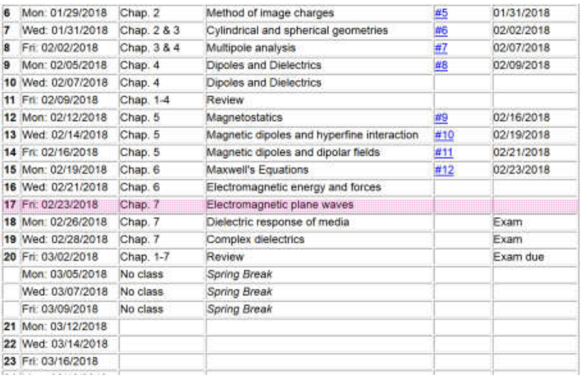
PHY 712 Electrodynamics
9-9:50 AM Olin 105

Plan for Lecture 16:

Read Chapter 7

1. Plane polarized electromagnetic waves
2. Reflectance and transmittance of electromagnetic waves – extension to anisotropy and complexity



6	Mon: 01/29/2018	Chap. 2	Method of image charges	#5	01/31/2018
7	Wed: 01/31/2018	Chap. 2 & 3	Cylindrical and spherical geometries	#6	02/02/2018
8	Fri: 02/02/2018	Chap. 3 & 4	Multipole analysis	#7	02/07/2018
9	Mon: 02/05/2018	Chap. 4	Dipoles and Dielectrics	#8	02/09/2018
10	Wed: 02/07/2018	Chap. 4	Dipoles and Dielectrics		
11	Fri: 02/09/2018	Chap. 1-4	Review		
12	Mon: 02/12/2018	Chap. 5	Magnetostatics	#9	02/16/2018
13	Wed: 02/14/2018	Chap. 5	Magnetic dipoles and hyperfine interaction	#10	02/19/2018
14	Fri: 02/16/2018	Chap. 5	Magnetic dipoles and dipolar fields	#11	02/21/2018
15	Mon: 02/19/2018	Chap. 6	Maxwell's Equations	#12	02/23/2018
16	Wed: 02/21/2018	Chap. 6	Electromagnetic energy and forces		
17	Fri: 02/23/2018	Chap. 7	Electromagnetic plane waves		
18	Mon: 02/26/2018	Chap. 7	Dielectric response of media		Exam
19	Wed: 02/28/2018	Chap. 7	Complex dielectrics		Exam
20	Fri: 03/02/2018	Chap. 1-7	Review		Exam due
	Mon: 03/05/2018	No class	*Spring Break*		
	Wed: 03/07/2018	No class	*Spring Break*		
	Fri: 03/09/2018	No class	*Spring Break*		
21	Mon: 03/12/2018				
22	Wed: 03/14/2018				
23	Fri: 03/16/2018				

Maxwell's equations

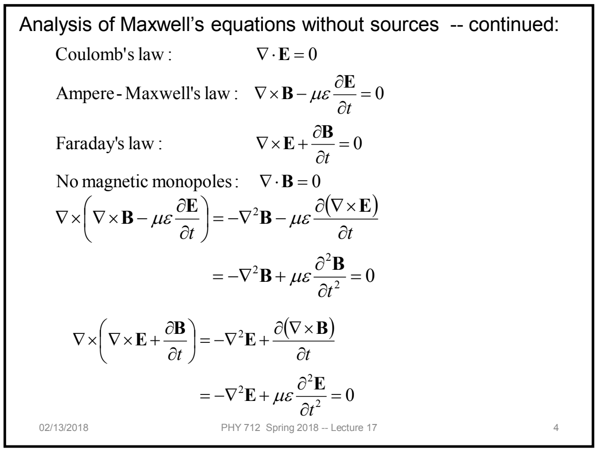
For linear isotropic media and no sources: $\mathbf{D}=\varepsilon\mathbf{E}; \quad \mathbf{B}=\mu\mathbf{H}$

Coulomb's law: $\nabla\cdot\mathbf{E}=0$

Ampere-Maxwell's law: $\nabla\times\mathbf{B}-\mu\varepsilon\frac{\partial\mathbf{E}}{\partial t}=0$

