

PHY 712 Electrodynamics
9-9:50 AM Olin 105

Plan for Lecture 18:

Read Chapter 7

1. Real and imaginary contributions to electromagnetic response
2. Frequency dependence of dielectric materials; Drude model
3. Kramers-Kronig relationships

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6	Mon: 01/29/2018	Chap. 2	Method of image charges	#5	01/31/2018
7	Wed: 01/31/2018	Chap. 2 & 3	Cylindrical and spherical geometries	#6	02/02/2018
8	Fri: 02/02/2018	Chap. 3 & 4	Multipole analysis	#7	02/07/2018
9	Mon: 02/05/2018	Chap. 4	Dipoles and Dielectrics	#8	02/09/2018
10	Wed: 02/07/2018	Chap. 4	Dipoles and Dielectrics		
11	Fri: 02/09/2018	Chap. 1-4	Review		
12	Mon: 02/12/2018	Chap. 5	Magnetostatics	#9	02/16/2018
13	Wed: 02/14/2018	Chap. 5	Magnetic dipoles and hyperfine interaction	#10	02/19/2018
14	Fri: 02/16/2018	Chap. 5	Magnetic dipoles and dipolar fields	#11	02/21/2018
15	Mon: 02/19/2018	Chap. 6	Maxwell's Equations	#12	02/23/2018
16	Wed: 02/21/2018	Chap. 6	Electromagnetic energy and forces		
17	Fri: 02/23/2018	Chap. 7	Electromagnetic plane waves		
18	Mon: 02/26/2018	Chap. 7	Dielectric response of media		Exam
19	Wed: 02/28/2018	Chap. 7	Complex dielectrics		Exam
20	Fri: 03/02/2018	Chap. 1-7	Review		Exam due
	Mon: 03/05/2018	No class	Spring Break		
	Wed: 03/07/2018	No class	Spring Break		
	Fri: 03/09/2018	No class	Spring Break		
21	Mon: 03/12/2018				
22	Wed: 03/14/2018				

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Maxwell's equations

For linear isotropic media and no sources: $\mathbf{D} = \epsilon\mathbf{E}$; $\mathbf{B} = \mu\mathbf{H}$

Coulomb's law: $\nabla \cdot \mathbf{E} = 0$

Ampere-Maxwell's law: $\nabla \times \mathbf{B} - \mu\epsilon \frac{\partial \mathbf{E}}{\partial t} = 0$

Faraday's law: $\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$

No magnetic monopoles: $\nabla \cdot \mathbf{B} = 0$

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Plane wave solutions to sourceless Maxwell's equations;
extension of analysis to complex dielectric functions

For simplicity assume that $\mu = \mu_0$

Suppose the dielectric function is complex :

$$\epsilon = \epsilon_R + i\epsilon_I \quad \frac{\epsilon}{\epsilon_0} = (n_R + in_I)^2 \equiv \alpha + i\beta$$

$$n_R = \left(\frac{\sqrt{\alpha^2 + \beta^2} + \alpha}{2} \right)^{1/2} \quad n_I = \left(\frac{\sqrt{\alpha^2 + \beta^2} - \alpha}{2} \right)^{1/2}$$

$$\mathbf{E}(\mathbf{r}, t) = \Re(\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r} - ct}) = \Re(\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r} - ct}) e^{-\frac{\omega}{c} n_I \mathbf{r}}$$

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Paul Karl Ludwig Drude 1863-1906



Scanned at the American Institute of Physics

http://photos.aip.org/history/Thumbnails/drude_paul_a1.jpg

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Drude model:

Vibrations of charged particles near equilibrium:

$$m\delta \ddot{\mathbf{r}} = q\mathbf{E}_0 e^{-i\omega t} - m\omega_0^2 \delta \mathbf{r} - m\gamma \dot{\delta \mathbf{r}}$$

For $\delta \mathbf{r} \equiv \delta \mathbf{r}_0 e^{-i\omega t}$, $\delta \mathbf{r}_0 = \frac{q\mathbf{E}_0}{m} \frac{1}{\omega_0^2 - \omega^2 - i\omega\gamma}$

