PHY 712 Electrodynamics
9-9:50 AM MWF Olin 105

Plan for Lecture 13:

Finish reading Chapter 5

1. **Recap of hyperfine interaction**
2. **Macroscopic magnetization density M**
3. **H field and its relation to B**
4. **Magnetic boundary values**

Colloquium: "Tailored Light-Matter Interactions in Scalable and Artificial Nanomaterials" – Wednesday, February 13, 2019, at 4:00 PM

Peijun Guo, PhD,
Named Fellowship-Enrico Fermi, Argonne National Laboratory
George P. Williams, Jr. Lecture Hall, (Olin 101)
Wednesday, February 13, 2019, at 4:00 PM

There will be a reception with refreshments at 3:30 PM in the lounge. All interested persons are cordially invited to attend.

ABSTRACT

The need for exquisite control of light is ubiquitous in energy-relevant applications, optoelectronics, and information science. In this talk, I will discuss how hybrid materials consisting of distinct sub-lattices and periodically nanostructured materials allow for dramatically enhanced light absorption, emission, and charge carrier generation at various time- and length-scales. I will first focus on hybrid organic-inorganic perovskites. These solution-

Course schedule for Spring 2019

(Preliminary schedule -- subject to frequent adjustment.)

	Lecture date	JDJ Reading	Topic	HW	Due date
1	Mon: 01/14/2019	Chap. 1 & Appen.	Introduction, units and Poisson equation	#1	01/23/2019
2	Wed: 01/16/2019	Chap. 1	Electrostatic energy calculations	#2	01/23/2019
3	Fri: 01/18/2019	Chap. 1	Electrostatic potentials and fields	#3	01/23/2019
	Mon: 01/21/2019	No class	Martin Luther King Holiday		
4	Wed: 01/23/2019	Chap. 1 - 3	Poisson's equation in 2 and 3 dimensions		
5	Fri: 01/25/2019	Chap. 1 - 3	Brief introduction to numerical methods	#4	01/28/2019
6	Mon: 01/28/2019	Chap. 2 & 3	Image charge constructions	#5	01/30/2019
7	Wed: 01/30/2019	Chap. 2 & 3	Cylindrical and spherical geometries		
8	Fri: 02/01/2019	Chap. 3 & 4	Spherical geometry and multipole moments	#6	02/04/2019
9	Mon: 02/04/2019	Chap. 4	Dipoles and Dielectrics	#7	02/06/2019
10	Wed: 02/06/2019	Chap. 4	Polarization and Dielectrics		
11	Fri: 02/08/2019	Chap. 5	Magnetostatics	#8	02/11/2019
12	Mon: 02/11/2019	Chap. 5	Magnetic dipoles and hyperfine interaction	#9	02/13/2019
13	Wed: 02/13/2019	Chap. 5	Magnetic dipoles and dipolar fields	#10	02/15/2019
14	Fri: 02/15/2019				
15	Mon: 02/18/2019				
16	Wed: 02/20/2019				

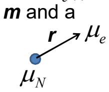
Summary of hyperfine interaction form:

Interactions between magnetic dipoles

Sources of magnetic dipoles and other sources of magnetism in an atom:

- Intrinsic magnetic moment of a nucleus μ_N
- Intrinsic magnetic moment of an electron μ_e
- Magnetic field due to electron orbital current $\mathbf{J}_e(\mathbf{r})$

Interaction energy between a magnetic dipole ***m*** and a magnetic field **B:** $E_{int} = -\mathbf{m}\cdot\mathbf{B}$



In this case: $E_{int} = -\mu_e \cdot \mathbf{B}_{\mu_N} - \mu_N \cdot \mathbf{B}_{\mathbf{J}_e}(0)$

$$\mathbf{B}_{\boldsymbol{\mu}_N}(\mathbf{r}) = \frac{\mu_0}{4\pi}\left\{\frac{3\hat{\mathbf{r}}(\boldsymbol{\mu}_N\cdot\hat{\mathbf{r}})-\boldsymbol{\mu}_N}{r^3} + \frac{8\pi}{3}\boldsymbol{\mu}_N\delta^3(\mathbf{r})\right\}$$

