# PHY 712 Electrodynamics
## 9-9:50 AM MWF Olin 105

## Plan for Lecture 16:

### Finish reading Chapter 6

1. Some details of Liénard-Wiechert results
2. Energy density and flux associated with electromagnetic fields
3. Time harmonic fields



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|---|---|---|---|---|---|
| 11 | Fri: 02/08/2019 | Chap. 5 | Magnetostatics | #8 | 02/11/2019 |
| 12 | Mon: 02/11/2019 | Chap. 5 | Magnetic dipoles and hyperfine interaction | #9 | 02/13/2019 |
| 13 | Wed: 02/13/2019 | Chap. 5 | Magnetic dipoles and dipolar fields | #10 | 02/15/2019 |
| 14 | Fri: 02/15/2019 | Chap. 6 | Maxwell's Equations | #11 | 02/18/2019 |
| 15 | Mon: 02/18/2019 | Chap. 6 | Electromagnetic energy and forces | #12 | 02/20/2019 |
| 16 | Wed: 02/20/2019 | Chap. 7 | Electromagnetic plane waves | #13 | 02/22/2019 |
| 17 | Fri: 02/22/2019 | | | | |
| 18 | Mon: 02/25/2019 | | | | |
| 19 | Wed: 02/27/2019 | | | | |
| 20 | Fri: 03/01/2019 | | | | |
| | Mon: 03/04/2019 | No class | *APS March Meeting* | Take Home Exam | |
| | Wed: 03/06/2019 | No class | *APS March Meeting* | Take Home Exam | |
| | Fri: 03/08/2019 | No class | *APS March Meeting* | Take Home Exam | |
| | Mon: 03/11/2019 | No class | *Spring Break* | | |
| | Wed: 03/13/2019 | No class | *Spring Break* | | |
| | Fri: 03/15/2019 | No class | *Spring Break* | | |
| 21 | Mon: 03/18/2019 | | | | |



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### Colloquium: "Plastic Nanoscale Optoelectronics Enabled Via Adhesion-Lithography" – Wednesday, February 20, 2019, at 4:00 PM

Thomas Anthopoulos, PhD,
Professor of Material Science and Engineering at King Abdullah University of Science and Technology
George P. Williams, Jr. Lecture Hall, (Olin 101)
Wednesday, February 20, 2019, at 4:00 PM

There will be a reception with refreshments at 3:30 PM in the lounge. All interested persons are cordially invited to attend.

Solution of Maxwell's equations in the Lorentz gauge -- continued

Liénard-Wiechert potentials and fields --
Determination of the scalar and vector potentials for a moving point particle (also see Landau and Lifshitz ***The Classical Theory of Fields***, Chapter 8.)

Consider the fields produced by the following source: a point charge $q$ moving on a trajectory $R_q(t)$.

$$\text{Charge density: } \rho(\mathbf{r},t) = q\delta^3(\mathbf{r}-\mathbf{R}_q(t))$$

$$\text{Current density: } \mathbf{J}(\mathbf{r},t) = q\dot{\mathbf{R}}_q(t)\delta^3(\mathbf{r}-\mathbf{R}_q(t)), \quad \text{where} \quad \dot{\mathbf{R}}_q(t) \equiv \frac{d\mathbf{R}_q(t)}{dt}.$$




Solution of Maxwell's equations in the Lorentz gauge -- continued

$$\Phi(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0}\int\int d^3r'dt'\frac{\rho(\mathbf{r}',t')}{|\mathbf{r}-\mathbf{r}']|}\delta\left(t'-(t-|\mathbf{r}-\mathbf{r}'|/c)\right)$$

$$\mathbf{A}(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0 c^2}\int\int d^3r'dt'\frac{\mathbf{J}(\mathbf{r}',t')}{|\mathbf{r}-\mathbf{r}'|}\delta\left(t'-(t-|\mathbf{r}-\mathbf{r}'|/c)\right).$$