Induced dipole:

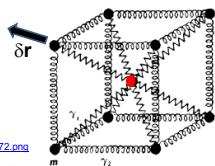
$$\mathbf{p} = q \delta \mathbf{r} = \frac{q^2 \mathbf{E}_0}{m} \frac{1}{\omega_0^2 - \omega^2 - i\omega\gamma} e^{-i\omega t}$$

Displacement field:

$$\mathbf{D} = \epsilon \mathbf{E} = \epsilon_0 \mathbf{E} + \mathbf{P}$$

$$\mathbf{P} = \sum_i \mathbf{p}_i \delta^3(\mathbf{r} - \mathbf{r}_i)$$

http://img.ftd.com/gspse/06/psed_0001_0012_0_img2972.png

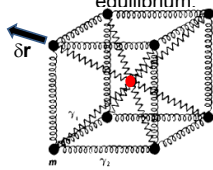


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Drude model:
Vibration of particle of charge q and mass m near equilibrium:



http://img.tfd.com/qgse/95/qsed_0001_0012_0_img2972.png

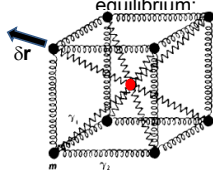
$$m\delta \ddot{\mathbf{r}} = q\mathbf{E}_0 e^{-i\omega t} - m\omega_0^2 \delta \mathbf{r} - m\gamma \delta \dot{\mathbf{r}}$$

Note that:

- $\gamma > 0$ represents dissipation of energy.
- ω_0 represents the natural frequency of the vibration; $\omega_0=0$ would represent a free (unbound) particle

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Drude model:
Vibration of particle of charge q and mass m near equilibrium:



http://img.tfd.com/qgse/95/qsed_0001_0012_0_img2972.png

$$m\delta \ddot{\mathbf{r}} = q\mathbf{E}_0 e^{-i\omega t} - m\omega_0^2 \delta \mathbf{r} - m\gamma \delta \dot{\mathbf{r}}$$

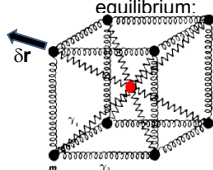
For $\delta \mathbf{r} \equiv \delta \mathbf{r}_0 e^{-i\omega t}$, $\delta \mathbf{r}_0 = \frac{q\mathbf{E}_0}{m} \frac{1}{\omega_0^2 - \omega^2 - i\omega\gamma}$

Induced dipole:

$$\mathbf{p} = q \delta \mathbf{r} = \frac{q^2 \mathbf{E}_0}{m} \frac{1}{\omega_0^2 - \omega^2 - i\omega\gamma} e^{-i\omega t}$$

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Drude model:
Vibration of particle of charge q and mass m near equilibrium:



http://img.tfd.com/qgse/95/qsed_0001_0012_0_img2972.png

$$m\delta \ddot{\mathbf{r}} = q\mathbf{E}_0 e^{-i\omega t} - m\omega_0^2 \delta \mathbf{r} - m\gamma \delta \dot{\mathbf{r}}$$

Displacement field:

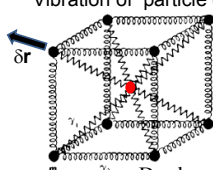
$$\mathbf{D} = \epsilon \mathbf{E} = \epsilon_0 \mathbf{E} + \mathbf{P}$$

$$\mathbf{P} = \sum_i \mathbf{p}_i \delta^3(\mathbf{r} - \mathbf{r}_i) \approx N \sum_i f_i \mathbf{p}_i$$

