PHY 712 Electrodynamics
9-9:50 AM Olin 105

Plan for Lecture 18:

Complete reading of Chapter 7

1. **Comments on reflectivity of plane waves**
2. **Summary of complex response functions for electromagnetic fields**

15	Mon: 02/18/2019	Chap. 6	Electromagnetic energy and forces	#12	02/20/2019
16	Wed: 02/20/2019	Chap. 7	Electromagnetic plane waves	#13	02/22/2019
17	Fri: 02/22/2019	Chap. 7	Electromagnetic plane waves	#14	02/25/2019
18	Mon: 02/25/2019	Chap. 7	Refractive index		
19	Wed: 02/27/2019				
20	Fri: 03/01/2019				
	Mon: 03/04/2019	No class	*APS March Meeting*	Take Home Exam	
	Wed: 03/06/2019	No class	*APS March Meeting*	Take Home Exam	
	Fri: 03/08/2019	No class	*APS March Meeting*	Take Home Exam	
	Mon: 03/11/2019	No class	*Spring Break*		
	Wed: 03/13/2019	No class	*Spring Break*		
	Fri: 03/15/2019	No class	*Spring Break*		
21	Mon: 03/18/2019				
22	Wed: 03/20/2019				
23	Fri: 03/22/2019				

Some comments on the Fresnel Equations

1. Different behaviors of *s* and *p* polarization
2. Brewster's angle
3. Total internal reflection

Review: Electromagnetic plane waves in isotropic medium with real permeability and permittivity: $\mu\ \varepsilon$.

$$\mathbf{E}(\mathbf{r},t)=\Re\left(\mathbf{E}_0 e^{i\frac{\omega}{c}\left(n\hat{\mathbf{k}}\cdot\mathbf{r}-ct\right)}\right) \qquad n^2=c^2\mu\varepsilon$$

$$\mathbf{B}(\mathbf{r},t)=\frac{n}{c}\hat{\mathbf{k}}\times\mathbf{E}(\mathbf{r},t)=\sqrt{\mu\varepsilon}\,\hat{\mathbf{k}}\times\mathbf{E}(\mathbf{r},t)$$

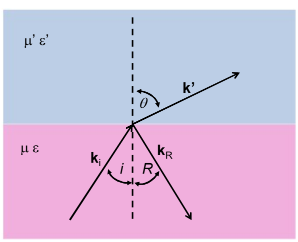
Poynting vector for plane electromagnetic waves:

$$\langle\mathbf{S}\rangle_{avg}=\frac{n|\mathbf{E}_0|^2}{2\mu c}\hat{\mathbf{k}}=\frac{1}{2}\sqrt{\frac{\varepsilon}{\mu}}|\mathbf{E}_0|^2\hat{\mathbf{k}}$$

Energy density for plane electromagnetic waves:

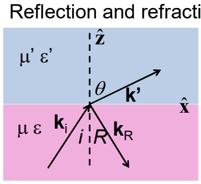
$$\langle u\rangle_{avg}=\frac{1}{2}\varepsilon|\mathbf{E}_0|^2$$

Review:
Reflection and refraction of plane electromagnetic waves at a plane interface between dielectrics (assumed to be lossless)



$$n'=\varepsilon'\mu'$$
$$n=\varepsilon\mu$$
$$i=R$$
$$n\sin i=n'\sin\theta$$
$$|\mathbf{k}_i|=|\mathbf{k}_R|=n\frac{\omega}{c}$$
$$|\mathbf{k}'|=n'\frac{\omega}{c}$$

Review:
Reflection and refraction between two isotropic media



Reflectance, transmittance:

$$\mathrm{R}=\frac{\mathbf{S}_R\cdot\hat{\mathbf{z}}}{\mathbf{S}_i\cdot\hat{\mathbf{z}}}=\left|\frac{E_{0R}}{E_{0i}}\right|^2 \qquad \mathrm{T}=\frac{\mathbf{S}'\cdot\hat{\mathbf{z}}}{\mathbf{S}_i\cdot\hat{\mathbf{z}}}=\left|\frac{E'_0}{E_{0i}}\right|^2\frac{n'}{n}\frac{\mu}{\mu'}\frac{\cos\theta}{\cos i}$$

