


Solution of Maxwell's equations in the Lorentz gauge -- continued

Liénard-Wiechert potentials and fields --
 Determination of the scalar and vector potentials for a moving point particle (also see Landau and Lifshitz *The Classical Theory of Fields*, Chapter 8.)

Consider the fields produced by the following source: a point charge q moving on a trajectory $\mathbf{R}_q(t)$.

Charge density: $\rho(\mathbf{r}, t) = q\delta^3(\mathbf{r} - \mathbf{R}_q(t))$

Current density: $\mathbf{J}(\mathbf{r}, t) = q\dot{\mathbf{R}}_q(t)\delta^3(\mathbf{r} - \mathbf{R}_q(t))$, where $\dot{\mathbf{R}}_q(t) \equiv \frac{d\mathbf{R}_q(t)}{dt}$.



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Solution of Maxwell's equations in the Lorentz gauge -- continued

$$\Phi(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \int \int d^3r' dt' \frac{\rho(\mathbf{r}', t')}{|\mathbf{r} - \mathbf{r}'|} \delta(t' - (t - |\mathbf{r} - \mathbf{r}'|/c))$$

$$\mathbf{A}(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0 c^2} \int \int d^3r' dt' \frac{\mathbf{J}(\mathbf{r}', t')}{|\mathbf{r} - \mathbf{r}'|} \delta(t' - (t - |\mathbf{r} - \mathbf{r}'|/c)).$$

We performing the integrations over first d^3r' and then dt' making use of the fact that for any function of t' ,

$$\int_{-\infty}^{\infty} dt' f(t') \delta(t' - (t - |\mathbf{r} - \mathbf{R}_q(t_r)|/c)) = \frac{f(t_r)}{1 - \frac{\dot{\mathbf{R}}_q(t_r) \cdot (\mathbf{r} - \mathbf{R}_q(t_r))}{c|\mathbf{r} - \mathbf{R}_q(t_r)|}}$$

where the "retarded time" is defined to be

$$t_r \equiv t - \frac{|\mathbf{r} - \mathbf{R}_q(t_r)|}{c}.$$

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Solution of Maxwell's equations in the Lorentz gauge -- continued

Resulting scalar and vector potentials:

$$\Phi(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0} \frac{1}{R - \frac{\mathbf{v} \cdot \mathbf{R}}{c}},$$

$$\mathbf{A}(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0 c^2} \frac{\mathbf{v}}{R - \frac{\mathbf{v} \cdot \mathbf{R}}{c}},$$

Notation: $\mathbf{R} \equiv \mathbf{r} - \mathbf{R}_q(t_r)$ $t_r \equiv t - \frac{|\mathbf{r} - \mathbf{R}_q(t_r)|}{c}$

$$\mathbf{v} \equiv \dot{\mathbf{R}}_q(t_r),$$

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Solution of Maxwell's equations in the Lorentz gauge -- continued

In order to find the electric and magnetic fields, we need to evaluate

$$\mathbf{E}(\mathbf{r}, t) = -\nabla\Phi(\mathbf{r}, t) - \frac{\partial\mathbf{A}(\mathbf{r}, t)}{\partial t}$$

$$\mathbf{B}(\mathbf{r}, t) = \nabla \times \mathbf{A}(\mathbf{r}, t)$$

The trick of evaluating these derivatives is that the retarded time t_r depends on position \mathbf{r} and on itself. We can show the following results using the shorthand notation:

$$\nabla_{t_r} = -\frac{\mathbf{R}}{c\left(R - \frac{\mathbf{v} \cdot \mathbf{R}}{c}\right)} \quad \text{and} \quad \frac{\partial t_r}{\partial t} = \frac{R}{R - \frac{\mathbf{v} \cdot \mathbf{R}}{c}}$$