$N \equiv$ number dipole/volume
 $f_i \equiv$ fraction of type i dipoles

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Drude model:
 Vibration of particle of charge q and mass m near equilibrium:



http://img.tfd.com/qqse/95/qqse/0001_0012_0_img2972.png

Drude model expression for permittivity:

$$\mathbf{D} = \epsilon \mathbf{E} = \epsilon_0 \mathbf{E} + \mathbf{P} = \epsilon_0 \mathbf{E} + N \sum_i f_i \mathbf{p}_i$$

$$\mathbf{p}_i = q_i \delta \mathbf{r} = \frac{q_i^2 \mathbf{E}_0}{m_i} \frac{1}{\omega_i^2 - \omega^2 - i\omega\gamma_i} e^{-i\omega t}$$

$$\epsilon \mathbf{E} = \epsilon_0 \mathbf{E}_0 e^{-i\omega t} \left(1 + N \sum_i f_i \frac{q_i^2}{\epsilon_0 m_i} \frac{1}{\omega_i^2 - \omega^2 - i\omega\gamma_i} \right)$$

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Drude model dielectric function:

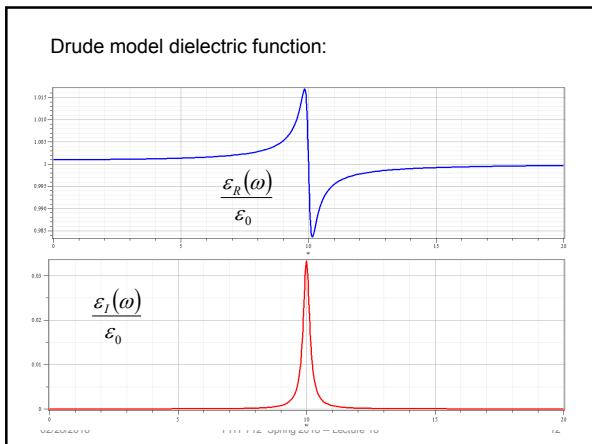
$$\frac{\epsilon(\omega)}{\epsilon_0} = 1 + N \sum_i f_i \frac{q_i^2}{\epsilon_0 m_i} \frac{1}{\omega_i^2 - \omega^2 - i\omega\gamma_i}$$

$$= \frac{\epsilon_R(\omega)}{\epsilon_0} + i \frac{\epsilon_I(\omega)}{\epsilon_0}$$

$$\frac{\epsilon_R(\omega)}{\epsilon_0} = 1 + N \sum_i f_i \frac{q_i^2}{\epsilon_0 m_i} \frac{\omega_i^2 - \omega^2}{(\omega_i^2 - \omega^2)^2 + \omega^2 \gamma_i^2}$$

$$\frac{\epsilon_I(\omega)}{\epsilon_0} = N \sum_i f_i \frac{q_i^2}{\epsilon_0 m_i} \frac{\omega \gamma_i}{(\omega_i^2 - \omega^2)^2 + \omega^2 \gamma_i^2}$$

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Drude model dielectric function – some analytic properties:

$$\frac{\epsilon(\omega)}{\epsilon_0} = 1 + N \sum_i f_i \frac{q_i^2}{\epsilon_0 m_i} \frac{1}{\omega_i^2 - \omega^2 - i\omega\gamma_i}$$

For $\omega \gg \omega_i$

$$\frac{\epsilon(\omega)}{\epsilon_0} \approx 1 - \frac{1}{\omega^2} \left(N \sum_i f_i \frac{q_i^2}{\epsilon_0 m_i} \right)$$

$$\equiv 1 - \frac{\omega_p^2}{\omega^2}$$

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Drude model dielectric function – some analytic properties:

$$\frac{\epsilon(\omega)}{\epsilon_0} = 1 + N \sum_i f_i \frac{q_i^2}{\epsilon_0 m_i} \frac{1}{\omega_i^2 - \omega^2 - i\omega\gamma_i}$$

For $\omega_0 = 0$ (representing a free particle of charge q_0 , mass m_0)

$$\frac{\epsilon(\omega)}{\epsilon_0} = 1 + N \sum_{i>0} f_i \frac{q_i^2}{\epsilon_0 m_i} \frac{1}{\omega_i^2 - \omega^2 - i\omega\gamma_i} + iNf_0 \frac{q_0^2}{\epsilon_0 m_0} \frac{1}{\omega(\gamma_0 - i\omega)}$$

$$\equiv \frac{\epsilon_b(\omega)}{\epsilon_0} + i \frac{\sigma(\omega)}{\epsilon_0 \omega}$$

