

PHY 712 Electrodynamics
9-9:50 AM Olin 105

Plan for Lecture 20:

Review Chapter 1-7 in Jackson

- 1. Brief review**
- 2. Comments on some homework problems**

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| | | | | | |
|----|-----------------|-------------|--|-----|------------|
| 8 | Fri: 02/02/2018 | Chap. 3 & 4 | Multipole analysis | #7 | 02/07/2018 |
| 9 | Mon: 02/05/2018 | Chap. 4 | Dipoles and Dielectrics | #8 | 02/09/2018 |
| 10 | Wed: 02/07/2018 | Chap. 4 | Dipoles and Dielectrics | | |
| 11 | Fri: 02/09/2018 | Chap. 1-4 | Review | | |
| 12 | Mon: 02/12/2018 | Chap. 5 | Magnetostatics | #9 | 02/16/2018 |
| 13 | Wed: 02/14/2018 | Chap. 5 | Magnetic dipoles and hyperfine interaction | #10 | 02/19/2018 |
| 14 | Fri: 02/16/2018 | Chap. 5 | Magnetic dipoles and dipolar fields | #11 | 02/21/2018 |
| 15 | Mon: 02/19/2018 | Chap. 6 | Maxwell's Equations | #12 | 02/23/2018 |
| 16 | Wed: 02/21/2018 | Chap. 6 | Electromagnetic energy and forces | | |
| 17 | Fri: 02/23/2018 | Chap. 7 | Electromagnetic plane waves | | |
| 18 | Mon: 02/26/2018 | Chap. 7 | Dielectric response of media | | Exam |
| 19 | Wed: 02/28/2018 | Chap. 7 | Complex dielectrics | | Exam |
| 20 | Fri: 03/02/2018 | Chap. 1-7 | Review | | Exam due |
| | Mon: 03/05/2018 | No class | Spring Break | | |
| | Wed: 03/07/2018 | No class | Spring Break | | |
| | Fri: 03/09/2018 | No class | Spring Break | | |
| 21 | Mon: 03/12/2018 | Chap. 8 | Wave guides | | |
| 22 | Wed: 03/14/2018 | | | | |
| 23 | Fri: 03/16/2018 | | | | |
| 24 | Mon: 03/19/2018 | | | | |
| 25 | Wed: 03/21/2018 | | | | |

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Review

Maxwell's equations

Coulomb's law : $\nabla \cdot \mathbf{D} = \rho_{free}$

Ampere - Maxwell's law : $\nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{J}_{free}$

Faraday's law : $\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$

No magnetic monopoles : $\nabla \cdot \mathbf{B} = 0$

For linear isotropic media and no sources: $\mathbf{D} = \epsilon \mathbf{E}; \mathbf{B} = \mu \mathbf{H}$

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Review -- continued

Maxwell's equations

Microscopic or vacuum form ($\mathbf{P} = 0$; $\mathbf{M} = 0$):

Coulomb's law : $\nabla \cdot \mathbf{E} = \rho / \epsilon_0$

Ampere - Maxwell's law : $\nabla \times \mathbf{B} - \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{J}$

Faraday's law : $\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$

No magnetic monopoles : $\nabla \cdot \mathbf{B} = 0$

$$\Rightarrow c^2 = \frac{1}{\epsilon_0 \mu_0}$$

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Review -- continued

$\nabla \cdot \mathbf{B} = 0 \Rightarrow \mathbf{B} = \nabla \times \mathbf{A}$

$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \Rightarrow \nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} \right) = 0$

$\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\nabla \Phi$

or $\mathbf{E} = -\nabla \Phi - \frac{\partial \mathbf{A}}{\partial t}$

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Review -- continued

Analysis of the scalar and vector potential equations :

$-\nabla^2 \Phi - \frac{\partial(\nabla \cdot \mathbf{A})}{\partial t} = \rho / \epsilon_0$

$\nabla \times (\nabla \times \mathbf{A}) + \frac{1}{c^2} \left(\frac{\partial(\nabla \Phi)}{\partial t} + \frac{\partial^2 \mathbf{A}}{\partial t^2} \right) = \mu_0 \mathbf{J}$

Lorentz gauge form -- require $\nabla \cdot \mathbf{A}_L + \frac{1}{c^2} \frac{\partial \Phi_L}{\partial t} = 0$

$-\nabla^2 \Phi_L + \frac{1}{c^2} \frac{\partial^2 \Phi_L}{\partial t^2} = \rho / \epsilon_0$

$-\nabla^2 \mathbf{A}_L + \frac{1}{c^2} \frac{\partial^2 \mathbf{A}_L}{\partial t^2} = \mu_0 \mathbf{J}$

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Review -- continued

When to solve equations using integral form versus differential form?