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Solution of Maxwell's equations in the Lorentz gauge -- continued

$$-\nabla\Phi(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0} \frac{1}{\left(R - \frac{\mathbf{v} \cdot \mathbf{R}}{c}\right)^3} \left[\mathbf{R} \left(1 - \frac{v^2}{c^2}\right) - \frac{v}{c} \left(R - \frac{\mathbf{v} \cdot \mathbf{R}}{c}\right) + \mathbf{R} \frac{\dot{\mathbf{v}} \cdot \mathbf{R}}{c^2} \right],$$

$$-\frac{\partial\mathbf{A}(\mathbf{r}, t)}{\partial t} = \frac{q}{4\pi\epsilon_0} \frac{1}{\left(R - \frac{\mathbf{v} \cdot \mathbf{R}}{c}\right)^3} \left[\frac{vR}{c} \left(\frac{v^2}{c^2} - \frac{\mathbf{v} \cdot \mathbf{R}}{Rc} - \frac{\dot{\mathbf{v}} \cdot \mathbf{R}}{c^2}\right) - \frac{vR}{c^2} \left(R - \frac{\mathbf{v} \cdot \mathbf{R}}{c}\right) \right],$$

$$\mathbf{E}(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0} \frac{1}{\left(R - \frac{\mathbf{v} \cdot \mathbf{R}}{c}\right)^3} \left[\left(R - \frac{vR}{c}\right) \left(1 - \frac{v^2}{c^2}\right) + \left(\mathbf{R} \times \left\{ \left(R - \frac{vR}{c}\right) \times \frac{\dot{\mathbf{v}}}{c^2} \right\} \right) \right],$$

$$\mathbf{B}(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0 c^2} \left[\frac{-\mathbf{R} \times \mathbf{v}}{\left(R - \frac{\mathbf{v} \cdot \mathbf{R}}{c}\right)^3} \left(1 - \frac{v^2}{c^2} + \frac{\dot{\mathbf{v}} \cdot \mathbf{R}}{c^2}\right) - \frac{\mathbf{R} \times \dot{\mathbf{v}} / c}{\left(R - \frac{\mathbf{v} \cdot \mathbf{R}}{c}\right)^2} \right] = \frac{\mathbf{R} \times \mathbf{E}(\mathbf{r}, t)}{cR}$$

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Maxwell's equations

Coulomb's law : $\nabla \cdot \mathbf{D} = \rho_{free}$

Ampere - Maxwell's law : $\nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{J}_{free}$

Faraday's law : $\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$

No magnetic monopoles : $\nabla \cdot \mathbf{B} = 0$

Energy analysis of electromagnetic fields and sources
Rate of work done on source $\mathbf{J}(\mathbf{r}, t)$ by electromagnetic field:

$$\frac{dW}{dt} = \frac{dE_{mech}}{dt} = \int d^3r \mathbf{E} \cdot \mathbf{J}_{free}$$

Expressing source current in terms of fields it produces:

$$\frac{dW}{dt} = \int d^3r \mathbf{E} \cdot \left(\nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} \right)$$

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Energy analysis of electromagnetic fields and sources -
 - continued

$$\frac{dW}{dt} = \int d^3r \mathbf{E} \cdot \mathbf{J}_{free} = \int d^3r \mathbf{E} \cdot \left(\nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} \right)$$

$$= - \int d^3r \left(\nabla \cdot (\mathbf{E} \times \mathbf{H}) + \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} + \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} \right)$$

Let $\mathbf{S} \equiv \mathbf{E} \times \mathbf{H}$ "Poynting vector"

$$u \equiv \frac{1}{2} (\mathbf{E} \cdot \mathbf{D} + \mathbf{H} \cdot \mathbf{B}) \quad \text{energy density}$$

$$\Rightarrow \frac{\partial u}{\partial t} + \nabla \cdot \mathbf{S} = -\mathbf{E} \cdot \mathbf{J}_{free}$$