Some details:

$\mathbf{D} = \epsilon_b \mathbf{E}$ $\mathbf{J} = \sigma \mathbf{E}$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} = (\sigma - i\omega\epsilon_b) \mathbf{E} = \epsilon \frac{\partial \mathbf{E}}{\partial t} = -i\omega \left(\epsilon_b + \frac{i\sigma}{\omega} \right) \mathbf{E}$$

$$\Rightarrow \sigma(\omega) = Nf_0 \frac{q_0^2}{m_0} \frac{1}{(\gamma_0 - i\omega)}$$

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Analytic properties of the dielectric function (in the Drude model or from “first principles” -- Kramers-Kronig transform

Consider Cauchy's integral formula for an analytic function $f(z)$:

$$\oint dz f(z) = 0 \quad f(\alpha) = \frac{1}{2\pi i} \oint_{\text{includes } \alpha} dz \frac{f(z)}{z-\alpha}$$

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Kramers-Kronig transform -- continued

$$f(\alpha) = \frac{1}{2\pi i} \oint_{\text{includes } \alpha} dz \frac{f(z)}{z-\alpha} = \frac{1}{2\pi i} \left(\int_{-\infty}^{\infty} dz_R \frac{f(z_R)}{z_R-\alpha} + \int_{\text{cut}} dz \frac{f(z)}{z-\alpha} \right)$$

=0

$$f(\alpha) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} dz_R \frac{f(z_R)}{z_R-\alpha} = \frac{1}{2\pi i} P \int_{-\infty}^{\infty} dz_R \frac{f(z_R)}{z_R-\alpha} + \frac{1}{2} f(\alpha)$$

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Kramers-Kronig transform -- continued

$$f(\alpha) = \frac{1}{2\pi i} P \int_{-\infty}^{\infty} dz_R \frac{f(z_R)}{z_R-\alpha} + \frac{1}{2} f(\alpha)$$

Suppose $f(z_R) = f_R(z_R) + if_I(z_R)$:

$$\Rightarrow \frac{1}{2} (f_R(\alpha) + if_I(\alpha)) = \frac{1}{2\pi i} P \int_{-\infty}^{\infty} dz_R \frac{f_R(z_R) + if_I(z_R)}{z_R-\alpha}$$

$$\Rightarrow f_R(\alpha) = \frac{1}{\pi} P \int_{-\infty}^{\infty} dz_R \frac{f_I(z_R)}{z_R-\alpha}$$

$$f_I(\alpha) = -\frac{1}{\pi} P \int_{-\infty}^{\infty} dz_R \frac{f_R(z_R)}{z_R-\alpha}$$

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Kramers-Kronig transform -- continued

$$f_R(\alpha) = \frac{1}{\pi} P \int_{-\infty}^{\infty} dz_R \frac{f_I(z_R)}{z_R-\alpha}$$

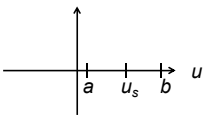
$$f_I(\alpha) = -\frac{1}{\pi} P \int_{-\infty}^{\infty} dz_R \frac{f_R(z_R)}{z_R-\alpha}$$

This Kramers - Kronig transform is useful for the dielectric function
 when $f(z_R) \Rightarrow \frac{\epsilon(\omega)}{\epsilon_0} - 1$

Must show that: 1. $f(z)$ is analytic for $z_i \geq 0$
 2. $f(z)$ vanishes for $z \rightarrow \infty$

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Some practical considerations



Principal parts integration :

$$P \int_a^b du g(u) = \lim_{\nu \rightarrow 0} \left(\int_a^{u_s-\nu} du g(u) + \int_{u_s+\nu}^b du g(u) \right)$$