Faraday's law: $\nabla\times\mathbf{E}+\frac{\partial\mathbf{B}}{\partial t}=0$

No magnetic monopoles: $\nabla\cdot\mathbf{B}=0$



Analysis of Maxwell's equations without sources -- continued:

$$\text{Coulomb's law:} \qquad \nabla\cdot\mathbf{E}=0$$

$$\text{Ampere-Maxwell's law:} \quad \nabla\times\mathbf{B}-\mu\varepsilon\frac{\partial\mathbf{E}}{\partial t}=0$$

$$\text{Faraday's law:} \qquad \nabla\times\mathbf{E}+\frac{\partial\mathbf{B}}{\partial t}=0$$

$$\text{No magnetic monopoles:} \quad \nabla\cdot\mathbf{B}=0$$

$$\nabla\times\left(\nabla\times\mathbf{B}-\mu\varepsilon\frac{\partial\mathbf{E}}{\partial t}\right)=-\nabla^2\mathbf{B}-\mu\varepsilon\frac{\partial(\nabla\times\mathbf{E})}{\partial t}$$

$$=-\nabla^2\mathbf{B}+\mu\varepsilon\frac{\partial^2\mathbf{B}}{\partial t^2}=0$$

$$\nabla\times\left(\nabla\times\mathbf{E}+\frac{\partial\mathbf{B}}{\partial t}\right)=-\nabla^2\mathbf{E}+\frac{\partial(\nabla\times\mathbf{B})}{\partial t}$$

$$=-\nabla^2\mathbf{E}+\mu\varepsilon\frac{\partial^2\mathbf{E}}{\partial t^2}=0$$

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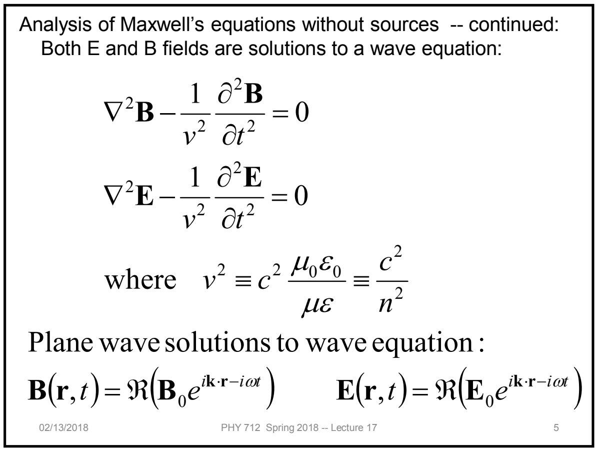

Analysis of Maxwell's equations without sources -- continued:
Both E and B fields are solutions to a wave equation:

$$\nabla^2\mathbf{B}-\frac{1}{v^2}\frac{\partial^2\mathbf{B}}{\partial t^2}=0$$

$$\nabla^2\mathbf{E}-\frac{1}{v^2}\frac{\partial^2\mathbf{E}}{\partial t^2}=0$$

$$\text{where} \quad v^2\equiv c^2\frac{\mu_0\varepsilon_0}{\mu\varepsilon}\equiv\frac{c^2}{n^2}$$

Plane wave solutions to wave equation:

$$\mathbf{B}(\mathbf{r},t)=\Re\left(\mathbf{B}_0e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\right) \qquad \mathbf{E}(\mathbf{r},t)=\Re\left(\mathbf{E}_0e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\right)$$

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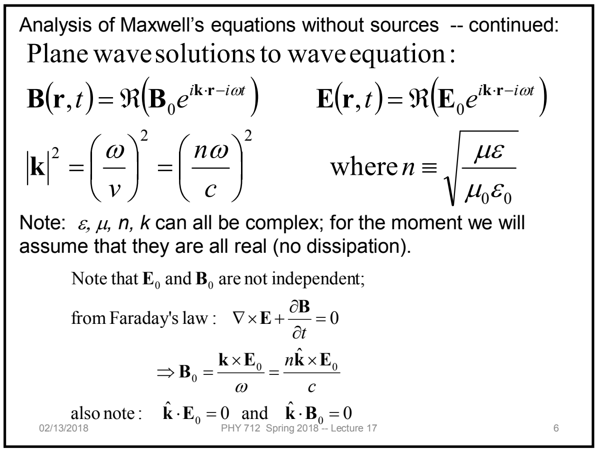

Analysis of Maxwell's equations without sources -- continued:

Plane wave solutions to wave equation:

$$\mathbf{B}(\mathbf{r},t)=\Re\left(\mathbf{B}_0e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\right) \qquad \mathbf{E}(\mathbf{r},t)=\Re\left(\mathbf{E}_0e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\right)$$

$$|\mathbf{k}|^2=\left(\frac{\omega}{v}\right)^2=\left(\frac{n\omega}{c}\right)^2 \qquad \text{where } n\equiv\sqrt{\frac{\mu\varepsilon}{\mu_0\varepsilon_0}}$$

Note: ε, μ, n, k can all be complex; for the moment we will assume that they are all real (no dissipation).