We performing the integrations over first $d^3r'$ and then $dt'$ making use of the fact that for any function of $t'$,

$$\int_{-\infty}^{\infty} dt'\, f(t')\delta\left(t'-(t-|\mathbf{r}-\mathbf{R}_q(t')|/c)\right) = \frac{f(t_r)}{1-\dfrac{\dot{\mathbf{R}}_q(t_r)\cdot(\mathbf{r}-\mathbf{R}_q(t_r))}{c|\mathbf{r}-\mathbf{R}_q(t_r)|}},$$

where the ``retarded time'' is defined to be

$$t_r \equiv t - \frac{|\mathbf{r}-\mathbf{R}_q(t_r)|}{c}.$$



Solution of Maxwell's equations in the Lorentz gauge -- continued

Resulting scalar and vector potentials:

$$\Phi(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0}\frac{1}{R-\dfrac{\mathbf{v}\cdot\mathbf{R}}{c}},$$

$$\mathbf{A}(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0 c^2}\frac{\mathbf{v}}{R-\dfrac{\mathbf{v}\cdot\mathbf{R}}{c}},$$

Notation: $\mathbf{R} \equiv \mathbf{r}-\mathbf{R}_q(t_r)$ $\quad t_r \equiv t - \dfrac{|\mathbf{r}-\mathbf{R}_q(t_r)|}{c}.$

$\mathbf{v} \equiv \dot{\mathbf{R}}_q(t_r),$

## Solution of Maxwell's equations in the Lorentz gauge -- continued

In order to find the electric and magnetic fields, we need to evaluate

$$\mathbf{E}(\mathbf{r},t) = -\nabla\Phi(\mathbf{r},t) - \frac{\partial \mathbf{A}(\mathbf{r},t)}{\partial t}$$

$$\mathbf{B}(\mathbf{r},t) = \nabla \times \mathbf{A}(\mathbf{r},t)$$

The trick of evaluating these derivatives is that the retarded time $t_r$ depends on position **r** and on itself. We can show the following results using the shorthand notation:

$$\nabla t_r = -\frac{\mathbf{R}}{c\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)} \quad \text{and} \quad \frac{\partial t_r}{\partial t} = \frac{R}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)}.$$

02/18/2019 PHY 712 Spring 2019 -- Lecture 15 7

## Solution of Maxwell's equations in the Lorentz gauge -- continued

$$-\nabla\Phi(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0}\frac{1}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left[\mathbf{R}\left(1 - \frac{v^2}{c^2}\right) - \frac{\mathbf{v}}{c}\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right) + \mathbf{R}\frac{\dot{\mathbf{v}}\cdot\mathbf{R}}{c^2}\right],$$

$$-\frac{\partial \mathbf{A}(\mathbf{r},t)}{\partial t} = \frac{q}{4\pi\epsilon_0}\frac{1}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left[\frac{\mathbf{v}R}{c}\left(\frac{v^2}{c^2} - \frac{\mathbf{v}\cdot\mathbf{R}}{Rc} - \frac{\dot{\mathbf{v}}\cdot R}{c^2}\right) - \frac{\dot{\mathbf{v}}R}{c^2}\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)\right].$$

$$\mathbf{E}(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0}\frac{1}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left[\left(\mathbf{R} - \frac{\mathbf{v}R}{c}\right)\left(1 - \frac{v^2}{c^2}\right) + \left(\mathbf{R}\times\left\{\left(\mathbf{R} - \frac{\mathbf{v}R}{c}\right)\times\frac{\dot{\mathbf{v}}}{c^2}\right\}\right)\right].$$

$$\mathbf{B}(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0 c^2}\left[\frac{-\mathbf{R}\times\mathbf{v}}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left(1 - \frac{v^2}{c^2} + \frac{\dot{\mathbf{v}}\cdot\mathbf{R}}{c^2}\right) - \frac{\mathbf{R}\times\dot{\mathbf{v}}/c}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^2}\right] = \frac{\mathbf{R}\times\mathbf{E}(\mathbf{r},t)}{cR}$$

02/18/2019 PHY 712 Spring 2019 -- Lecture 15 8

## Maxwell's equations

Coulomb's law: $\nabla\cdot\mathbf{D} = \rho_{free}$

Ampere-Maxwell's law: $\nabla\times\mathbf{H} - \frac{\partial\mathbf{D}}{\partial t} = \mathbf{J}_{free}$

Faraday's law: $\nabla\times\mathbf{E} + \frac{\partial\mathbf{B}}{\partial t} = 0$

No magnetic monopoles: $\nabla\cdot\mathbf{B} = 0$

Energy analysis of electromagnetic fields and sources

Rate of work done on source $\mathbf{J}(\mathbf{r},t)$ by electromagnetic field:

$$\frac{dW_{mech}}{dt} \equiv \frac{dE_{mech}}{dt} = \int d^3r\ \mathbf{E}\cdot\mathbf{J}_{free}$$