Examples from electrostatic and magnetostatic cases:

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int d^3r' \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$$

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int d^3r' \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$$

Useful for spatially confined sources.

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Review -- continued

Useful identity:

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \sum_{lm} \frac{4\pi}{2l+1} \frac{r_{<}^l}{r_{>}^{l+1}} Y_{lm}(\theta, \phi) Y_{lm}^*(\theta', \phi')$$

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Review -- continued

Some spherical harmonic functions:

$$Y_{00}(\hat{\mathbf{r}}) = \frac{1}{\sqrt{4\pi}}$$

$$Y_{1(\pm 1)}(\hat{\mathbf{r}}) = \mp \sqrt{\frac{3}{8\pi}} \sin \theta e^{\pm i\phi}$$

$$Y_{10}(\hat{\mathbf{r}}) = \sqrt{\frac{3}{4\pi}} \cos \theta$$

$$Y_{2(\pm 2)}(\hat{\mathbf{r}}) = \sqrt{\frac{15}{32\pi}} \sin^2 \theta e^{\pm 2i\phi}$$

$$Y_{2(\pm 1)}(\hat{\mathbf{r}}) = \mp \sqrt{\frac{15}{8\pi}} \sin \theta \cos \theta e^{\pm i\phi}$$

$$Y_{20}(\hat{\mathbf{r}}) = \sqrt{\frac{5}{4\pi}} \left(\frac{3}{2} \cos^2 \theta - \frac{1}{2} \right)$$

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Review -- continued

General form of electrostatic potential with boundary value $r \rightarrow \infty$, for isolated charge density $\rho(\mathbf{r})$:

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int d^3r' \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$$

$$= \frac{1}{4\pi\epsilon_0} \int d^3r' \rho(\mathbf{r}') \left(\sum_{lm} \frac{4\pi}{2l+1} \frac{r_{<}^l}{r_{>}^{l+1}} Y_{lm}(\theta, \varphi) Y_{lm}^*(\theta', \varphi') \right)$$

Suppose that $\rho(\mathbf{r}) = \sum_{lm} \rho_{lm}(r) Y_{lm}(\theta, \varphi)$

$$\Rightarrow \Phi(\mathbf{r}) = \frac{1}{\epsilon_0} \sum_{lm} \frac{1}{2l+1} Y_{lm}(\theta, \varphi) \left(\frac{1}{r^{l+1}} \int_0^r r'^{2+l} dr' \rho_{lm}(r') + r^l \int_r^\infty r'^{1-l} dr' \rho_{lm}(r') \right)$$

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Review -- continued

Hyperfine interaction energy:

$$E_{int} \equiv H_{HF} = -\mu_e \cdot \mathbf{B}_{\mu_N} - \mu_N \cdot \mathbf{B}_{\mathbf{I}_e}(0)$$

Putting all of the terms together:

$$H_{HF} = -\frac{\mu_0}{4\pi} \left(\left\langle \frac{3(\mu_N \cdot \hat{\mathbf{r}})(\mu_e \cdot \hat{\mathbf{r}}) - \mu_N \cdot \mu_e}{r^3} + \frac{8\pi}{3} \mu_N \cdot \mu_e \delta^3(\mathbf{r}) \right\rangle + \frac{e}{m_e} \left\langle \frac{\mathbf{L} \cdot \mu_N}{r^3} \right\rangle \right)$$

In this expression the brackets $\langle \rangle$ indicate evaluating the expectation value relative to the electronic state.