Note that $\mathbf{E}_0$ and $\mathbf{B}_0$ are not independent;

from Faraday's law: $\nabla\times\mathbf{E}+\frac{\partial\mathbf{B}}{\partial t}=0$

$$\Rightarrow \mathbf{B}_0=\frac{\mathbf{k}\times\mathbf{E}_0}{\omega}=\frac{n\hat{\mathbf{k}}\times\mathbf{E}_0}{c}$$

also note: $\hat{\mathbf{k}}\cdot\mathbf{E}_0=0$ and $\hat{\mathbf{k}}\cdot\mathbf{B}_0=0$

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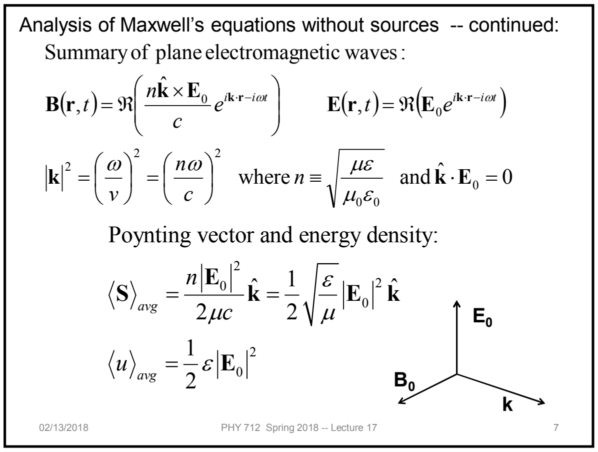


Analysis of Maxwell's equations without sources -- continued:

Summary of plane electromagnetic waves:

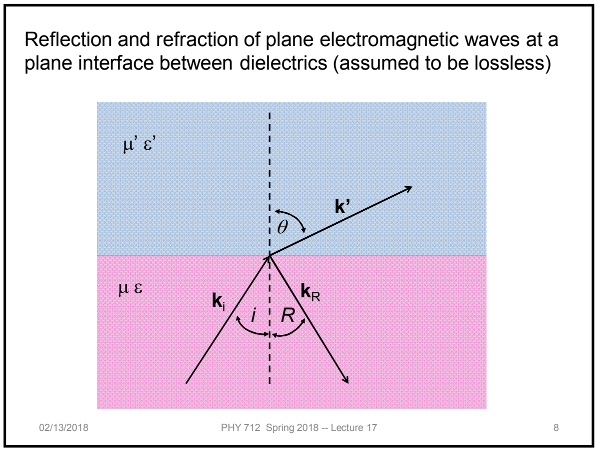
$$\mathbf{B}(\mathbf{r},t)=\Re\left(\frac{n\hat{\mathbf{k}}\times\mathbf{E}_0}{c}e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\right)\qquad\mathbf{E}(\mathbf{r},t)=\Re\left(\mathbf{E}_0e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\right)$$

$$|\mathbf{k}|^2=\left(\frac{\omega}{v}\right)^2=\left(\frac{n\omega}{c}\right)^2\quad\text{where } n\equiv\sqrt{\frac{\mu\varepsilon}{\mu_0\varepsilon_0}}\quad\text{and } \hat{\mathbf{k}}\cdot\mathbf{E}_0=0$$

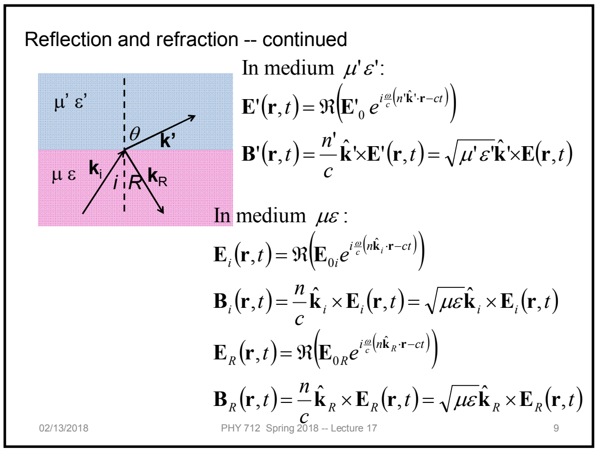
Poynting vector and energy density:

$$\langle\mathbf{S}\rangle_{avg}=\frac{n|\mathbf{E}_0|^2}{2\mu c}\hat{\mathbf{k}}=\frac{1}{2}\sqrt{\frac{\varepsilon}{\mu}}|\mathbf{E}_0|^2\,\hat{\mathbf{k}}$$

$$\langle u\rangle_{avg}=\frac{1}{2}\varepsilon|\mathbf{E}_0|^2$$



Reflection and refraction of plane electromagnetic waves at a plane interface between dielectrics (assumed to be lossless)



Reflection and refraction -- continued

In medium $\mu'\varepsilon'$:

$$\mathbf{E}'(\mathbf{r},t)=\Re\left(\mathbf{E}'_0\,e^{i\frac{\omega}{c}\left(n'\hat{\mathbf{k}}'\cdot\mathbf{r}-ct\right)}\right)$$

$$\mathbf{B}'(\mathbf{r},t)=\frac{n'}{c}\hat{\mathbf{k}}'\times\mathbf{E}'(\mathbf{r},t)=\sqrt{\mu'\varepsilon'}\hat{\mathbf{k}}'\times\mathbf{E}(\mathbf{r},t)$$