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Energy analysis of electromagnetic fields and sources -
 - continued

$$\frac{dE_{mech}}{dt} \equiv \int d^3r \mathbf{E} \cdot \mathbf{J}_{free}$$

Electromagnetic energy density: $u \equiv \frac{1}{2} (\mathbf{E} \cdot \mathbf{D} + \mathbf{H} \cdot \mathbf{B})$

$$E_{field} \equiv \int d^3r u(\mathbf{r}, t)$$

Poynting vector: $\mathbf{S} \equiv \mathbf{E} \times \mathbf{H}$

From the previous energy analysis: $\frac{\partial u}{\partial t} + \nabla \cdot \mathbf{S} = -\mathbf{E} \cdot \mathbf{J}_{free}$

$$\Rightarrow \frac{dE_{mech}}{dt} + \frac{dE_{field}}{dt} = - \int d^3r \nabla \cdot \mathbf{S}(\mathbf{r}, t) = - \oint d^2r \hat{\mathbf{r}} \cdot \mathbf{S}(\mathbf{r}, t)$$

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Momentum analysis of electromagnetic fields and sources

$$\frac{d\mathbf{P}_{mech}}{dt} \equiv \int d^3r (\rho \mathbf{E} + \mathbf{J} \times \mathbf{B})$$

Follows by analogy with Lorentz force:

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

$$\mathbf{P}_{field} = \epsilon_0 \int d^3r (\mathbf{E} \times \mathbf{B})$$

Expression for vacuum fields:

$$\left(\frac{d\mathbf{P}_{mech}}{dt} + \frac{d\mathbf{P}_{field}}{dt} \right)_i = \sum_j \int d^3r \frac{\partial T_{ij}}{\partial r_j}$$

Maxwell stress tensor:

$$T_{ij} \equiv \epsilon_0 \left(E_i E_j + c^2 B_i B_j - \delta_{ij} \frac{1}{2} (\mathbf{E} \cdot \mathbf{E} + c^2 \mathbf{B} \cdot \mathbf{B}) \right)$$

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Comment on treatment of time-harmonic fields
 Fourier transformation in time domain :

$$\mathbf{E}(\mathbf{r}, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \tilde{\mathbf{E}}(\mathbf{r}, \omega) e^{-i\omega t}$$

$$\tilde{\mathbf{E}}(\mathbf{r}, \omega) = \int_{-\infty}^{\infty} dt \mathbf{E}(\mathbf{r}, t) e^{i\omega t}$$

Note that $\mathbf{E}(\mathbf{r}, t)$ is real $\Rightarrow \tilde{\mathbf{E}}(\mathbf{r}, \omega) = \tilde{\mathbf{E}}^*(\mathbf{r}, -\omega)$

These relations and the notion of the superposition principle, lead to the common treatment :

$$\mathbf{E}(\mathbf{r}, t) = \Re \left(\tilde{\mathbf{E}}(\mathbf{r}, \omega) e^{-i\omega t} \right) \equiv \frac{1}{2} \left(\tilde{\mathbf{E}}(\mathbf{r}, \omega) e^{-i\omega t} + \tilde{\mathbf{E}}^*(\mathbf{r}, \omega) e^{i\omega t} \right)$$

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Comment on treatment of time-harmonic fields -- continued
 Equations for time harmonic fields :

$$\mathbf{E}(\mathbf{r}, t) = \Re \left(\tilde{\mathbf{E}}(\mathbf{r}, \omega) e^{-i\omega t} \right) \equiv \frac{1}{2} \left(\tilde{\mathbf{E}}(\mathbf{r}, \omega) e^{-i\omega t} + \tilde{\mathbf{E}}^*(\mathbf{r}, \omega) e^{i\omega t} \right)$$

Equations	in time domain	in frequency domain
Coulomb's law :	$\nabla \cdot \mathbf{D} = \rho_{free}$	$\nabla \cdot \tilde{\mathbf{D}} = \tilde{\rho}_{free}$
Ampere - Maxwell's law :	$\nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{J}_{free}$	$\nabla \times \tilde{\mathbf{H}} + i\omega \tilde{\mathbf{D}} = \tilde{\mathbf{J}}_{free}$
Faraday's law :	$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$	$\nabla \times \tilde{\mathbf{E}} - i\omega \tilde{\mathbf{B}} = 0$
No magnetic monopoles :	$\nabla \cdot \mathbf{B} = 0$	$\nabla \cdot \tilde{\mathbf{B}} = 0$

Note -- in all of these, the real part is taken at the end of the calculation.