Hyperfine interaction energy: -- continued

$$E_{int} = -\mu_e \cdot \mathbf{B}_{\mu_N} - \mu_N \cdot \mathbf{B}_{\mathbf{J}_e}(0)$$

Evaluation of the magnetic field at the nucleus due to the electron current density:

The vector potential associated with an electron in a bound state of an atom as described by a quantum mechanical wavefunction $\psi_{nlm_l}(\mathbf{r})$ can be written:

$$\mathbf{A}_{\mathbf{J}_e}(\mathbf{r}) = -\frac{\mu_0}{4\pi}\frac{e\hbar}{m_e} m_l \int d^3r' \frac{\hat{\mathbf{z}}\times\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}\frac{\left|\psi_{nlm_l}(\mathbf{r}')\right|^2}{r'^2\sin^2\theta'}$$

We want to evaluate the magnetic field $\mathbf{B} = \nabla\times\mathbf{A}$ in the vicinity of the nucleus $(\mathbf{r}\to 0)$.

Hyperfine interaction energy: -- continued

$$\mathbf{B}_{\mathbf{J}_e}(0) = \nabla\times\mathbf{A}_{\mathbf{J}_e}\Big|_{\mathbf{r}\to 0} = -\frac{\mu_0}{4\pi}\frac{e\hbar}{m_e} m_l \int d^3r' \nabla\times\frac{\hat{\mathbf{z}}\times\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}\frac{\left|\psi_{nlm_l}(\mathbf{r}')\right|^2}{r'^2\sin^2\theta'}\Bigg|_{\mathbf{r}\to 0}$$

$$\mathbf{B}_{\mathbf{o}}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{e\hbar}{m_e} m_l \int d^3r' \frac{(\mathbf{r}-\mathbf{r}')\times(\hat{\mathbf{z}}\times\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\frac{\left|\psi_{nlm_l}(\mathbf{r}')\right|^2}{r'^2\sin^2\theta'}\Bigg|_{\mathbf{r}\to 0}$$

$$\mathbf{B}_{\mathbf{o}}(\mathbf{0}) = -\frac{\mu_0}{4\pi}\frac{e\hbar}{m_e} m_l \int d^3r' \frac{\mathbf{r}'\times(\hat{\mathbf{z}}\times\mathbf{r}')}{r'^3}\frac{\left|\psi_{nlm_l}(\mathbf{r}')\right|^2}{r'^2\sin^2\theta'}$$

$$\hat{\mathbf{r}}'\times(\hat{\mathbf{z}}\times\hat{\mathbf{r}}') = \hat{\mathbf{z}}(\mathbf{1}-\cos^2\boldsymbol{\theta}') - \hat{\mathbf{x}}\cos\theta'\sin\theta'\cos\phi' - \hat{\mathbf{y}}\cos\theta'\sin\theta'\sin\phi').$$

$$\mathbf{B}_{\mathbf{o}}(\mathbf{0}) = -\frac{\mu_0}{4\pi}\frac{e\hbar}{m_e} m_l \int d^3r' \frac{\hat{\mathbf{z}}r'^2\sin^2\theta'}{r'^3}\frac{\left|\psi_{nlm_l}(\mathbf{r}')\right|^2}{r'^2\sin^2\theta'} = -\frac{\mu_0}{4\pi}\frac{e\hbar}{m_e} m_l\,\hat{\mathbf{z}}\int d^3r' \frac{\left|\psi_{nlm_l}(\mathbf{r}')\right|^2}{r'^3}$$

$$= -\frac{\mu_0}{4\pi}\frac{e\hbar}{m_e} m_l\,\hat{\mathbf{z}}\left\langle\frac{1}{r'^3}\right\rangle$$