Note that $\mathrm{R}+\mathrm{T}=1$

For s-polarization (E perpendicular to plane of incidence)

$$\frac{E_{0R}}{E_{0i}} = \frac{n\cos i - \frac{\mu}{\mu'} n'\cos\theta}{n\cos i + \frac{\mu}{\mu'} n'\cos\theta} \qquad \frac{E'_0}{E_{0i}} = \frac{2n\cos i}{n\cos i + \frac{\mu}{\mu'} n'\cos\theta}$$

Note that: $n'\cos\theta = \sqrt{n'^2 - n^2\sin^2 i}$

For p-polarization (E in plane of incidence)

$$\frac{E_{0R}}{E_{0i}} = \frac{\frac{\mu}{\mu'} n'^2 \cos i - nn'\cos\theta}{\frac{\mu}{\mu'} n'^2 \cos i + nn'\cos\theta} \qquad \frac{E'_0}{E_{0i}} = \frac{2nn'\cos i}{\frac{\mu}{\mu'} n'^2 \cos i + nn'\cos\theta}$$

Note that: $n'\cos\theta = \sqrt{n'^2 - n^2\sin^2 i}$

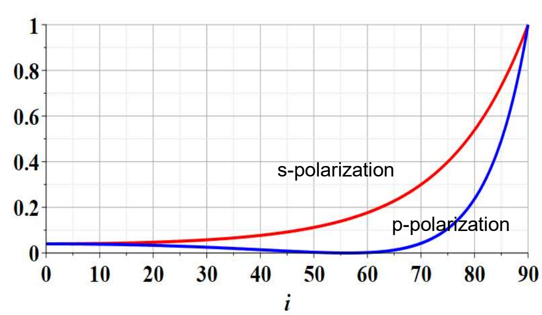
Reflectance for s-polarization

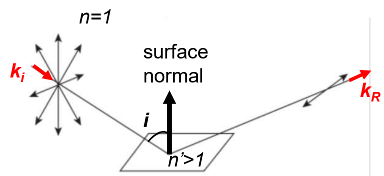
$$R_s = \left|\frac{E_{0R}}{E_{0i}}\right|^2 = \left|\frac{n\cos i - \frac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin^2 i}}{n\cos i + \frac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin^2 i}}\right|^2$$

Reflectance for p-polarization

$$R_p = \left|\frac{E_{0R}}{E_{0i}}\right|^2 = \left|\frac{\frac{\mu}{\mu'} n'\cos i - \frac{n}{n'}\sqrt{n'^2 - n^2\sin^2 i}}{\frac{\mu}{\mu'} n'\cos i + \frac{n}{n'}\sqrt{n'^2 - n^2\sin^2 i}}\right|^2$$

Example for $\mu = \mu'$; $n = 1$ and $n' = 1.5$



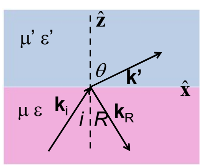


Polarization due to reflection from a refracting surface

Brewster's angle: for $i = i_B$, $R_p(i_B) = 0$

$$R_p = \left|\frac{E_{0R}}{E_{0i}}\right|^2 = \left|\frac{\frac{\mu}{\mu'}n'\cos i - \frac{n}{n'}\sqrt{n'^2 - n^2\sin^2 i}}{\frac{\mu}{\mu'}n'\cos i + \frac{n}{n'}\sqrt{n'^2 - n^2\sin^2 i}}\right|^2 \qquad \text{For } \mu' = \mu, \quad i_B = \tan^{-1}\left(\frac{n'}{n}\right)$$