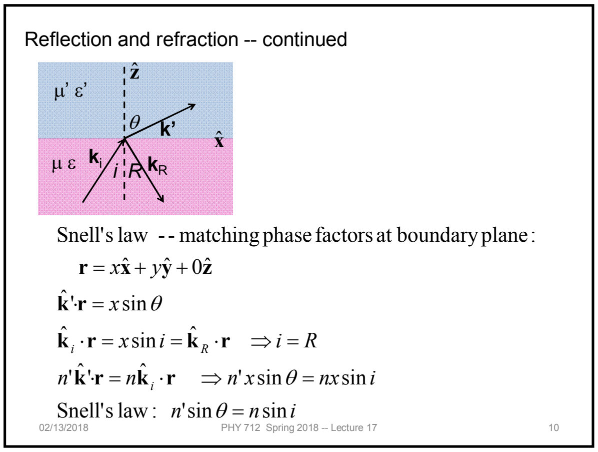
In medium $\mu\varepsilon$:

$$\mathbf{E}_i(\mathbf{r},t)=\Re\left(\mathbf{E}_{0i}e^{i\frac{\omega}{c}\left(n\hat{\mathbf{k}}_i\cdot\mathbf{r}-ct\right)}\right)$$

$$\mathbf{B}_i(\mathbf{r},t)=\frac{n}{c}\hat{\mathbf{k}}_i\times\mathbf{E}_i(\mathbf{r},t)=\sqrt{\mu\varepsilon}\hat{\mathbf{k}}_i\times\mathbf{E}_i(\mathbf{r},t)$$

$$\mathbf{E}_R(\mathbf{r},t)=\Re\left(\mathbf{E}_{0R}e^{i\frac{\omega}{c}\left(n\hat{\mathbf{k}}_R\cdot\mathbf{r}-ct\right)}\right)$$

$$\mathbf{B}_R(\mathbf{r},t)=\frac{n}{c}\hat{\mathbf{k}}_R\times\mathbf{E}_R(\mathbf{r},t)=\sqrt{\mu\varepsilon}\hat{\mathbf{k}}_R\times\mathbf{E}_R(\mathbf{r},t)$$



Reflection and refraction -- continued

Snell's law -- matching phase factors at boundary plane:

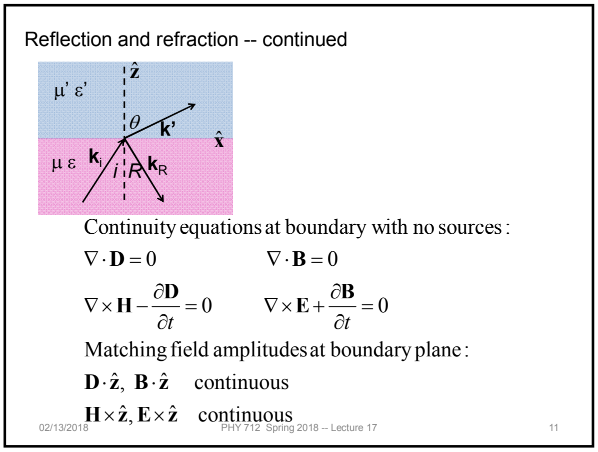
$$\mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + 0\hat{\mathbf{z}}$$

$$\hat{\mathbf{k}}'\cdot\mathbf{r} = x\sin\theta$$

$$\hat{\mathbf{k}}_i\cdot\mathbf{r} = x\sin i = \hat{\mathbf{k}}_R\cdot\mathbf{r} \quad \Rightarrow i = R$$

$$n'\hat{\mathbf{k}}'\cdot\mathbf{r} = n\hat{\mathbf{k}}_i\cdot\mathbf{r} \quad \Rightarrow n'x\sin\theta = nx\sin i$$

Snell's law: $n'\sin\theta = n\sin i$



Reflection and refraction -- continued

Continuity equations at boundary with no sources:

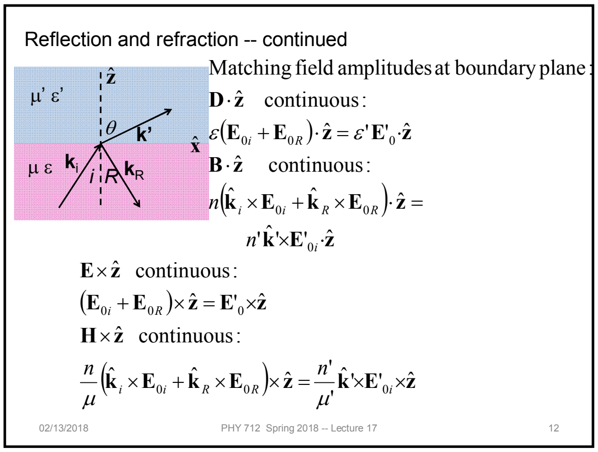
$$\nabla\cdot\mathbf{D} = 0 \qquad \nabla\cdot\mathbf{B} = 0$$

$$\nabla\times\mathbf{H} - \frac{\partial\mathbf{D}}{\partial t} = 0 \qquad \nabla\times\mathbf{E} + \frac{\partial\mathbf{B}}{\partial t} = 0$$

Matching field amplitudes at boundary plane:

$\mathbf{D}\cdot\hat{\mathbf{z}}$, $\mathbf{B}\cdot\hat{\mathbf{z}}$ continuous

$\mathbf{H}\times\hat{\mathbf{z}}$, $\mathbf{E}\times\hat{\mathbf{z}}$ continuous



Reflection and refraction -- continued

Matching field amplitudes at boundary plane:

$\mathbf{D}\cdot\hat{\mathbf{z}}$ continuous:

$$\varepsilon\left(\mathbf{E}_{0i} + \mathbf{E}_{0R}\right)\cdot\hat{\mathbf{z}} = \varepsilon'\mathbf{E}'_0\cdot\hat{\mathbf{z}}$$

$\mathbf{B}\cdot\hat{\mathbf{z}}$ continuous:

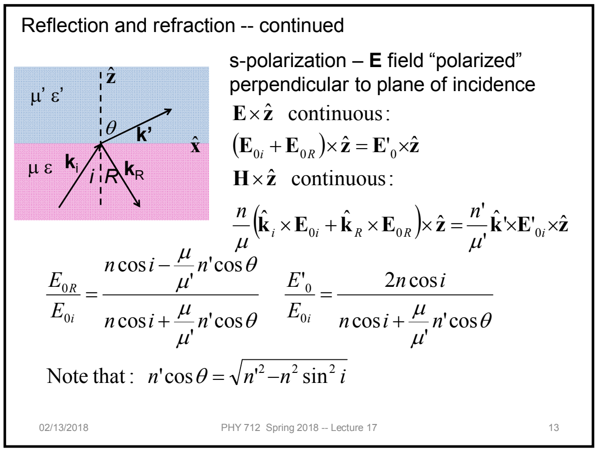
$$n\left(\hat{\mathbf{k}}_i\times\mathbf{E}_{0i} + \hat{\mathbf{k}}_R\times\mathbf{E}_{0R}\right)\cdot\hat{\mathbf{z}} = n'\hat{\mathbf{k}}'\times\mathbf{E}'_{0i}\cdot\hat{\mathbf{z}}$$

$\mathbf{E}\times\hat{\mathbf{z}}$ continuous:

$$\left(\mathbf{E}_{0i} + \mathbf{E}_{0R}\right)\times\hat{\mathbf{z}} = \mathbf{E}'_0\times\hat{\mathbf{z}}$$

$\mathbf{H}\times\hat{\mathbf{z}}$ continuous:

$$\frac{n}{\mu}\left(\hat{\mathbf{k}}_i\times\mathbf{E}_{0i} + \hat{\mathbf{k}}_R\times\mathbf{E}_{0R}\right)\times\hat{\mathbf{z}} = \frac{n'}{\mu'}\hat{\mathbf{k}}'\times\mathbf{E}'_{0i}\times\hat{\mathbf{z}}$$



Reflection and refraction -- continued

s-polarization – **E** field "polarized" perpendicular to plane of incidence

$\mathbf{E}\times\hat{\mathbf{z}}$ continuous:

$$(\mathbf{E}_{0i}+\mathbf{E}_{0R})\times\hat{\mathbf{z}}=\mathbf{E}'_0\times\hat{\mathbf{z}}$$

$\mathbf{H}\times\hat{\mathbf{z}}$ continuous:

$$\frac{n}{\mu}\left(\hat{\mathbf{k}}_i\times\mathbf{E}_{0i}+\hat{\mathbf{k}}_R\times\mathbf{E}_{0R}\right)\times\hat{\mathbf{z}}=\frac{n'}{\mu'}\hat{\mathbf{k}}'\times\mathbf{E}'_{0i}\times\hat{\mathbf{z}}$$

$$\frac{E_{0R}}{E_{0i}}=\frac{n\cos i-\frac{\mu}{\mu'}n'\cos\theta}{n\cos i+\frac{\mu}{\mu'}n'\cos\theta}\qquad\frac{E'_0}{E_{0i}}=\frac{2n\cos i}{n\cos i+\frac{\mu}{\mu'}n'\cos\theta}$$