Hyperfine interaction energy: -- continued

$$E_{int} \equiv H_{HF} = -\mu_e \cdot \mathbf{B}_{\mu_N} - \mu_N \cdot \mathbf{B}_{\mathbf{J_e}}(0)$$

Putting all of the terms together:

$$H_{\mathrm{HF}} = -\frac{\mu_0}{4\pi}\left(\left\langle \frac{3(\boldsymbol{\mu}_\mathbf{N}\cdot\hat{\mathbf{r}})(\boldsymbol{\mu}_\mathbf{e}\cdot\hat{\mathbf{r}}) - \boldsymbol{\mu}_\mathbf{N}\cdot\boldsymbol{\mu}_\mathbf{e}}{r^3} + \frac{8\pi}{3}\boldsymbol{\mu}_\mathbf{N}\cdot\boldsymbol{\mu}_\mathbf{e}\delta^3(\mathbf{r})\right\rangle + \frac{e}{m_e}\left\langle\frac{\mathbf{L}\cdot\boldsymbol{\mu}_\mathbf{N}}{r^3}\right\rangle\right).$$

In this expression the brackets $\langle\ \rangle$ indicate evaluating the expectation value relative to the electronic state.

Macroscopic dipolar effects --
Magnetic dipole moment

$$\mathbf{m} = \frac{1}{2}\int d^3r\ \mathbf{r}\times\mathbf{J}(\mathbf{r})$$

Note that the intrinsic spin of elementary particles is associated with a magnetic dipole moment, but we often do not have a detailed knowledge of **J**(**r**).

Vector potential for magnetic dipole moment

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\mathbf{r}}{|\mathbf{r}|^3}$$

Macroscopic magnetization

$$\mathbf{M}(\mathbf{r}) = \sum_i \mathbf{m}_i\delta^3(\mathbf{r}-\mathbf{r}_i)$$

Vector potential due to "free" current $\mathbf{J}_{\mathbf{free}}(\mathbf{r})$ and macroscopic magnetization **M**(**r**). Note: the designation $\mathbf{J}_{\mathbf{free}}(\mathbf{r})$ implies that this current does not also contribute to the magnetization density.

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int d^3r'\left(\frac{\mathbf{J}_{free}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} + \frac{\mathbf{M}(\mathbf{r}')\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\right)$$

Vector potential contributions from macroscopic magnetization -- continued

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int d^3r' \left(\frac{\mathbf{J}_{free}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} + \frac{\mathbf{M}(\mathbf{r}')\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} \right)$$

Note that :

$$\frac{\mathbf{M}(\mathbf{r}')\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} = \mathbf{M}(\mathbf{r}')\times\nabla'\frac{1}{|\mathbf{r}-\mathbf{r}'|}$$

$$= -\nabla'\times\left(\frac{\mathbf{M}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\right) + \frac{\nabla'\times\mathbf{M}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$$

$$\Rightarrow \mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int d^3r' \frac{\mathbf{J}_{free}(\mathbf{r}') + \nabla'\times\mathbf{M}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$$

Vector potential contributions from macroscopic magnetization -- continued

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int d^3r' \frac{\mathbf{J}_{free}(\mathbf{r}') + \nabla'\times\mathbf{M}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$$

Note that for the case that $\nabla\cdot\mathbf{A} = 0$:

$$\nabla\times\mathbf{B}(\mathbf{r}) = \nabla\times(\nabla\times\mathbf{A}(\mathbf{r})) = -\nabla^2\mathbf{A}(\mathbf{r})$$

$$= \frac{\mu_0}{4\pi}\int d^3r' \left(4\pi\delta^3(\mathbf{r}-\mathbf{r}')\right)\left(\mathbf{J}_{free}(\mathbf{r}') + \nabla'\times\mathbf{M}(\mathbf{r}')\right)$$

$$= \mu_0\left(\mathbf{J}_{free}(\mathbf{r}) + \nabla\times\mathbf{M}(\mathbf{r})\right)$$

$$\Rightarrow \nabla\times\left(\mathbf{B}(\mathbf{r}) - \mu_0\mathbf{M}(\mathbf{r})\right) = \mu_0\mathbf{J}_{free}(\mathbf{r})$$