Reflection and refraction between two isotropic media -- continued



For each wave:

$$\mathbf{E}(\mathbf{r},t) = \Re\left(\mathbf{E}_0 e^{i\frac{\omega}{c}\left(n\hat{\mathbf{k}}\cdot\mathbf{r} - ct\right)}\right) \qquad n^2 = c^2\mu\varepsilon$$

$$\mathbf{B}(\mathbf{r},t) = \frac{n}{c}\hat{\mathbf{k}}\times\mathbf{E}(\mathbf{r},t) = \sqrt{\mu\varepsilon}\,\hat{\mathbf{k}}\times\mathbf{E}(\mathbf{r},t)$$

Matching condition at interface:

$$n'\cos\theta = \sqrt{n'^2 - n^2\sin^2 i}$$

Total internal reflection:

If $n > n'$, for $i > i_0 \equiv \sin^{-1}\left(\frac{n'}{n}\right)$,

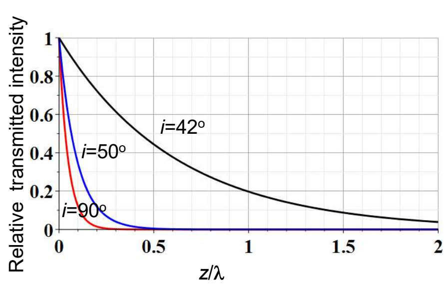
refracted field no longer propagates in medium $\mu'\varepsilon'$

$$n'\cos\theta = i\sqrt{n^2\sin^2 i - n'^2} = i\,n\sqrt{\frac{\sin^2 i}{\sin^2 i_0} - 1}$$

$$\mathbf{E}'(\mathbf{r},t) = e^{-\left(\frac{n\omega}{c}\sqrt{\frac{\sin^2 i}{\sin^2 i_0}-1}\right)z}\,\Re\left(\mathbf{E}'_0\, e^{i\frac{\omega}{c}\left(n\hat{\mathbf{k}}_\parallel\cdot\mathbf{r} - ct\right)}\right)$$

Example of total internal reflection

$n'=1$ and $n=1.5$ ➔ $i_0 = \sin^{-1}(1/1.5) = 41.81°$



Transmitted illumination confined within a few wavelengths of the surface.

TIRF (total internal reflection fluorescence)

www.nikon.com/products/microscope-solutions/bioscience.../nikon_note_10_lr.pdf

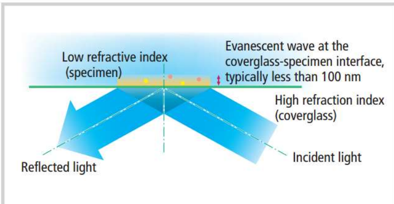


Figure 1: Creation of an evanescent wave at the coverglass-specimen interface

Design of TIRF device using laser and high power lens

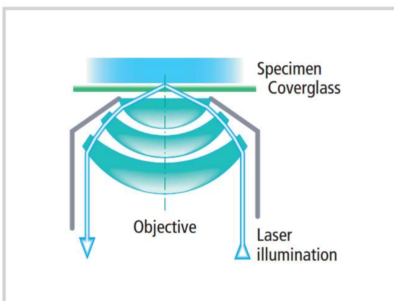


Figure 2: Through-the-lens laser TIRF.

PMC full text: Curr Protoc Cytom. Author manuscript; available in PMC 2015 Aug 18.
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Curr Protoc Cytom. 2009 Oct; 0 12: Unit12.18.
doi: 10.1002/0471142956.cy1218s50
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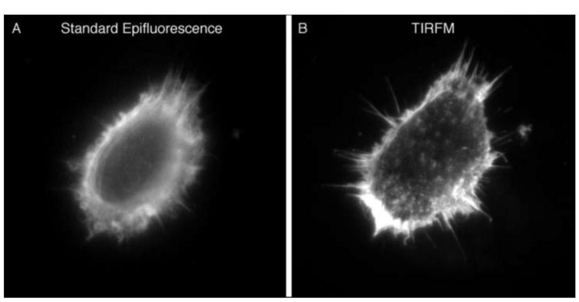
Figure 1