Expressing source current in terms of fields it produces:

$$\frac{dW_{mech}}{dt} = \int d^3r\ \mathbf{E}\cdot\left(\nabla\times\mathbf{H} - \frac{\partial\mathbf{D}}{\partial t}\right)$$

02/18/2019 PHY 712 Spring 2019 -- Lecture 15 9

Energy analysis of electromagnetic fields and sources - continued

$$\frac{dW_{mech}}{dt} = \int d^3r \;\; \mathbf{E}\cdot\mathbf{J}_{free} = \int d^3r \;\; \mathbf{E}\cdot\left(\nabla\times\mathbf{H} - \frac{\partial\mathbf{D}}{\partial t}\right)$$

$$= -\int d^3r \;\; \left(\nabla\cdot\left(\mathbf{E}\times\mathbf{H}\right) + \mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial t} + \mathbf{H}\cdot\frac{\partial\mathbf{B}}{\partial t}\right)$$

Let $\mathbf{S}\equiv\mathbf{E}\times\mathbf{H}$ "Poynting vector"

$u \equiv \frac{1}{2}\left(\mathbf{E}\cdot\mathbf{D} + \mathbf{H}\cdot\mathbf{B}\right)$ energy density

$$\Rightarrow \frac{\partial u}{\partial t} + \nabla\cdot\mathbf{S} = -\mathbf{E}\cdot\mathbf{J}_{free}$$

02/18/2019 PHY 712 Spring 2019 -- Lecture 15 10

Energy analysis of electromagnetic fields and sources - continued

$$\frac{dE_{mech}}{dt} \equiv \int d^3r \;\; \mathbf{E}\cdot\mathbf{J}_{free}$$

Electromagnetic energy density: $u \equiv \frac{1}{2}\left(\mathbf{E}\cdot\mathbf{D} + \mathbf{H}\cdot\mathbf{B}\right)$

$E_{field} \equiv \int d^3r \;\; u(\mathbf{r},t)$

Poynting vector: $\mathbf{S}\equiv\mathbf{E}\times\mathbf{H}$

From the previous energy analysis: $\frac{\partial u}{\partial t} + \nabla\cdot\mathbf{S} = -\mathbf{E}\cdot\mathbf{J}_{free}$

$$\Rightarrow \frac{dE_{mech}}{dt} + \frac{dE_{field}}{dt} = -\int d^3r \;\; \nabla\cdot\mathbf{S}(\mathbf{r},t) = -\oint d^2r \; \hat{\mathbf{r}}\cdot\mathbf{S}(\mathbf{r},t)$$

02/18/2019 PHY 712 Spring 2019 -- Lecture 15 11

Momentum analysis of electromagnetic fields and sources

$$\frac{d\mathbf{P}_{mech}}{dt} \equiv \int d^3r \;\; \left(\rho\mathbf{E} + \mathbf{J}\times\mathbf{B}\right)$$

Follows by analogy with Lorentz force:

$$\mathbf{F} = q\left(\mathbf{E} + \mathbf{v}\times\mathbf{B}\right)$$

$$\mathbf{P}_{field} = \varepsilon_0 \int d^3r \;\; \left(\mathbf{E}\times\mathbf{B}\right)$$

Expression for vacuum fields:

$$\left(\frac{d\mathbf{P}_{mech}}{dt} + \frac{d\mathbf{P}_{field}}{dt}\right)_i = \sum_j \int d^3r \, \frac{\partial T_{ij}}{\partial r_j}$$

Maxwell stress tensor:

$$T_{ij} \equiv \varepsilon_0\left(E_iE_j + c^2B_iB_j - \delta_{ij}\frac{1}{2}\left(\mathbf{E}\cdot\mathbf{E} + c^2\mathbf{B}\cdot\mathbf{B}\right)\right)$$

02/18/2019 PHY 712 Spring 2019 -- Lecture 15 12

Comment on treatment of time-harmonic fields

Fourier transformation in time domain :

$$\mathbf{E}(\mathbf{r},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, \tilde{\mathbf{E}}(\mathbf{r},\omega)\, e^{-i\omega t}$$

$$\tilde{\mathbf{E}}(\mathbf{r},\omega) = \int_{-\infty}^{\infty} dt\, \mathbf{E}(\mathbf{r},t)\, e^{i\omega t}$$

Note that $\mathbf{E}(\mathbf{r},t)$ is real $\Rightarrow \tilde{\mathbf{E}}(\mathbf{r},\omega) = \tilde{\mathbf{E}}^*(\mathbf{r},-\omega)$