Example :

$$P \int_a^b du \frac{1}{u-u_s} = \lim_{\nu \rightarrow 0} \left(\int_a^{u_s-\nu} du \frac{1}{u-u_s} + \int_{u_s+\nu}^b du \frac{1}{u-u_s} \right)$$

$$= \lim_{\nu \rightarrow 0} \left(\ln \left(\frac{\nu}{u_s-a} \right) + \ln \left(\frac{b-u_s}{\nu} \right) \right) = \ln \left(\frac{b-u_s}{u_s-a} \right)$$

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More practical considerations

For dielectric function $\epsilon(\omega)$:

$$\epsilon(-\omega) = \epsilon^*(\omega)$$

$$\Rightarrow \epsilon_R(-\omega) = \epsilon_R(\omega)$$

$$\Rightarrow \epsilon_I(-\omega) = -\epsilon_I(\omega)$$

Analytic properties the dielectric function which justify the treatment of $f(z) \Rightarrow \frac{\epsilon(z)}{\epsilon_0} - 1$

Must show that :

- $f(z)$ is analytic for $z \geq 0$
- $f(z)$ vanishes for $z \rightarrow \infty$ (for $z \geq 0$)

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Analysis for Drude model dielectric function:

$$\frac{\epsilon(\omega)}{\epsilon_0} = 1 + N \sum_i f_i \frac{q_i^2}{\epsilon_0 m_i} \frac{1}{\omega_i^2 - \omega^2 - i\omega\gamma_i}$$

Let $f(z) = \frac{\epsilon(z)}{\epsilon_0} - 1 = N \sum_i f_i \frac{q_i^2}{\epsilon_0 m_i} \frac{1}{\omega_i^2 - z^2 - iz\gamma_i}$

For $|z| \gg \omega_i$

$$f(z) \approx -\frac{1}{z^2} \left(N \sum_i f_i \frac{q_i^2}{\epsilon_0 m_i} \right) \Rightarrow \text{vanishes at large } z$$

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Analysis for Drude model dielectric function – continued --
 Analytic properties:

$$f(z) = \frac{\epsilon(z)}{\epsilon_0} - 1 = N \sum_i f_i \frac{q_i^2}{\epsilon_0 m_i \omega_i^2 - z^2 - iz\gamma_i}$$

$f(z)$ has poles z_p at $\omega_i^2 - z_p^2 - iz_p\gamma_i = 0$

$$z_p = -i \frac{\gamma_i}{2} \pm \sqrt{\omega_i^2 - \left(\frac{\gamma_i}{2}\right)^2}$$

Note that $\Im(z_p) \leq 0 \Rightarrow f(z)$ is analytic for $\Im(z_p) > 0$

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Kramers-Kronig transform – for dielectric function:

$$\frac{\epsilon_r(\omega)}{\epsilon_0} - 1 = \frac{1}{\pi} P \int_{-\infty}^{\infty} d\omega' \frac{\epsilon_i(\omega')}{\epsilon_0} \frac{1}{\omega' - \omega}$$

$$\frac{\epsilon_i(\omega)}{\epsilon_0} = -\frac{1}{\pi} P \int_{-\infty}^{\infty} d\omega' \left(\frac{\epsilon_r(\omega')}{\epsilon_0} - 1 \right) \frac{1}{\omega' - \omega}$$

with $\epsilon_r(-\omega) = \epsilon_r(\omega)$; $\epsilon_i(-\omega) = -\epsilon_i(\omega)$

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Further comments on analytic behavior of dielectric function

"Causal" relationship between **E** and **D** fields:

$$\mathbf{D}(\mathbf{r}, t) = \epsilon_0 \left\{ \mathbf{E}(\mathbf{r}, t) + \int_0^{\infty} d\tau G(\tau) \mathbf{E}(\mathbf{r}, t - \tau) \right\}$$

$$\frac{\epsilon(\omega)}{\epsilon_0} - 1 = \int_0^{\infty} d\tau G(\tau) e^{i\omega\tau}$$

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