Special case: normal incidence (i=0, θ=0)

$$\frac{E_{0R}}{E_{0i}} = \frac{\frac{\mu}{\mu'}n'-n}{\frac{\mu}{\mu'}n'+n} \qquad \frac{E'_0}{E_{0i}} = \frac{2n}{\frac{\mu}{\mu'}n'+n}$$

Reflectance, transmittance:

$$\mathrm{R} = \left|\frac{E_{0R}}{E_{0i}}\right|^2 = \left|\frac{\frac{\mu}{\mu'}n'-n}{\frac{\mu}{\mu'}n'+n}\right|^2$$

$$\mathrm{T} = \left|\frac{E'_0}{E_{0i}}\right|^2 \frac{n'}{n}\frac{\mu}{\mu'} = \left|\frac{2n}{\frac{\mu}{\mu'}n'+n}\right|^2 \frac{n'}{n}\frac{\mu}{\mu'}$$

02/25/2019 PHY 712 Spring 2019 -- Lecture 18 16

Extension to complex refractive index $n = n_R + i\, n_I$

Suppose $\mu = \mu'$, n = real, $n' = n'_R + i n'_I$

Reflectance at normal incidence:

$$\mathrm{R} = \left|\frac{E_{0R}}{E_{0i}}\right|^2 = \left|\frac{\frac{\mu}{\mu'}n'-n}{\frac{\mu}{\mu'}n'+n}\right|^2 = \frac{(n'_R-n)^2+(n'_I)^2}{(n'_R+n)^2+(n'_I)^2}$$

Note that for $n'_I >> |n'_R \pm n|$:

$$\mathrm{R} \approx 1$$

02/25/2019 PHY 712 Spring 2019 -- Lecture 18 17

Origin of imaginary contributions to permittivity --
Review: Drude model dielectric function:

$$\frac{\varepsilon(\omega)}{\varepsilon_0} = 1 + N\sum_i f_i \frac{q_i^2}{\varepsilon_0 m_i}\frac{1}{\omega_i^2-\omega^2-i\omega\gamma_i}$$

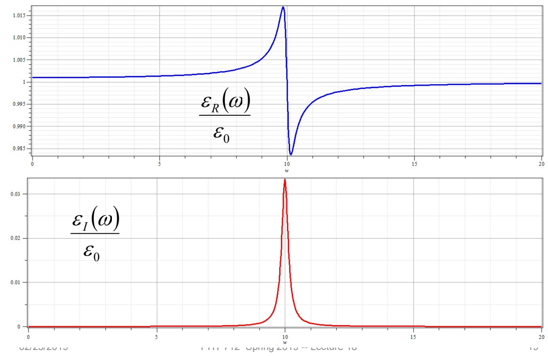
$$= \frac{\varepsilon_R(\omega)}{\varepsilon_0} + i\frac{\varepsilon_I(\omega)}{\varepsilon_0}$$

$$\frac{\varepsilon_R(\omega)}{\varepsilon_0} = 1 + N\sum_i f_i \frac{q_i^2}{\varepsilon_0 m_i}\frac{\omega_i^2-\omega^2}{(\omega_i^2-\omega^2)^2+\omega^2\gamma_i^2}$$

$$\frac{\varepsilon_I(\omega)}{\varepsilon_0} = N\sum_i f_i \frac{q_i^2}{\varepsilon_0 m_i}\frac{\omega\gamma_i}{(\omega_i^2-\omega^2)^2+\omega^2\gamma_i^2}$$

02/25/2019 PHY 712 Spring 2019 -- Lecture 18 18

Drude model dielectric function:



Drude model dielectric function – some analytic properties:

$$\frac{\varepsilon(\omega)}{\varepsilon_0} = 1 + N\sum_i f_i \frac{q_i^2}{\varepsilon_0 m_i} \frac{1}{\omega_i^2 - \omega^2 - i\omega\gamma_i}$$

$$\text{For } \omega >> \omega_i \quad \frac{\varepsilon(\omega)}{\varepsilon_0} \approx 1 - \frac{1}{\omega^2}\left(N\sum_i f_i \frac{q_i^2}{\varepsilon_0 m_i}\right)$$

$$\equiv 1 - \frac{\omega_P^2}{\omega^2}$$

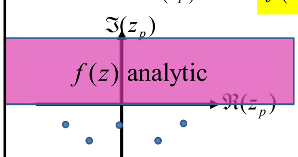
Analysis for Drude model dielectric function – continued -- Analytic properties:

$$f(z) = \frac{\varepsilon(z)}{\varepsilon_0} - 1 = N\sum_i f_i \frac{q_i^2}{\varepsilon_0 m_i} \frac{1}{\omega_i^2 - z^2 - iz\gamma_i}$$

$f(z)$ has poles z_P at $\omega_i^2 - z_P{}^2 - iz_P\gamma_i = 0$

$$z_P = -i\frac{\gamma_i}{2} \pm \sqrt{\omega_i^2 - \left(\frac{\gamma_i}{2}\right)^2}$$

Note that $\Im(z_P) \leq 0 \;\Rightarrow f(z)$ is analytic for $\Im(z_P) > 0$



Because of these analytic properties, Cauchy's integral theorem results in:

Kramers-Kronig transform – for dielectric function:

$$\frac{\varepsilon_R(\omega)}{\varepsilon_0} - 1 = \frac{1}{\pi} P \int_{-\infty}^{\infty} d\omega' \frac{\varepsilon_I(\omega')}{\varepsilon_0} \frac{1}{\omega' - \omega}$$

$$\frac{\varepsilon_I(\omega)}{\varepsilon_0} = -\frac{1}{\pi} P \int_{-\infty}^{\infty} d\omega' \left(\frac{\varepsilon_R(\omega')}{\varepsilon_0} - 1 \right) \frac{1}{\omega' - \omega}$$

with $\varepsilon_R(-\omega) = \varepsilon_R(\omega); \quad \varepsilon_I(-\omega) = -\varepsilon_I(\omega)$

Further comments on analytic behavior of dielectric function

"Causal" relationship between **E** and **D** fields:

$$\mathbf{D}(\mathbf{r},t) = \varepsilon_0 \left\{ \mathbf{E}(\mathbf{r},t) + \int_0^{\infty} d\tau \, G(\tau) \mathbf{E}(\mathbf{r}, t-\tau) \right\}$$

$$\frac{\varepsilon(\omega)}{\varepsilon_0} - 1 = \int_0^{\infty} d\tau \, G(\tau) e^{i\omega\tau}$$

Some details: Consider a convolution integral such as

$f(t) = \int_{-\infty}^{\infty} g(t') h(t-t') dt'$ where the functions $f(t), g(t),$ and $h(t)$

are all well-defined functions with Fourier transforms such as

$$\tilde{f}(\omega) = \int_{-\infty}^{\infty} f(t') e^{i\omega t'} dt' \qquad f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(\omega) e^{-i\omega t} d\omega$$

It follows that: $\tilde{f}(\omega) = \tilde{g}(\omega) \tilde{h}(\omega)$

Further comments on analytic behavior of dielectric function

"Causal" relationship between **E** and **D** fields:

$$\mathbf{D}(\mathbf{r},t) = \varepsilon_0 \left\{ \mathbf{E}(\mathbf{r},t) + \int_0^{\infty} d\tau \; G(\tau) \mathbf{E}(\mathbf{r}, t-\tau) \right\}$$

$$G(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\frac{\varepsilon(\omega)}{\varepsilon_0} - 1 \right) e^{-i\omega\tau} d\omega \qquad \tilde{G}(\omega) = \frac{\varepsilon(\omega)}{\varepsilon_0} - 1 = \int_0^{\infty} d\tau \; G(\tau) e^{i\omega\tau}$$