Magnetic field contributions

$$\nabla\times\left(\mathbf{B}(\mathbf{r}) - \mu_0\mathbf{M}(\mathbf{r})\right) = \mu_0\mathbf{J}_{free}(\mathbf{r})$$

Define the magnetic flux density :

$$\mu_0\mathbf{H}(\mathbf{r}) \equiv \mathbf{B}(\mathbf{r}) - \mu_0\mathbf{M}(\mathbf{r})$$

$$\Rightarrow \nabla\times\mathbf{H}(\mathbf{r}) = \mathbf{J}_{free}(\mathbf{r})$$

$$\nabla\times\left(\mathbf{B}(\mathbf{r}) - \mu_0\mathbf{M}(\mathbf{r})\right) = \mu_0\mathbf{J}_{free}(\mathbf{r})$$

Note that $\mathbf{B}(\mathbf{r}) \equiv$ the magnetic flux density

Define $\mathbf{H}(\mathbf{r}) \equiv$ the magnetic field

$$\mu_0\mathbf{H}(\mathbf{r}) \equiv \mathbf{B}(\mathbf{r}) - \mu_0\mathbf{M}(\mathbf{r})$$

$$\Rightarrow \nabla\times\mathbf{H}(\mathbf{r}) = \mathbf{J}_{free}(\mathbf{r})$$

Energy associated with magnetic fields

Note: We previously used without proof --

the force on a magnetic dipole $\mathbf{m}$ in an external $\mathbf{B}$ field is:

$$\mathbf{F} = \nabla\left(\mathbf{m}\cdot\mathbf{B}\right)$$

This implies that energy associated with aligning a magnetic dipole $\mathbf{m}$ in an external $\mathbf{B}$ field is given by:

$$U = -\mathbf{m}\cdot\mathbf{B}$$

Macroscopic energies --

It can be shown that: $\quad W_B = \frac{1}{2}\int d^3r\ \mathbf{B}(\mathbf{r})\cdot\mathbf{H}(\mathbf{r})$

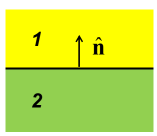
In analogy to: $\quad W_E = \frac{1}{2}\int d^3r\ \mathbf{E}(\mathbf{r})\cdot\mathbf{D}(\mathbf{r})$

Summary of equations of magnetostatics:

$$\nabla\times\mathbf{B}(\mathbf{r}) = \mu_0\mathbf{J}_{total}(\mathbf{r})$$
$$\nabla\times\mathbf{H}(\mathbf{r}) = \mathbf{J}_{free}(\mathbf{r})$$
$$\mathbf{B}(\mathbf{r}) = \mu_0(\mathbf{H}(\mathbf{r}) + \mathbf{M}(\mathbf{r}))$$
$$\nabla\cdot\mathbf{B}(\mathbf{r}) = 0$$

For the case that $\mathbf{J}_{free}(\mathbf{r}) = 0$:

$$\nabla\times\mathbf{H}(\mathbf{r}) = 0$$
$$\nabla\cdot\mathbf{B}(\mathbf{r}) = 0$$

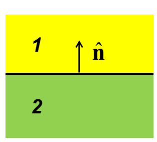


For the case that $\mathbf{J}_{free}(\mathbf{r}) = 0$:

$$\nabla\times\mathbf{H}(\mathbf{r}) = 0$$
$$\nabla\cdot\mathbf{B}(\mathbf{r}) = 0$$