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Comment on treatment of time-harmonic fields -- continued
 Equations for time harmonic fields :

$$\mathbf{E}(\mathbf{r}, t) = \Re \left(\tilde{\mathbf{E}}(\mathbf{r}, \omega) e^{-i\omega t} \right) \equiv \frac{1}{2} \left(\tilde{\mathbf{E}}(\mathbf{r}, \omega) e^{-i\omega t} + \tilde{\mathbf{E}}^*(\mathbf{r}, \omega) e^{i\omega t} \right)$$

Poynting vector : $\mathbf{S}(\mathbf{r}, t) = \mathbf{E}(\mathbf{r}, t) \times \mathbf{H}(\mathbf{r}, t)$

$$\mathbf{S}(\mathbf{r}, t) = \frac{1}{4} \left(\tilde{\mathbf{E}}(\mathbf{r}, \omega) e^{-i\omega t} + \tilde{\mathbf{E}}^*(\mathbf{r}, \omega) e^{i\omega t} \right) \times \left(\tilde{\mathbf{H}}(\mathbf{r}, \omega) e^{-i\omega t} + \tilde{\mathbf{H}}^*(\mathbf{r}, \omega) e^{i\omega t} \right)$$

$$= \frac{1}{4} \left(\tilde{\mathbf{E}}(\mathbf{r}, \omega) \times \tilde{\mathbf{H}}^*(\mathbf{r}, \omega) + \tilde{\mathbf{E}}^*(\mathbf{r}, \omega) \times \tilde{\mathbf{H}}(\mathbf{r}, \omega) \right)$$

$$+ \frac{1}{4} \left(\tilde{\mathbf{E}}(\mathbf{r}, \omega) \times \tilde{\mathbf{H}}(\mathbf{r}, \omega) e^{-2i\omega t} + \tilde{\mathbf{E}}^*(\mathbf{r}, \omega) \times \tilde{\mathbf{H}}^*(\mathbf{r}, \omega) e^{2i\omega t} \right)$$

$$\langle \mathbf{S}(\mathbf{r}, t) \rangle_{t_{avg}} = \Re \left(\frac{1}{2} \left(\tilde{\mathbf{E}}(\mathbf{r}, \omega) \times \tilde{\mathbf{H}}^*(\mathbf{r}, \omega) \right) \right)$$

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Summary and review

Maxwell's equations

Coulomb's law : $\nabla \cdot \mathbf{D} = \rho_{free}$

Ampere - Maxwell's law : $\nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{J}_{free}$

Faraday's law : $\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$

No magnetic monopoles : $\nabla \cdot \mathbf{B} = 0$

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Maxwell's equations

For linear isotropic media -- $\mathbf{D} = \epsilon \mathbf{E}$; $\mathbf{B} = \mu \mathbf{H}$
and no sources :

Coulomb's law : $\nabla \cdot \mathbf{E} = 0$

Ampere - Maxwell's law : $\nabla \times \mathbf{B} - \mu \epsilon \frac{\partial \mathbf{E}}{\partial t} = 0$

Faraday's law : $\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$

No magnetic monopoles : $\nabla \cdot \mathbf{B} = 0$

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Analysis of Maxwell's equations without sources -- continued:

Coulomb's law : $\nabla \cdot \mathbf{E} = 0$

Ampere - Maxwell's law : $\nabla \times \mathbf{B} - \mu \epsilon \frac{\partial \mathbf{E}}{\partial t} = 0$