These relations and the notion of the superposition principle, lead to the common treatment :

$$\mathbf{E}(\mathbf{r},t) = \Re\left(\tilde{\mathbf{E}}(\mathbf{r},\omega)e^{-i\omega t}\right) \equiv \frac{1}{2}\left(\tilde{\mathbf{E}}(\mathbf{r},\omega)e^{-i\omega t} + \tilde{\mathbf{E}}^*(\mathbf{r},\omega)e^{i\omega t}\right)$$



Comment on treatment of time-harmonic fields -- continued

Equations for time harmonic fields :

$$\mathbf{E}(\mathbf{r},t) = \Re\left(\tilde{\mathbf{E}}(\mathbf{r},\omega)e^{-i\omega t}\right) \equiv \frac{1}{2}\left(\tilde{\mathbf{E}}(\mathbf{r},\omega)e^{-i\omega t} + \tilde{\mathbf{E}}^*(\mathbf{r},\omega)e^{i\omega t}\right)$$

| Equations | in time domain | in frequency domain |
|---|---|---|
| Coulomb's law : | $\nabla\cdot\mathbf{D} = \rho_{free}$ | $\nabla\cdot\tilde{\mathbf{D}} = \tilde{\rho}_{free}$ |
| Ampere - Maxwell's law : | $\nabla\times\mathbf{H} - \frac{\partial\mathbf{D}}{\partial t} = \mathbf{J}_{free}$ | $\nabla\times\tilde{\mathbf{H}} + i\omega\tilde{\mathbf{D}} = \tilde{\mathbf{J}}_{free}$ |
| Faraday's law : | $\nabla\times\mathbf{E} + \frac{\partial\mathbf{B}}{\partial t} = 0$ | $\nabla\times\tilde{\mathbf{E}} - i\omega\tilde{\mathbf{B}} = 0$ |
| No magnetic monopoles : | $\nabla\cdot\mathbf{B} = 0$ | $\nabla\cdot\tilde{\mathbf{B}} = 0$ |

Note -- in all of these, the real part is taken at the end of the calculation.



Comment on treatment of time-harmonic fields -- continued

Equations for time harmonic fields :

$$\mathbf{E}(\mathbf{r},t) = \Re\left(\tilde{\mathbf{E}}(\mathbf{r},\omega)e^{-i\omega t}\right) \equiv \frac{1}{2}\left(\tilde{\mathbf{E}}(\mathbf{r},\omega)e^{-i\omega t} + \tilde{\mathbf{E}}^*(\mathbf{r},\omega)e^{i\omega t}\right)$$

Poynting vector : $\mathbf{S}(\mathbf{r},t) = \mathbf{E}(\mathbf{r},t)\times\mathbf{H}(\mathbf{r},t)$

$$\mathbf{S}(\mathbf{r},t) = \frac{1}{4}\left(\tilde{\mathbf{E}}(\mathbf{r},\omega)e^{-i\omega t} + \tilde{\mathbf{E}}^*(\mathbf{r},\omega)e^{i\omega t}\right)\times\left(\tilde{\mathbf{H}}(\mathbf{r},\omega)e^{-i\omega t} + \tilde{\mathbf{H}}^*(\mathbf{r},\omega)e^{i\omega t}\right)$$

$$= \frac{1}{4}\left(\tilde{\mathbf{E}}(\mathbf{r},\omega)\times\tilde{\mathbf{H}}^*(\mathbf{r},\omega) + \tilde{\mathbf{E}}^*(\mathbf{r},\omega)\times\tilde{\mathbf{H}}(\mathbf{r},\omega)\right)$$

$$+ \frac{1}{4}\left(\tilde{\mathbf{E}}(\mathbf{r},\omega)\times\tilde{\mathbf{H}}(\mathbf{r},\omega)e^{-2i\omega t} + \tilde{\mathbf{E}}^*(\mathbf{r},\omega)\times\tilde{\mathbf{H}}^*(\mathbf{r},\omega)e^{2i\omega t}\right)$$

$$\langle\mathbf{S}(\mathbf{r},t)\rangle_{t\,\mathrm{avg}} = \Re\left(\frac{1}{2}\left(\tilde{\mathbf{E}}(\mathbf{r},\omega)\times\tilde{\mathbf{H}}^*(\mathbf{r},\omega)\right)\right)$$