At boundary:

$$\mathbf{H}_1\times\hat{\mathbf{n}} = \mathbf{H}_2\times\hat{\mathbf{n}}$$
$$\mathbf{B}_1\cdot\hat{\mathbf{n}} = \mathbf{B}_2\cdot\hat{\mathbf{n}}$$



Example magnetostatic boundary value problem

$\mathbf{M}_0$

$$\mathbf{M}(\mathbf{r}) = \begin{cases} M_0\hat{\mathbf{z}} & r \le a \\ 0 & r > a \end{cases}$$

$$\nabla \times \mathbf{H}(\mathbf{r}) = 0 \qquad \Rightarrow \mathbf{H}(\mathbf{r}) = -\nabla \Phi_H(\mathbf{r})$$
$$\mathbf{B}(\mathbf{r}) = \mu_0(\mathbf{H}(\mathbf{r}) + \mathbf{M}(\mathbf{r}))$$
$$\nabla \cdot \mathbf{B}(\mathbf{r}) = 0 = \mu_0 \nabla \cdot (\mathbf{H}(\mathbf{r}) + \mathbf{M}(\mathbf{r}))$$
$$\Rightarrow \nabla^2 \Phi_H(\mathbf{r}) = \nabla \cdot \mathbf{M}(\mathbf{r})$$

Example magnetostatic boundary value problem -- continued

$\mathbf{M}_0$

$$\mathbf{M}(\mathbf{r}) = \begin{cases} M_0\hat{\mathbf{z}} & r \le a \\ 0 & r > a \end{cases}$$

$$\nabla^2 \Phi_H(\mathbf{r}) = \nabla \cdot \mathbf{M}(\mathbf{r})$$
$$\Rightarrow \Phi_H(\mathbf{r}) = -\frac{1}{4\pi}\int d^3r' \frac{\nabla' \cdot \mathbf{M}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$$
$$= -\frac{1}{4\pi}\int d^3r' \left[\nabla' \cdot \left(\frac{\mathbf{M}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} \right) - \mathbf{M}(\mathbf{r}') \cdot \nabla' \frac{1}{|\mathbf{r}-\mathbf{r}'|} \right]$$
$$= -\frac{1}{4\pi} \nabla \cdot \int d^3r' \frac{\mathbf{M}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$$

Example magnetostatic boundary value problem -- continued

$\mathbf{M}_0$

$$\mathbf{M}(\mathbf{r}) = \begin{cases} M_0\hat{\mathbf{z}} & r \le a \\ 0 & r > a \end{cases} \qquad \Phi_H(\mathbf{r}) = -\frac{1}{4\pi} \nabla \cdot \int d^3r' \frac{\mathbf{M}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$$

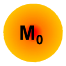
For this example:

$$\Phi_H(\mathbf{r}) = -\frac{M_0}{4\pi}\frac{\partial}{\partial z}\left(4\pi \int_0^a r'^2\, dr' \frac{1}{r_>}\right)$$

For $r \le a$: $\quad \Phi_H(\mathbf{r}) = -M_0 \frac{\partial}{\partial z}\left(\frac{a^2}{2} - \frac{r^2}{6}\right) = \frac{M_0 z}{3}$

For $r > a$: $\quad \Phi_H(\mathbf{r}) = -M_0 \frac{\partial}{\partial z}\left(\frac{a^3}{3r}\right) = \frac{M_0 a^3 z}{3r^3}$

Example magnetostatic boundary value problem -- continued



$$\mathbf{M}(\mathbf{r}) = \begin{cases} M_0\hat{\mathbf{z}} & r \le a \\ 0 & r > a \end{cases}$$

For $r \le a$: $\Phi_H(\mathbf{r}) = \frac{M_0 z}{3}$ $\quad \mathbf{H}(\mathbf{r}) = -\nabla\Phi_H(\mathbf{r}) = -\frac{M_0}{3}\hat{\mathbf{z}}$