Faraday's law : $\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$

No magnetic monopoles : $\nabla \cdot \mathbf{B} = 0$

$$\begin{aligned} \nabla \times \left(\nabla \times \mathbf{B} - \mu \epsilon \frac{\partial \mathbf{E}}{\partial t} \right) &= -\nabla^2 \mathbf{B} - \mu \epsilon \frac{\partial (\nabla \times \mathbf{E})}{\partial t} \\ &= -\nabla^2 \mathbf{B} + \mu \epsilon \frac{\partial^2 \mathbf{B}}{\partial t^2} = 0 \end{aligned}$$

$$\begin{aligned} \nabla \times \left(\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} \right) &= -\nabla^2 \mathbf{E} + \frac{\partial (\nabla \times \mathbf{B})}{\partial t} \\ &= -\nabla^2 \mathbf{E} + \mu \epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0 \end{aligned}$$

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Analysis of Maxwell's equations without sources -- continued:
Both E and B fields are solutions to a wave equation:

$$\nabla^2 \mathbf{B} - \frac{1}{v^2} \frac{\partial^2 \mathbf{B}}{\partial t^2} = 0$$

$$\nabla^2 \mathbf{E} - \frac{1}{v^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0$$

where $v^2 \equiv c^2 \frac{\mu_0 \epsilon_0}{\mu \epsilon} \equiv \frac{c^2}{n^2}$

Plane wave solutions to wave equation :

$$\mathbf{B}(\mathbf{r}, t) = \Re(\mathbf{B}_0 e^{i\mathbf{k} \cdot \mathbf{r} - i\omega t}) \quad \mathbf{E}(\mathbf{r}, t) = \Re(\mathbf{E}_0 e^{i\mathbf{k} \cdot \mathbf{r} - i\omega t})$$

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Analysis of Maxwell's equations without sources -- continued:
Plane wave solutions to wave equation :

$$\mathbf{B}(\mathbf{r}, t) = \Re(\mathbf{B}_0 e^{i\mathbf{k} \cdot \mathbf{r} - i\omega t}) \quad \mathbf{E}(\mathbf{r}, t) = \Re(\mathbf{E}_0 e^{i\mathbf{k} \cdot \mathbf{r} - i\omega t})$$

$$|\mathbf{k}|^2 = \left(\frac{\omega}{v}\right)^2 = \left(\frac{n\omega}{c}\right)^2 \quad \text{where } n \equiv \sqrt{\frac{\mu \epsilon}{\mu_0 \epsilon_0}}$$

Note: ϵ, μ, n, k can all be complex; for the moment we will assume that they are all real (no dissipation).

Note that \mathbf{E}_0 and \mathbf{B}_0 are not independent;
from Faraday's law : $\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$

$$\Rightarrow \mathbf{B}_0 = \frac{\mathbf{k} \times \mathbf{E}_0}{\omega} = \frac{n \hat{\mathbf{k}} \times \mathbf{E}_0}{c}$$

also note : $\hat{\mathbf{k}} \cdot \mathbf{E}_0 = 0$ and $\hat{\mathbf{k}} \cdot \mathbf{B}_0 = 0$

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Analysis of Maxwell's equations without sources -- continued:
Summary of plane electromagnetic waves :

$$\mathbf{B}(\mathbf{r}, t) = \Re\left(\frac{n \hat{\mathbf{k}} \times \mathbf{E}_0}{c} e^{i\mathbf{k} \cdot \mathbf{r} - i\omega t}\right) \quad \mathbf{E}(\mathbf{r}, t) = \Re(\mathbf{E}_0 e^{i\mathbf{k} \cdot \mathbf{r} - i\omega t})$$

$$|\mathbf{k}|^2 = \left(\frac{\omega}{v}\right)^2 = \left(\frac{n\omega}{c}\right)^2 \quad \text{where } n \equiv \sqrt{\frac{\mu \epsilon}{\mu_0 \epsilon_0}} \quad \text{and } \hat{\mathbf{k}} \cdot \mathbf{E}_0 = 0$$