Summary and review

# Maxwell's equations

Coulomb's law: $\nabla \cdot \mathbf{D} = \rho_{free}$

Ampere-Maxwell's law: $\nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{J}_{free}$

Faraday's law: $\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$

No magnetic monopoles: $\nabla \cdot \mathbf{B} = 0$



# Maxwell's equations

For linear isotropic media -- $\mathbf{D} = \varepsilon \mathbf{E}$; $\mathbf{B} = \mu \mathbf{H}$
and no sources:

Coulomb's law: $\nabla \cdot \mathbf{E} = 0$

Ampere-Maxwell's law: $\nabla \times \mathbf{B} - \mu\varepsilon \frac{\partial \mathbf{E}}{\partial t} = 0$

Faraday's law: $\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$

No magnetic monopoles: $\nabla \cdot \mathbf{B} = 0$



Analysis of Maxwell's equations without sources -- continued:

Coulomb's law: $\nabla \cdot \mathbf{E} = 0$

Ampere-Maxwell's law: $\nabla \times \mathbf{B} - \mu\varepsilon \frac{\partial \mathbf{E}}{\partial t} = 0$

Faraday's law: $\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$

No magnetic monopoles: $\nabla \cdot \mathbf{B} = 0$

$$\nabla \times \left( \nabla \times \mathbf{B} - \mu\varepsilon \frac{\partial \mathbf{E}}{\partial t} \right) = -\nabla^2 \mathbf{B} - \mu\varepsilon \frac{\partial (\nabla \times \mathbf{E})}{\partial t}$$

$$= -\nabla^2 \mathbf{B} + \mu\varepsilon \frac{\partial^2 \mathbf{B}}{\partial t^2} = 0$$

$$\nabla \times \left( \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} \right) = -\nabla^2 \mathbf{E} + \frac{\partial (\nabla \times \mathbf{B})}{\partial t}$$

$$= -\nabla^2 \mathbf{E} + \mu\varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0$$

Analysis of Maxwell's equations without sources -- continued:
Both E and B fields are solutions to a wave equation:

$$\nabla^2\mathbf{B} - \frac{1}{v^2}\frac{\partial^2\mathbf{B}}{\partial t^2} = 0$$

$$\nabla^2\mathbf{E} - \frac{1}{v^2}\frac{\partial^2\mathbf{E}}{\partial t^2} = 0$$

$$\text{where} \quad v^2 \equiv c^2\frac{\mu_0\varepsilon_0}{\mu\varepsilon} \equiv \frac{c^2}{n^2}$$

Plane wave solutions to wave equation:

$$\mathbf{B}(\mathbf{r},t) = \Re\left(\mathbf{B}_0 e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\right) \qquad \mathbf{E}(\mathbf{r},t) = \Re\left(\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\right)$$



Analysis of Maxwell's equations without sources -- continued:
Plane wave solutions to wave equation:

$$\mathbf{B}(\mathbf{r},t) = \Re\left(\mathbf{B}_0 e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\right) \qquad \mathbf{E}(\mathbf{r},t) = \Re\left(\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\right)$$

$$|\mathbf{k}|^2 = \left(\frac{\omega}{v}\right)^2 = \left(\frac{n\omega}{c}\right)^2 \qquad \text{where } n \equiv \sqrt{\frac{\mu\varepsilon}{\mu_0\varepsilon_0}}$$

Note: $\varepsilon$, $\mu$, $n$, $k$ can all be complex; for the moment we will assume that they are all real (no dissipation).

Note that $\mathbf{E}_0$ and $\mathbf{B}_0$ are not independent;

from Faraday's law: $\nabla\times\mathbf{E} + \frac{\partial\mathbf{B}}{\partial t} = 0$

$$\Rightarrow \mathbf{B}_0 = \frac{\mathbf{k}\times\mathbf{E}_0}{\omega} = \frac{n\hat{\mathbf{k}}\times\mathbf{E}_0}{c}$$

also note: $\hat{\mathbf{k}}\cdot\mathbf{E}_0 = 0$ and $\hat{\mathbf{k}}\cdot\mathbf{B}_0 = 0$