For $r > a$: $\Phi_H(\mathbf{r}) = \frac{M_0 a^3 z}{3r^3}$ $\quad \mathbf{H}(\mathbf{r}) = -\nabla\Phi_H(\mathbf{r}) = -\frac{M_0 a^3}{3}\left(\frac{\hat{\mathbf{z}}}{r^3} - \frac{3z\mathbf{r}}{r^5}\right)$

$$\mathbf{B}(\mathbf{r}) = \mu_0\left(\mathbf{H}(\mathbf{r}) + \mathbf{M}(\mathbf{r})\right)$$

For $r \le a$: $\mathbf{H}(\mathbf{r}) = -\frac{M_0\hat{\mathbf{z}}}{3}$ $\quad \mathbf{B}(\mathbf{r}) = \mu_0\frac{2M_0\hat{\mathbf{z}}}{3}$

For $r > a$: $\mathbf{H}(\mathbf{r}) = -\frac{M_0 a^3}{3}\left(\frac{\hat{\mathbf{z}}}{r^3} - \frac{3z\mathbf{r}}{r^5}\right)$

$$\mathbf{B}(\mathbf{r}) = -\mu_0\frac{M_0 a^3}{3}\left(\frac{\hat{\mathbf{z}}}{r^3} - \frac{3z\mathbf{r}}{r^5}\right)$$

Check boundary values:

For $r \le a$: $\mathbf{H}(\mathbf{r}) = -\frac{M_0\hat{\mathbf{z}}}{3}$ $\quad \mathbf{H}(a\hat{\mathbf{r}}) \times \hat{\mathbf{r}} = -\frac{M_0}{3}\hat{\mathbf{z}} \times \hat{\mathbf{r}}$

For $r > a$: $\mathbf{H}(\mathbf{r}) = -\frac{M_0 a^3}{3}\left(\frac{\hat{\mathbf{z}}}{r^3} - \frac{3z\mathbf{r}}{r^5}\right)$

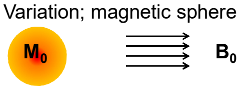
$$\mathbf{H}(a\hat{\mathbf{r}}) \times \hat{\mathbf{r}} = -\frac{M_0 a^3}{3}\frac{\hat{\mathbf{z}} \times \hat{\mathbf{r}}}{a^3}$$

For $r \le a$: $\mathbf{B}(\mathbf{r}) = \mu_0\frac{2M_0\hat{\mathbf{z}}}{3}$ $\quad \mathbf{B}(a\hat{\mathbf{r}}) \cdot \hat{\mathbf{r}} = \mu_0\frac{2M_0}{3}\hat{\mathbf{z}} \cdot \hat{\mathbf{r}}$

For $r > a$: $\mathbf{B}(\mathbf{r}) = -\mu_0\frac{M_0 a^3}{3}\left(\frac{\hat{\mathbf{z}}}{r^3} - \frac{3z\mathbf{r}}{r^5}\right)$

$$\mathbf{B}(a\hat{\mathbf{r}}) \cdot \hat{\mathbf{r}} = -\mu_0\frac{M_0 a^3}{3}\hat{\mathbf{z}} \cdot \hat{\mathbf{r}}\left(\frac{1}{a^3} - \frac{3a^2}{a^5}\right)$$

Variation; magnetic sphere plus external field $\mathbf{B}_0$



$$\mathbf{M}(\mathbf{r}) = \begin{cases} \mathbf{M}_0 & r \le a \\ 0 & r > a \end{cases}$$

By superposition :