Poynting vector for plane electromagnetic waves :

$$\langle \mathbf{S} \rangle_{\text{avg}} = \frac{1}{2} \Re\left(\mathbf{E}_0 e^{i\mathbf{k} \cdot \mathbf{r} - i\omega t} \times \frac{1}{\mu} \left(\frac{n \hat{\mathbf{k}} \times \mathbf{E}_0}{c} e^{i\mathbf{k} \cdot \mathbf{r} - i\omega t}\right)^*\right)$$

$$= \frac{n |\mathbf{E}_0|^2}{2 \mu c} \hat{\mathbf{k}} = \frac{1}{2} \sqrt{\frac{\epsilon}{\mu}} |\mathbf{E}_0|^2 \hat{\mathbf{k}}$$

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Analysis of Maxwell's equations without sources -- continued:

Transverse Electric and Magnetic (TEM) waves

Summary of plane electromagnetic waves:

$$\mathbf{B}(\mathbf{r}, t) = \Re\left\{\frac{n\hat{\mathbf{k}} \times \mathbf{E}_0}{c} e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t}\right\} \quad \mathbf{E}(\mathbf{r}, t) = \Re\left\{\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t}\right\}$$

$$|\mathbf{k}|^2 = \left(\frac{\omega}{v}\right)^2 = \left(\frac{n\omega}{c}\right)^2 \quad \text{where } n \equiv \sqrt{\frac{\mu\epsilon}{\mu_0\epsilon_0}} \quad \text{and } \hat{\mathbf{k}} \cdot \mathbf{E}_0 = 0$$

Energy density for plane electromagnetic waves:

$$\langle u \rangle_{\text{avg}} = \frac{1}{4} \Re\left\{\epsilon \mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t} \cdot \left(\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t}\right)^*\right\} +$$

$$\frac{1}{4} \Re\left\{\frac{1}{\mu} \frac{n\hat{\mathbf{k}} \times \mathbf{E}_0}{c} e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t} \cdot \left(\frac{n\hat{\mathbf{k}} \times \mathbf{E}_0}{c} e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t}\right)^*\right\}$$

$$= \frac{1}{2} \epsilon |\mathbf{E}_0|^2$$

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More general result (example found in waveguides)

1. Suppose that an electromagnetic wave of pure (real) frequency ω is traveling along the z-axis of a wave guide having a square cross section with side dimension a composed of a medium having a real permittivity constant ϵ and a real permeability constant μ . Suppose that the wave is known to have the form:

$$\mathbf{E}(\mathbf{r}, t) = \Re\left\{H_0 e^{i(kz - \omega t)} \left[(i\mu\omega) \frac{a}{\pi} \sin\left(\frac{\pi x}{a}\right) \hat{\mathbf{y}} \right]\right\}$$

$$\mathbf{H}(\mathbf{r}, t) = \Re\left\{H_0 e^{i(kz - \omega t)} \left[-ik \frac{a}{\pi} \sin\left(\frac{\pi x}{a}\right) \hat{\mathbf{x}} + \cos\left(\frac{\pi x}{a}\right) \hat{\mathbf{z}} \right]\right\}$$

Here H_0 denotes a real amplitude, and the parameter k is assumed to be real and equal to

$$k = \sqrt{\mu\epsilon\omega^2 - \left(\frac{\pi}{a}\right)^2}, \quad \text{for } \mu\epsilon\omega^2 > \left(\frac{\pi}{a}\right)^2$$

a. Show that this wave satisfies the sourceless Maxwell's equations.
 b. Find the form of the time-averaged Poynting vector

$$\langle \mathbf{S} \rangle_{\text{avg}} \equiv \frac{1}{2} \Re\left\{\mathbf{E}(\mathbf{r}, t) \times \mathbf{H}^*(\mathbf{r}, t)\right\}$$

for this electromagnetic wave.

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