Note that: $n'\cos\theta=\sqrt{n'^2-n^2\sin^2 i}$

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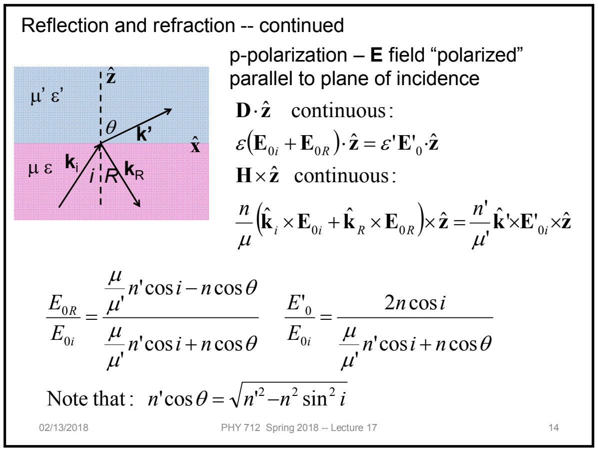

Reflection and refraction -- continued

p-polarization – **E** field "polarized" parallel to plane of incidence

$\mathbf{D}\cdot\hat{\mathbf{z}}$ continuous:

$$\varepsilon(\mathbf{E}_{0i}+\mathbf{E}_{0R})\cdot\hat{\mathbf{z}}=\varepsilon'\mathbf{E}'_0\cdot\hat{\mathbf{z}}$$

$\mathbf{H}\times\hat{\mathbf{z}}$ continuous:

$$\frac{n}{\mu}\left(\hat{\mathbf{k}}_i\times\mathbf{E}_{0i}+\hat{\mathbf{k}}_R\times\mathbf{E}_{0R}\right)\times\hat{\mathbf{z}}=\frac{n'}{\mu'}\hat{\mathbf{k}}'\times\mathbf{E}'_{0i}\times\hat{\mathbf{z}}$$

$$\frac{E_{0R}}{E_{0i}}=\frac{\frac{\mu}{\mu'}n'\cos i-n\cos\theta}{\frac{\mu}{\mu'}n'\cos i+n\cos\theta}\qquad\frac{E'_0}{E_{0i}}=\frac{2n\cos i}{\frac{\mu}{\mu'}n'\cos i+n\cos\theta}$$

Note that: $n'\cos\theta=\sqrt{n'^2-n^2\sin^2 i}$

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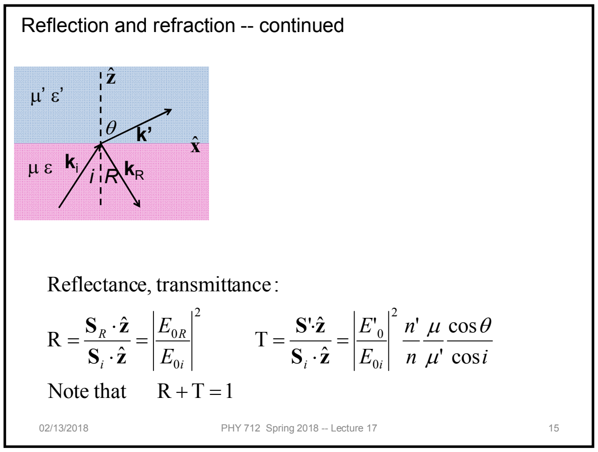

Reflection and refraction -- continued

Reflectance, transmittance:

$$\mathrm{R}=\frac{\mathbf{S}_R\cdot\hat{\mathbf{z}}}{\mathbf{S}_i\cdot\hat{\mathbf{z}}}=\left|\frac{E_{0R}}{E_{0i}}\right|^2\qquad\mathrm{T}=\frac{\mathbf{S}'\cdot\hat{\mathbf{z}}}{\mathbf{S}_i\cdot\hat{\mathbf{z}}}=\left|\frac{E'_0}{E_{0i}}\right|^2\frac{n'}{n}\frac{\mu}{\mu'}\frac{\cos\theta}{\cos i}$$

Note that $\mathrm{R}+\mathrm{T}=1$

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For s-polarization

$$\frac{E_{0R}}{E_{0i}} = \frac{n\cos i - \frac{\mu}{\mu'} n'\cos\theta}{n\cos i + \frac{\mu}{\mu'} n'\cos\theta} \qquad \frac{E'_0}{E_{0i}} = \frac{2n\cos i}{n\cos i + \frac{\mu}{\mu'} n'\cos\theta}$$

Note that: $n'\cos\theta = \sqrt{n'^2 - n^2 \sin^2 i}$

For p-polarization

$$\frac{E_{0R}}{E_{0i}} = \frac{\frac{\mu}{\mu'} n'\cos i - n\cos\theta}{\frac{\mu}{\mu'} n'\cos i + n\cos\theta} \qquad \frac{E'_0}{E_{0i}} = \frac{2n\cos i}{\frac{\mu}{\mu'} n'\cos i + n\cos\theta}$$