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Example of use of spherical harmonic expansion:
 Problem #4.9 in Jackson (not assigned this year)
 A point charge q is located in free space a distance d from the center of a dielectric sphere of radius a ($a < d$) and dielectric constant ϵ/ϵ_0 . Find the electrostatic potential. For $r \leq a$:

$$\Phi(\mathbf{r}) = \sum_l A_l r^l P_l(\cos \theta)$$

For $r \geq a$:

$$\Phi(\mathbf{r}) = \sum_l B_l \frac{1}{r^{l+1}} P_l(\cos \theta) + \frac{q}{4\pi\epsilon_0} \frac{1}{|\mathbf{r} - d\hat{\mathbf{z}}|}$$

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Review of HW -- continued

For $r \leq a$:

$$\Phi(\mathbf{r}) = \sum_l A_l r^l P_l(\cos \theta)$$

For $r \geq a$:

$$\Phi(\mathbf{r}) = \sum_l B_l \frac{1}{r^{l+1}} P_l(\cos \theta) + \frac{q}{4\pi\epsilon_0} \frac{1}{|\mathbf{r} - d\hat{\mathbf{z}}|}$$

In order to match BC's at $r = a$:

$$\frac{1}{|\mathbf{r} - d\hat{\mathbf{z}}|} = \sum_{l=0}^{r'_{<}} \frac{r'^l}{r'^{l+1}} P_l(\cos \theta) = \sum_{l=0}^{a'} \frac{a^l}{d^{l+1}} P_l(\cos \theta)$$

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For $r \leq a$:

$$\Phi(\mathbf{r}) = \sum_l A_l r^l P_l(\cos \theta)$$

For $r \geq a$:

$$\Phi(\mathbf{r}) = \sum_l B_l \frac{1}{r^{l+1}} P_l(\cos \theta) + \frac{q}{4\pi\epsilon_0} \frac{1}{|\mathbf{r} - d\hat{\mathbf{z}}|}$$

Boundary conditions:

$\mathbf{D} \cdot \hat{\mathbf{r}}|_{r=a}$ = continuous $\mathbf{E} \cdot \hat{\boldsymbol{\theta}}|_{r=a}$ = continuous

$$\epsilon \frac{\partial \Phi_{in}(r)}{\partial r} \Big|_{r=a} = \epsilon_0 \frac{\partial \Phi_{out}(r)}{\partial r} \Big|_{r=a} \quad \frac{\partial \Phi_{in}(r)}{\partial \theta} \Big|_{r=a} = \frac{\partial \Phi_{out}(r)}{\partial \theta} \Big|_{r=a}$$

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Boundary conditions:

$\mathbf{D} \cdot \hat{\mathbf{r}}|_{r=a}$ = continuous $\mathbf{E} \cdot \hat{\boldsymbol{\theta}}|_{r=a}$ = continuous

$$\epsilon \frac{\partial \Phi_{in}(r)}{\partial r} \Big|_{r=a} = \epsilon_0 \frac{\partial \Phi_{out}(r)}{\partial r} \Big|_{r=a} \quad \frac{\partial \Phi_{in}(r)}{\partial \theta} \Big|_{r=a} = \frac{\partial \Phi_{out}(r)}{\partial \theta} \Big|_{r=a}$$


Equality for each l :

$$\epsilon l a^{l-1} A_l = -\frac{\epsilon_0(l+1)}{a^{l+2}} B_l + \frac{q}{4\pi} \frac{a^{l-1}}{d^{l+1}}$$

$$a^l A_l = +\frac{1}{a^{l+1}} B_l + \frac{q}{4\pi} \frac{a^l}{d^{l+1}}$$

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Comment on HW #9




1. Consider an infinitely long wire with radius a , oriented along the z axis. There is a steady uniform current inside the wire. Specifically the current is along the z -axis with the magnitude of J_0 for $\rho \leq a$ and zero for $\rho > a$, where ρ denotes the radial parameter of the natural cylindrical coordinates of the system.


- Find the vector potential (\mathbf{A}) for all ρ .
- Find the magnetic flux field (\mathbf{B}) for all ρ .

Solution to problem using PHY 114 ideas
In this case, it is convenient to solve part b first.