For $r \le a$:

$$\mathbf{B}(\mathbf{r}) = \mathbf{B}_0 + \mu_0\frac{2}{3}\mathbf{M}_0$$

$$\mathbf{H}(\mathbf{r}) = \frac{1}{\mu_0}\mathbf{B}_0 - \frac{1}{3}\mathbf{M}_0$$

$$\mathbf{B}(\mathbf{r}) + 2\mu_0\mathbf{H}(\mathbf{r}) = 3\mathbf{B}_0$$

For an isotropic "paramagnetic" material, $\mathbf{B}(\mathbf{r}) = \mu\mathbf{H}(\mathbf{r})$

$$\mathbf{M}_0 = \frac{3}{\mu_0}\left(\frac{\mu - \mu_0}{\mu + 2\mu_0}\right)\mathbf{B}_0$$

Summary of equations of magnetostatics:

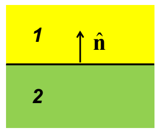
$$\nabla \times \mathbf{B}(\mathbf{r}) = \mu_0 \mathbf{J}_{total}(\mathbf{r})$$
$$\nabla \times \mathbf{H}(\mathbf{r}) = \mathbf{J}_{free}(\mathbf{r})$$
$$\mathbf{B}(\mathbf{r}) = \mu_0 (\mathbf{H}(\mathbf{r}) + \mathbf{M}(\mathbf{r}))$$
$$\nabla \cdot \mathbf{B}(\mathbf{r}) = 0$$

For the case that $\mathbf{J}_{free}(\mathbf{r})$:

$$\nabla \times \mathbf{H}(\mathbf{r}) = 0$$
$$\nabla \cdot \mathbf{B}(\mathbf{r}) = 0$$

At boundary:

$$\mathbf{H}_1 \times \hat{\mathbf{n}} = \mathbf{H}_2 \times \hat{\mathbf{n}}$$
$$\mathbf{B}_1 \cdot \hat{\mathbf{n}} = \mathbf{B}_2 \cdot \hat{\mathbf{n}}$$



Magnetism in materials

$$\mathbf{B}(\mathbf{r}) = \mu_0 (\mathbf{H}(\mathbf{r}) + \mathbf{M}(\mathbf{r}))$$

For materials with linear magnetism:

$$\mathbf{B} = \mu \mathbf{H}$$
$$\mu > \mu_0 \quad \Rightarrow \text{paramagnetic material}$$
$$\mu < \mu_0 \quad \Rightarrow \text{diamagnetic material}$$

For ferromagnetic, antiferromagnetic materials

$$\mathbf{B} = f(\mathbf{H}) \quad \text{(with hysteresis)}$$

https://en.wikipedia.org/wiki/Permeability_(electromagnetism)

Magnetic susceptibility and permeability data for selected materials

Medium	Susceptibility, volumetric, SI, χ_m	Permeability, μ (H/m)	Relative permeability, max., μ/μ_0	Magnetic field
Metglas 2714A (annealed)		1.26×10^0	1 000 000[10]	At 0.5 T
Iron (99.95% pure Fe annealed in H)		2.5×10^{-1}	200 000[11]	
NANOPERM®		1.0×10^{-1}	80 000[12]	At 0.5 T
Mu-metal		2.5×10^{-2}	20 000[13]	At 0.002 T
Mu-metal		6.3×10^{-2}	50 000[14]	
Cobalt-iron (high permeability strip material)		2.3×10^{-2}	18 000[15]	
Permalloy	8000	1.0×10^{-2}	8000[13]	At 0.002 T

Mumetal Magnetic Shielding

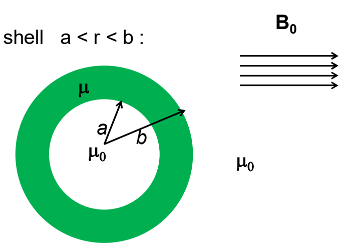
Mu metal is a soft ferromagnetic alloy that has extremely high initial and maximum magnetic permeability. It is used in electric transformer, storage disks, magnetic phonographs, resonance devices and superconducting circuits.

Mumetal alloy generally attributes relative permeability about 80,000 to 100,000 than the normal steel alloy. It is also called as soft magnetic alloy and offers low magnetic anisotropy and magnetostriction providing low corceivity to saturate the low magnetic fields. It provides nominal hysteresis loses when the alloy is employed in the AC magnetic circuits.