Note that: $n'\cos\theta = \sqrt{n'^2 - n^2 \sin^2 i}$

Special case: normal incidence (i=0, θ=0)

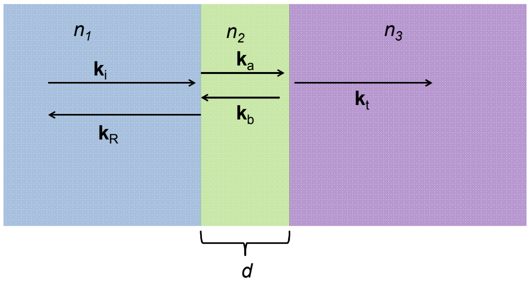
$$\frac{E_{0R}}{E_{0i}} = \frac{\frac{\mu}{\mu'} n' - n}{\frac{\mu}{\mu'} n' + n} \qquad \frac{E'_0}{E_{0i}} = \frac{2n}{\frac{\mu}{\mu'} n' + n}$$

Reflectance, transmittance:

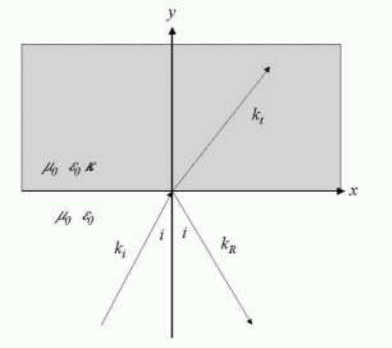
$$\mathrm{R} = \left|\frac{E_{0R}}{E_{0i}}\right|^2 = \left|\frac{\frac{\mu}{\mu'} n' - n}{\frac{\mu}{\mu'} n' + n}\right|^2$$

$$\mathrm{T} = \left|\frac{E'_0}{E_{0i}}\right|^2 \frac{n'}{n}\frac{\mu}{\mu'} = \left|\frac{2n}{\frac{\mu}{\mu'} n' + n}\right|^2 \frac{n'}{n}\frac{\mu}{\mu'}$$

Multilayer dielectrics (Problem #7.2)



Extension of analysis to anisotropic media --



Consider the problem of determining the reflectance from an anisotropic medium with isotropic permeability μ_0 and anisotropic permittivity $\varepsilon_0 \boldsymbol{\kappa}$ where:

$$\boldsymbol{\kappa} \equiv \begin{pmatrix} \kappa_{xx} & 0 & 0 \\ 0 & \kappa_{yy} & 0 \\ 0 & 0 & \kappa_{zz} \end{pmatrix}$$

By assumption, the wave vector in the medium is confined to the *x-y* plane and will be denoted by

$$\mathbf{k}_t \equiv \frac{\omega}{c}(n_x\hat{\mathbf{x}} + n_y\hat{\mathbf{y}}), \text{where } n_x \text{ and } n_y \text{ are to be determined.}$$

The electric field inside the medium is given by:

$$\mathbf{E} = (E_x\hat{\mathbf{x}} + E_y\hat{\mathbf{y}} + E_z\hat{\mathbf{z}})\mathrm{e}^{i\frac{\omega}{c}(n_x x + n_y y) - i\omega t}.$$

Inside the anisotropic medium, Maxwell's equations are:

$$\nabla \cdot \mathbf{H} = 0 \qquad \nabla \cdot \boldsymbol{\kappa} \cdot \mathbf{E} = 0$$
$$\nabla \times \mathbf{E} - i\omega\mu_0\mathbf{H} = 0 \qquad \nabla \times \mathbf{H} + i\omega\epsilon_0\boldsymbol{\kappa} \cdot \mathbf{E} = 0$$

After some algebra, the equation for **E** is:

$$\begin{pmatrix} \kappa_{xx} - n_y^2 & n_x n_y & 0 \\ n_x n_y & \kappa_{yy} - n_x^2 & 0 \\ 0 & 0 & \kappa_{zz} - (n_x^2 + n_y^2) \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = 0.$$

From **E, H** can be determined from

$$\mathbf{H} = \frac{1}{\mu_0 c}\left\{E_z(n_y\hat{\mathbf{x}} - n_x\hat{\mathbf{y}}) + (E_y n_x - E_x n_y)\hat{\mathbf{z}}\right\}\mathrm{e}^{i\frac{\omega}{c}(n_x x + n_y y) - i\omega t}.$$

The fields for the incident and reflected waves are the same as for the isotropic case.

$$\mathbf{k}_i = \frac{\omega}{c}(\sin i\hat{\mathbf{x}} + \cos i\hat{\mathbf{y}}),$$

$$\mathbf{k}_R = \frac{\omega}{c}(\sin i\hat{\mathbf{x}} - \cos i\hat{\mathbf{y}}).$$

Note that, consistent with Snell's law: $n_x = \sin i$

Continuity conditions at the *y=0* plane must be applied for the following fields:

$$\mathbf{H}(x,0,z,t),\ E_x(x,0,z,t),\ E_z(x,0,z,t),\ \text{and}\ D_y(x,0,z,t).$$

There will be two different solutions, depending of the polarization of the incident field.