For $\displaystyle \frac{\varepsilon(\omega)}{\varepsilon_0} - 1 = \frac{N}{\varepsilon_0} \sum_i f_i \frac{q_i^2}{m_i} \frac{1}{\omega_i^2 - \omega^2 - i\omega\gamma_i}$

$$G(\tau) = \frac{N}{\varepsilon_0} \sum_i f_i \frac{q_i^2}{m_i} e^{-\gamma_i \tau/2} \frac{\sin(\nu_i \tau)}{\nu_i} \Theta(\tau)$$

where $\nu_i \equiv \sqrt{\omega_i^2 - \gamma_i^2/4}$

Some details

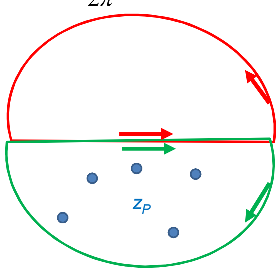
$$G(\tau)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{\varepsilon(\omega)}{\varepsilon_0}-1\right)e^{-i\omega\tau}d\omega=\frac{1}{2\pi}\oint f(z)e^{-iz\tau}dz$$

Let $f(z)=\frac{\varepsilon(z)}{\varepsilon_0}-1=N\sum_i f_i\frac{q_i^2}{\varepsilon_0 m_i}\frac{1}{\omega_i^2-z^2-iz\gamma_i}$

$f(z)$ has poles z_P at $\omega_i^2-z_P{}^2-iz_P\gamma_i=0$

$$z_P=-i\frac{\gamma_i}{2}\pm\sqrt{\omega_i^2-\left(\frac{\gamma_i}{2}\right)^2}\quad\text{or}\quad z_P=-i\left(\frac{\gamma_i}{2}\pm\sqrt{\left(\frac{\gamma_i}{2}\right)^2-\omega_i^2}\right)$$

$$G(\tau)=\frac{1}{2\pi}\oint f(z)e^{-iz\tau}dz=i\sum_P \mathrm{Res}(z_P)$$



For $\tau>0$, $e^{-iz\tau}\to\infty$ no

For $\tau<0$, $e^{-iz\tau}\to 0$ ok

$$G(\tau)=0\quad\text{for } \tau<0$$

For $\tau>0$, $e^{-iz\tau}\to 0$ ok

For $\tau<0$, $e^{-iz\tau}\to\infty$ no

$$G(\tau)=\frac{1}{2\pi}\oint f(z)e^{-iz\tau}dz=i\sum_P \mathrm{Res}(z_P)$$

Let $f(z)=\frac{\varepsilon(z)}{\varepsilon_0}-1=N\sum_i f_i\frac{q_i^2}{\varepsilon_0 m_i}\frac{1}{\omega_i^2-z^2-iz\gamma_i}$

$f(z)$ has poles z_P at $\omega_i^2-z_P{}^2-iz_P\gamma_i=0$

$$z_P=-i\frac{\gamma_i}{2}\pm\sqrt{\omega_i^2-\left(\frac{\gamma_i}{2}\right)^2}\quad\text{or}\quad z_P=-i\left(\frac{\gamma_i}{2}\pm\sqrt{\left(\frac{\gamma_i}{2}\right)^2-\omega_i^2}\right)$$

$$G(\tau)=\frac{N}{\varepsilon_0}\sum_i f_i\frac{q_i^2}{m_i}e^{-\gamma_i\tau/2}\frac{\sin(\nu_i\tau)}{\nu_i}\Theta(\tau)$$

where $\nu_i\equiv\sqrt{\omega_i^2-\gamma_i^2/4}$ assuming $\omega_i^2-\gamma_i^2/4\geq 0$