Electrodynamics – PHY712

Lecture 9 – Dipole fields

Reference: Chap. 4 in J. D. Jackson's textbook.

The dipole moment is defined by

$$\mathbf{p} = \int d^3 r \rho(r) \mathbf{r},\tag{1}$$

with the corresponding potential

$$\Phi(r) = \frac{1}{4\pi\varepsilon_0} \frac{\mathbf{p} \cdot \hat{\mathbf{r}}}{r^2},\tag{2}$$

1

and electrostatic field

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \left\{ \frac{3\hat{\mathbf{r}}(\mathbf{p}\cdot\hat{\mathbf{r}}) - \mathbf{p}}{r^3} - \frac{4\pi}{3}\mathbf{p}\ \delta^3(\mathbf{r}) \right\}.$$
 (3)



"Justification" of surprizing δ -function term in dipole electric field

We note that Eq. (3) is poorly defined as $r \to 0$, and consider the value of a small integral of $\mathbf{E}(\mathbf{r})$ about zero. (For this purpose, we are supposing that the dipole \mathbf{p} is located at $\mathbf{r} = \mathbf{0}$.) In this case we will approximate

$$\mathbf{E}(\mathbf{r} \approx \mathbf{0}) \approx \left(\int_{\text{sphere}} \mathbf{E}(\mathbf{r}) d^3 r \right) \delta^3(\mathbf{r}). \tag{4}$$

First we note that

$$\int_{r \le R} \mathbf{E}(\mathbf{r}) d^3 r = -R^2 \int_{r=R} \Phi(\mathbf{r}) \hat{\mathbf{r}} d\mathbf{\Omega}.$$
 (5)



δ -function contribution dipole electric field – continued

$$\int_{r \le R} \mathbf{E}(\mathbf{r}) d^3 r = -R^2 \int_{r=R} \Phi(\mathbf{r}) \hat{\mathbf{r}} d\mathbf{\Omega}.$$
 (6)

This result follows from the Divergence theorm:

$$\int_{\text{vol}} \nabla \cdot \mathcal{V} d^3 r = \int_{\text{surface}} \mathcal{V} \cdot d\mathbf{A}.$$
 (7)

In our case, this theorem can be used to prove Eq. (6) for each cartesian coordinate if we choose $\mathcal{V} \equiv \hat{\mathbf{x}} \Phi(\mathbf{r})$ for the *x*- component *etc*.:

$$\int_{r \leq R} \nabla \Phi(\mathbf{r}) d^3 r = \mathbf{\hat{x}} \int_{r \leq R} \nabla \cdot (\mathbf{\hat{x}}\Phi) d^3 r + \mathbf{\hat{y}} \int_{r \leq R} \nabla \cdot (\mathbf{\hat{y}}\Phi) d^3 r + \mathbf{\hat{z}} \int_{r \leq R} \nabla \cdot (\mathbf{\hat{z}}\Phi) d^3 r, \quad (8)$$

which is equal to

$$\int_{r=R} \Phi(\mathbf{r}) R^2 d\Omega \left((\mathbf{\hat{x}} \cdot \mathbf{\hat{r}}) \mathbf{\hat{x}} + (\mathbf{\hat{y}} \cdot \mathbf{\hat{r}}) \mathbf{\hat{y}} + (\mathbf{\hat{z}} \cdot \mathbf{\hat{r}}) \mathbf{\hat{z}} \right) = \int_{r=R} \Phi(\mathbf{r}) R^2 d\Omega \mathbf{\hat{r}}.$$
 (9)

Thus,

$$\mathbf{E}_{\leq R} \mathbf{E}(\mathbf{r}) d^3 r = -\int_{r \leq R} \nabla \Phi(\mathbf{r}) d^3 r = -R^2 \int_{r=R} \Phi(\mathbf{r}) \hat{\mathbf{r}} d\mathbf{\Omega}.$$
 (10)



δ -function contribution dipole electric field – continued

$$\int_{r \le R} \mathbf{E}(\mathbf{r}) d^3 r = -R^2 \int_{r=R} \Phi(\mathbf{r}) \hat{\mathbf{r}} d\mathbf{\Omega}.$$
 (11)

Now, we notice that the electrostatic potential can be determined from the charge density $\rho(\mathbf{r})$ according to:

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int d^3r' \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} = \frac{1}{4\pi\epsilon_0} \sum_{lm} \frac{4\pi}{2l+1} \int d^3r' \rho(\mathbf{r}') \frac{r_{<}^l}{r_{>}^{l+1}} Y_{lm}^*(\hat{\mathbf{r}}) Y_{lm}(\hat{\mathbf{r}'}).$$
(12)

We also note that the unit vector can be written in terms of spherical harmonic functions:

$$\hat{\mathbf{r}} = \begin{cases} \sin(\theta)\cos(\phi)\hat{\mathbf{x}} + \sin(\theta)\sin(\phi)\hat{\mathbf{y}} + \cos(\theta)\hat{\mathbf{z}} \\ \sqrt{\frac{4\pi}{3}}\left(Y_{1-1}(\hat{\mathbf{r}})\frac{\hat{\mathbf{x}} + i\hat{\mathbf{y}}}{\sqrt{2}} + Y_{11}(\hat{\mathbf{r}})\frac{-\hat{\mathbf{x}} + i\hat{\mathbf{y}}}{\sqrt{2}} + Y_{10}(\hat{\mathbf{r}})\hat{\mathbf{z}} \right) \end{cases}$$



δ -function contribution dipole electric field – continued

Therefore, when we evaluate the integral over solid angle Ω in Eq. (11), only the l = 1 term contributes and the effect of the integration reduced to the expression:

$$-R^2 \int_{r=R} \Phi(\mathbf{r}) \hat{\mathbf{r}} d\mathbf{\Omega} = -\frac{1}{4\pi\epsilon_0} \frac{4\pi R^2}{3} \int d^3 r' \rho(\mathbf{r}') \frac{r_{<}}{r_{>}^2} \hat{\mathbf{r}'}.$$
 (13)

The choice of $r_{<}$ and $r_{>}$ is a choice between the integration variable r' and the sphere radius R. If the sphere encloses the charge distribution $\rho(\mathbf{r}')$, then $r_{<} = r'$ and $r_{>} = R$ so that Eq. (13) becomes

$$-R^{2}\int_{r=R}\Phi(\mathbf{r})\hat{\mathbf{r}}d\mathbf{\Omega} = -\frac{1}{4\pi\epsilon_{0}}\frac{4\pi R^{2}}{3}\frac{1}{R^{2}}\int d^{3}r'\rho(\mathbf{r}')r'\hat{\mathbf{r}'} \equiv -\frac{\mathbf{p}}{3\varepsilon_{0}}.$$
 (14)

If the charge distribution $\rho(\mathbf{r}')$ lies outside of the sphere, then $r_> = r'$ and $r_< = R$ so that Eq. (13) becomes

$$-R^2 \int_{r=R} \Phi(\mathbf{r}) \hat{\mathbf{r}} d\mathbf{\Omega} = -\frac{1}{4\pi\epsilon_0} \frac{4\pi R^2}{3} R \int d^3 r' \frac{\rho(\mathbf{r}')}{{r'}^2} \hat{\mathbf{r}'} \equiv \frac{4\pi R^3}{3} \mathbf{E}(0).$$
(15)