Solution for s-polarization

$$E_x = E_y = 0 \qquad \Rightarrow n_y^2 = \kappa_{zz} - n_x^2$$

$$\mathbf{E} = E_z\hat{\mathbf{z}}e^{i\frac{\omega}{c}(n_x x + n_y y) - i\omega t} \qquad \mathbf{H} = \frac{1}{\mu_0 c}\left\{E_z(n_y\hat{\mathbf{x}} - n_x\hat{\mathbf{y}})\right\}e^{i\frac{\omega}{c}(n_x x + n_y y) - i\omega t}$$

E_z must be determined from the continuity conditions:

$$E_0 + E_0'' = E_z \qquad (E_0 - E_0'')\cos i = E_z n_y \qquad (E_0 + E_0'')\sin i = E_z n_x$$

$$\frac{E_0''}{E_0} = \frac{\cos i - n_y}{\cos i + n_y}.$$

Solution for p-polarization

$$E_z = 0 \qquad \Rightarrow n_y^2 = \frac{\kappa_{xx}}{\kappa_{yy}}(\kappa_{yy} - n_x^2).$$

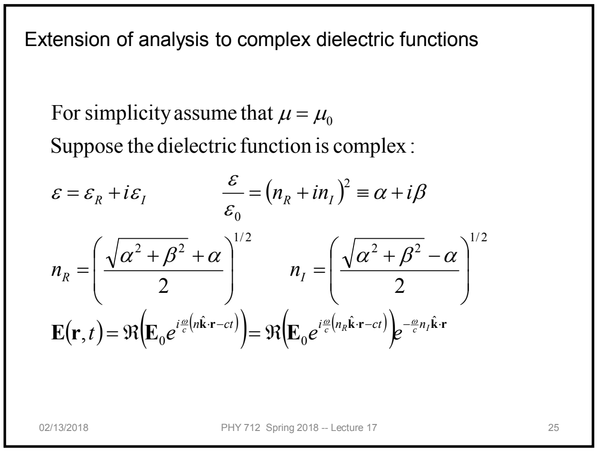
$$\mathbf{E} = E_x\left(\hat{\mathbf{x}} - \frac{\kappa_{xx}n_x}{\kappa_{yy}n_y}\hat{\mathbf{y}}\right)e^{i\frac{\omega}{c}(n_x x + n_y y) - i\omega t}.$$

$$\mathbf{H} = -\frac{E_x}{\mu_0 c}\frac{\kappa_{xx}}{n_y}\hat{\mathbf{z}}e^{i\frac{\omega}{c}(n_x x + n_y y) - i\omega t}.$$

E_x must be determined from the continuity conditions:

$$(E_0 - E_0'')\cos i = E_x \qquad (E_0 + E_0'') = \frac{\kappa_{xx}}{n_y}E_x \qquad (E_0 + E_0'')\sin i = \frac{\kappa_{xx}n_x}{n_y}E_x.$$

$$\frac{E_0''}{E_0} = \frac{\kappa_{xx}\cos i - n_y}{\kappa_{xx}\cos i + n_y}.$$



Extension of analysis to complex dielectric functions

For simplicity assume that $\mu = \mu_0$

Suppose the dielectric function is complex :

$$\varepsilon = \varepsilon_R + i\varepsilon_I \qquad \frac{\varepsilon}{\varepsilon_0} = \left(n_R + in_I\right)^2 \equiv \alpha + i\beta$$

$$n_R = \left(\frac{\sqrt{\alpha^2+\beta^2}+\alpha}{2}\right)^{1/2} \qquad n_I = \left(\frac{\sqrt{\alpha^2+\beta^2}-\alpha}{2}\right)^{1/2}$$

$$\mathbf{E}(\mathbf{r},t) = \Re\left(\mathbf{E}_0 e^{i\frac{\omega}{c}\left(n\hat{\mathbf{k}}\cdot\mathbf{r}-ct\right)}\right) = \Re\left(\mathbf{E}_0 e^{i\frac{\omega}{c}\left(n_R\hat{\mathbf{k}}\cdot\mathbf{r}-ct\right)}\right) e^{-\frac{\omega}{c}n_I\hat{\mathbf{k}}\cdot\mathbf{r}}$$