Analysis of Maxwell's equations without sources -- continued:
Summary of plane electromagnetic waves:

$$\mathbf{B}(\mathbf{r},t) = \Re\left(\frac{n\hat{\mathbf{k}}\times\mathbf{E}_0}{c} e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\right) \qquad \mathbf{E}(\mathbf{r},t) = \Re\left(\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\right)$$

$$|\mathbf{k}|^2 = \left(\frac{\omega}{v}\right)^2 = \left(\frac{n\omega}{c}\right)^2 \quad \text{where } n \equiv \sqrt{\frac{\mu\varepsilon}{\mu_0\varepsilon_0}} \quad \text{and } \hat{\mathbf{k}}\cdot\mathbf{E}_0 = 0$$

Poynting vector for plane electromagnetic waves:

$$\langle\mathbf{S}\rangle_{avg} = \frac{1}{2}\Re\left(\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t} \times \frac{1}{\mu}\left(\frac{n\hat{\mathbf{k}}\times\mathbf{E}_0}{c} e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\right)^*\right)$$

$$= \frac{n|\mathbf{E}_0|^2}{2\mu c}\hat{\mathbf{k}} = \frac{1}{2}\sqrt{\frac{\varepsilon}{\mu}}|\mathbf{E}_0|^2\hat{\mathbf{k}}$$

Note that:

$$\mathbf{E}_0 \times \left(\hat{\mathbf{k}}\times\mathbf{E}_0\right) = \hat{\mathbf{k}}\left(\mathbf{E}_0\cdot\mathbf{E}_0\right) - \mathbf{E}_0\left(\hat{\mathbf{k}}\cdot\mathbf{E}_0\right) = \hat{\mathbf{k}}|\mathbf{E}_0|^2$$

## Analysis of Maxwell's equations without sources -- continued:

### Transverse Electric and Magnetic (TEM) waves

Summary of plane electromagnetic waves:

$$\mathbf{B}(\mathbf{r},t)=\Re\left(\frac{n\hat{\mathbf{k}}\times\mathbf{E}_0}{c}e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\right) \qquad \mathbf{E}(\mathbf{r},t)=\Re\left(\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\right)$$

$$|\mathbf{k}|^2=\left(\frac{\omega}{v}\right)^2=\left(\frac{n\omega}{c}\right)^2 \quad \text{where } n\equiv\sqrt{\frac{\mu\varepsilon}{\mu_0\varepsilon_0}} \quad \text{and } \hat{\mathbf{k}}\cdot\mathbf{E}_0=0$$

Energy density for plane electromagnetic waves:

$$\langle u\rangle_{avg}=\frac{1}{4}\Re\left(\varepsilon\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\cdot\left(\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\right)^*\right)+$$

$$\frac{1}{4}\Re\left(\frac{1}{\mu}\frac{n\hat{\mathbf{k}}\times\mathbf{E}_0}{c}e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\cdot\left(\frac{n\hat{\mathbf{k}}\times\mathbf{E}_0}{c}e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\right)^*\right)$$

$$=\frac{1}{2}\varepsilon|\mathbf{E}_0|^2$$



## More general result (example found in waveguides)

1. Suppose that an electromagnetic wave of pure (real) frequency $\omega$ is traveling along the $z$-axis of a wave guide having a square cross section with side dimension $a$ composed of a medium having a real permittivity constant $\epsilon$ and a real permeability constant $\mu$. Suppose that the wave is known to have the form:

$$\mathbf{E}(\mathbf{r},t)=\Re\left\{H_0 e^{ikz-i\omega t}(i\mu\omega)\frac{a}{\pi}\sin\left(\frac{\pi x}{a}\right)\hat{\mathbf{y}}\right\}$$

$$\mathbf{H}(\mathbf{r},t)=\Re\left\{H_0 e^{ikz-i\omega t}\left[-ik\frac{a}{\pi}\sin\left(\frac{\pi x}{a}\right)\hat{\mathbf{x}}+\cos\left(\frac{\pi x}{a}\right)\hat{\mathbf{z}}\right]\right\}.$$

Here $H_0$ denotes a real amplitude, and the parameter $k$ is assumed to be real and equal to

$$k\equiv\sqrt{\mu\epsilon\omega^2-\left(\frac{\pi}{a}\right)^2},$$

for $\mu\epsilon\omega^2>\left(\frac{\pi}{a}\right)^2$.

(a) Show that this wave satisfies the sourceless Maxwell's equations.

(b) Find the form of the time-averaged Poynting vector

$$\langle\mathbf{S}\rangle_{avg}\equiv\frac{1}{2}\Re\left\{\mathbf{E}(\mathbf{r},t)\times\mathbf{H}^*(\mathbf{r},t)\right\}$$

for this electromagnetic wave.