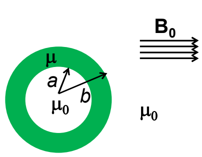
Composed of 80% Ni, 15% Fe, 5% Mo+other materials

Example: permalloy, mumetal $\mu/\mu_0 \sim 10^4$

Spherical shell $a < r < b$:



Example: permalloy, mumetal $\mu/\mu_0 \sim 10^4$ -- continued



For this case:

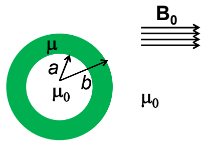
$$\nabla \times \mathbf{H}(\mathbf{r}) = 0$$
$$\nabla \cdot \mathbf{B}(\mathbf{r}) = 0$$
$$\mathbf{B}(\mathbf{r}) = \mu \mathbf{H}(\mathbf{r})$$

Continuity at boundaries:

$\mathbf{H} \times \hat{\mathbf{n}} =$ continuous

$\mathbf{B} \cdot \hat{\mathbf{n}} =$ continuous

Example: permalloy, mumetal $\mu/\mu_0 \sim 10^4$ -- continued



$$\text{Let:}\quad \mathbf{H}(\mathbf{r}) = -\nabla\Phi_H(\mathbf{r})$$

$$\nabla\cdot\mathbf{B}(\mathbf{r}) = 0 \qquad \Rightarrow \nabla^2\Phi_H(\mathbf{r}) = 0$$

$$\text{For}\quad 0 \le r \le a \qquad \Phi_H(\mathbf{r}) = \sum_l \delta_l r^l P_l(\cos\theta)$$

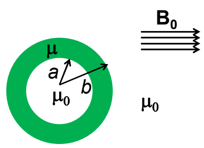
$$\text{For}\quad a \le r \le b \qquad \Phi_H(\mathbf{r}) = \sum_l \left(\beta_l r^l + \frac{\gamma_l}{r^{l+1}}\right) P_l(\cos\theta)$$

$$\text{For}\quad r \ge b \qquad \Phi_H(\mathbf{r}) = -\frac{B_0}{\mu_0} r\cos\theta + \sum_l \frac{\alpha_l}{r^{l+1}} P_l(\cos\theta)$$

Example: permalloy, mumetal $\mu/\mu_0 \sim 10^4$ -- continued

Applying boundary conditions

(only $l = 1$ terms contribute):



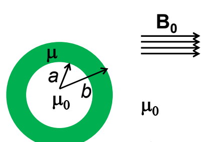
$$\text{At}\quad r = a \qquad \delta_1 = \frac{\mu}{\mu_0}\left(\beta_1 - 2\frac{\gamma_1}{a^3}\right)$$

$$a\delta_1 = a\beta_1 + \frac{\gamma_1}{a^2}$$

$$\text{At}\quad r = b \qquad \frac{\mu}{\mu_0}\left(\beta_1 - 2\frac{\gamma_1}{b^3}\right) = -\frac{B_0}{\mu_0} - 2\frac{\alpha_1}{b^3}$$

$$b\beta_1 + \frac{\gamma_1}{b^2} = -b\frac{B_0}{\mu_0} + \frac{\alpha_1}{b^2}$$

Example: permalloy, mumetal $\mu/\mu_0 \sim 10^4$ -- continued



When the dust clears:

$$\delta_1 = \left(\frac{-9\mu/\mu_0}{(2\mu/\mu_0 + 1)(\mu/\mu_0 + 2) - 2(a/b)^3(\mu/\mu_0 - 1)^2}\right)\frac{B_0}{\mu_0}$$

$$\approx \frac{1}{\mu/\mu_0}\left(\frac{-9/2}{\left(1-(a/b)^3\right)}\frac{B_0}{\mu_0}\right)$$