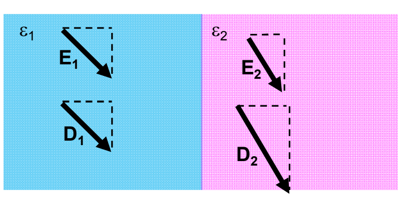
For $\rho_{mono}(\mathbf{r}) = 0$

$\nabla\cdot\mathbf{D}(\mathbf{r}) = 0$ and $\nabla\times\mathbf{E}(\mathbf{r}) = 0$

$\Rightarrow$ At a surface between two dielectrics, in terms of surface normal $\hat{\mathbf{r}}$:

$\hat{\mathbf{r}}\cdot\mathbf{D}(\mathbf{r}) = \text{continuous} = \hat{\mathbf{r}}\times\mathbf{E}(\mathbf{r})$

Boundary value problems in the presence of dielectrics – example:



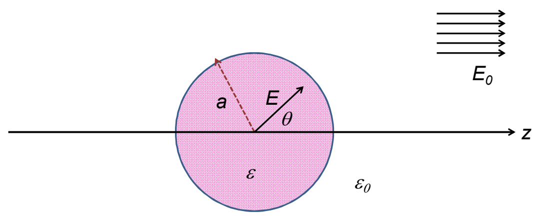
For $\frac{\varepsilon_2}{\varepsilon_1} = 2$

For isotropic dielectrics:

$$D_{1n} = D_{2n} \qquad \varepsilon_1 E_{1n} = \varepsilon_2 E_{2n}$$

$$E_{1t} = E_{2t} \qquad \frac{D_{1t}}{\varepsilon_1} = \frac{D_{2t}}{\varepsilon_2}$$

Boundary value problems in the presence of dielectrics – example:



$$\nabla \cdot \mathbf{D}(\mathbf{r}) = 0 \quad \text{and} \quad \nabla \times \mathbf{E}(\mathbf{r}) = 0 \quad \text{At } r = a: \quad \varepsilon \frac{\partial \Phi_<(\mathbf{r})}{\partial r} = \varepsilon_0 \frac{\partial \Phi_>(\mathbf{r})}{\partial r}$$

$$\text{For } r \le a \quad \mathbf{D}(\mathbf{r}) = -\varepsilon \nabla \Phi(\mathbf{r}) \qquad \frac{\partial \Phi_<(\mathbf{r})}{\partial \theta} = \frac{\partial \Phi_>(\mathbf{r})}{\partial \theta}$$

$$\text{For } r > a \quad \mathbf{D}(\mathbf{r}) = -\varepsilon_0 \nabla \Phi(\mathbf{r})$$

Boundary value problems in the presence of dielectrics – example -- continued:

$$\Phi_<(\mathbf{r}) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) \qquad \text{At } r = a: \quad \varepsilon \frac{\partial \Phi_<(\mathbf{r})}{\partial r} = \varepsilon_0 \frac{\partial \Phi_>(\mathbf{r})}{\partial r}$$

$$\frac{\partial \Phi_<(\mathbf{r})}{\partial \theta} = \frac{\partial \Phi_>(\mathbf{r})}{\partial \theta}$$

$$\Phi_>(\mathbf{r}) = \sum_{l=0}^{\infty} \left(B_l r^l + \frac{C_l}{r^{l+1}} \right) P_l(\cos\theta) \qquad \text{For } r \to \infty \quad \Phi_>(\mathbf{r}) = -E_0 r \cos\theta$$

Solution -- only $l = 1$ contributes

$$B_1 = -E_0$$

$$A_1 = -\left(\frac{3}{2 + \varepsilon / \varepsilon_0} \right) E_0 \qquad C_1 = \left(\frac{\varepsilon / \varepsilon_0 - 1}{2 + \varepsilon / \varepsilon_0} \right) a^3 E_0$$

Boundary value problems in the presence of dielectrics – example -- continued:

$$\Phi_<(\mathbf{r}) = -\left(\frac{3}{2 + \varepsilon / \varepsilon_0} \right) E_0 \, r \cos\theta$$

$$\Phi_>(\mathbf{r}) = -\left(r - \left(\frac{\varepsilon / \varepsilon_0 - 1}{2 + \varepsilon / \varepsilon_0} \right) \frac{a^3}{r^2} \right) E_0 \cos\theta$$