Top view
for $\rho < a$




Top view
for $\rho > a$




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Comment on HW #11 -- continued

Top view
for $\rho < a$



Top view
for $\rho > a$



$$\oint \mathbf{B} \cdot d\ell = \mu_0 \int \mathbf{J} \cdot d\mathbf{A}$$

$$2\pi\rho B = \mu_0 J_0 \pi \rho^2$$

$$B = \frac{\mu_0 J_0 \rho}{2}$$

$$\mathbf{B} = \frac{\mu_0 J_0 \rho}{2} \hat{\phi} = \nabla \times \mathbf{A}$$

$$\mathbf{A} = -\frac{\mu_0 J_0 (\rho^2 - a^2)}{4} \hat{z}$$

$$\oint \mathbf{B} \cdot d\ell = \mu_0 \int \mathbf{J} \cdot d\mathbf{A}$$

$$2\pi\rho B = \mu_0 J_0 \pi a^2$$


$$B = \frac{\mu_0 J_0 a^2}{2\rho}$$

$$\mathbf{B} = \frac{\mu_0 J_0 a^2}{2\rho} \hat{\phi} = \nabla \times \mathbf{A}$$

$$\mathbf{A} = -\frac{\mu_0 J_0 a^2 \ln(\rho/a)}{2} \hat{z}$$

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Comment on HW #9 -- continued
Alternative treatment using differential equations:



$$-\nabla^2 \mathbf{A} = \begin{cases} \mu_0 J_0 \hat{z} & \text{for } \rho \leq a \\ 0 & \text{for } \rho > a \end{cases}$$

$$-\frac{1}{\rho} \frac{\partial}{\partial \rho} \rho \frac{\partial A_z(\rho)}{\partial \rho} = \begin{cases} \mu_0 J_0 & \text{for } \rho \leq a \\ 0 & \text{for } \rho > a \end{cases}$$

$$A_z(\rho) = \begin{cases} -\frac{\mu_0 J_0 \rho^2}{4} + C_1 & \text{for } \rho \leq a \\ C_2 + C_3 \ln(\rho) & \text{for } \rho > a \end{cases}$$

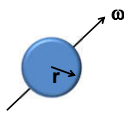
Choosing constants from continuity requirements:

$$A_z(\rho) = \begin{cases} -\frac{\mu_0 J_0 \rho^2}{4} + \frac{\mu_0 J_0 a^2}{4} & \text{for } \rho \leq a \\ -\frac{\mu_0 J_0 a^2}{2} \ln(\rho/a) & \text{for } \rho > a \end{cases}$$

$$\mathbf{B} = -\frac{\partial A_z(\rho)}{\partial \rho} \hat{\phi}$$

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Comment on HW #10



A sphere of radius a carries a uniform surface charge distribution σ . The sphere is rotated about a diameter with constant angular velocity $\boldsymbol{\omega}$. Find the vector potential \mathbf{A} and magnetic field \mathbf{B} both inside and outside the sphere.

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int d^3r' \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$$

$$\mathbf{J}(\mathbf{r}') = \begin{cases} \sigma \delta(r'-a) \boldsymbol{\omega} \times \mathbf{r}' & \text{for } r' \leq a \\ 0 & \text{otherwise} \end{cases}$$

Note that: $\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \sum_m \frac{4\pi}{2l+1} \frac{r'^l}{r^{l+1}} Y_{lm}(\hat{\mathbf{r}}) Y_{lm}^*(\hat{\mathbf{r}}')$

and: $\int d\Omega' \sum_m Y_{lm}(\hat{\mathbf{r}}) Y_{lm}^*(\hat{\mathbf{r}}') \mathbf{r}' = \frac{r'}{r} \delta_{l1}$.

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Comment on HW #10 -- continued

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int d^3r' \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} = \frac{\mu_0 \sigma \boldsymbol{\omega} \times \mathbf{r}}{4\pi r} \frac{4\pi}{3} \int_0^a r'^3 dr' \delta(r'-a) \frac{r'_z}{r'^2}$$

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 \sigma}{3} \boldsymbol{\omega} \times \mathbf{r} \begin{cases} a & \text{for } r \leq a \\ \frac{a^4}{r^3} & \text{for } r > a \end{cases}$$

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 \sigma}{3} \begin{cases} 2\boldsymbol{\omega} a & \text{for } r \leq a \\ \frac{a^4}{r^3} (3(\hat{\mathbf{r}} \cdot \boldsymbol{\omega}) \hat{\mathbf{r}} - \boldsymbol{\omega}) & \text{for } r > a \end{cases}$$

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