

PHY 712 Electrodynamics
9-9:50 AM MWF Olin 105

Plan for Lecture 21:
Sources of radiation

Start reading Chap. 9

A. Electromagnetic waves due to specific sources

B. Dipole radiation patterns

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16	Wed: 02/20/2019	Chap. 7	Electromagnetic plane waves	#13	02/22/2019
17	Fri: 02/22/2019	Chap. 7	Electromagnetic plane waves	#14	02/25/2019
18	Mon: 02/25/2019	Chap. 7	Refractive index		
19	Wed: 02/27/2019	Chap. 8	EM waves in wave guides		
20	Fri: 03/01/2019	Chap. 1-8	Review		
	Mon: 03/04/2019	No class	APS March Meeting	Take Home Exam	
	Wed: 03/06/2019	No class	APS March Meeting	Take Home Exam	
	Fri: 03/08/2019	No class	APS March Meeting	Take Home Exam	
	Mon: 03/11/2019	No class	Spring Break		
	Wed: 03/13/2019	No class	Spring Break		
	Fri: 03/15/2019	No class	Spring Break		
21	Mon: 03/18/2019	Chap. 9	Radiation from localized oscillating sources	#15	3/22/2019
22	Wed: 03/20/2019				
23	Fri: 03/22/2019				
24	Mon: 03/25/2019				
25	Wed: 03/27/2019				
26	Fri: 03/29/2019				

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Wake Forest College & Graduate School of Arts and Sciences

WFU Physics

WFU Physics People Events and News Undergraduate Graduate Research Resources

Events

Colloquium: "Calculus Meets Electromagnetism and Thermodynamics: A Tale of Two Disciplines" - Wednesday, March 20, 2019, at 4:00 PM
 Speaker: David Dray, PhD, Department of Mathematics, Oregon State University
 George P. Williams, Jr., Professor, Conoco Phillips School of Geology and Geophysics, University of Oklahoma
 TIME: Wednesday, March 20, 2019, at 4:00 PM. There will be a reception with...

Career Advising Event - 11:30am - 1:30pm
 Wednesday, March 27, 2019, from 12:00 - 1:00 PM
 WFU Physics Career Advising Event
 SPEAKER: Dr. Heather Belle, Assistant Professor, Conoco Phillips School of Geology and Geophysics, University of Oklahoma
 TIME: Wednesday, March 27, 2019, from 12:00 - 1:00 PM

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Maxwell's equations

Microscopic or vacuum form ($\mathbf{P} = 0$; $\mathbf{M} = 0$):

Coulomb's law : $\nabla \cdot \mathbf{E} = \rho / \epsilon_0$

Ampere - Maxwell's law : $\nabla \times \mathbf{B} - \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{J}$

Faraday's law : $\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$

No magnetic monopoles : $\nabla \cdot \mathbf{B} = 0$

$$\Rightarrow c^2 = \frac{1}{\epsilon_0 \mu_0}$$

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Formulation of Maxwell's equations in terms of vector and scalar potentials

$$\nabla \cdot \mathbf{B} = 0 \quad \Rightarrow \quad \mathbf{B} = \nabla \times \mathbf{A}$$

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \quad \Rightarrow \quad \nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} \right) = 0$$

$$\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\nabla \Phi$$

or $\mathbf{E} = -\nabla \Phi - \frac{\partial \mathbf{A}}{\partial t}$

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Formulation of Maxwell's equations in terms of vector and scalar potentials -- continued

Lorentz gauge form -- require: $\nabla \cdot \mathbf{A}_L + \frac{1}{c^2} \frac{\partial \Phi_L}{\partial t} = 0$

$$-\nabla^2 \Phi_L + \frac{1}{c^2} \frac{\partial^2 \Phi_L}{\partial t^2} = \rho / \epsilon_0$$

$$-\nabla^2 \mathbf{A}_L + \frac{1}{c^2} \frac{\partial^2 \mathbf{A}_L}{\partial t^2} = \mu_0 \mathbf{J}$$

General equation form :

$$\left(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \Psi = -4\pi f$$

$$\Psi(\mathbf{r}, t) = \begin{cases} \Phi(\mathbf{r}, t) \\ A_x(\mathbf{r}, t) \\ A_y(\mathbf{r}, t) \\ A_z(\mathbf{r}, t) \end{cases} \quad f(\mathbf{r}, t) = \begin{cases} \rho(\mathbf{r}, t) / (4\pi\epsilon_0) \\ \mu_0 J_x(\mathbf{r}, t) / (4\pi) \\ \mu_0 J_y(\mathbf{r}, t) / (4\pi) \\ \mu_0 J_z(\mathbf{r}, t) / (4\pi) \end{cases}$$

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Solution of Maxwell's equations in the Lorentz gauge -- continued

$$G(\mathbf{r}, t; \mathbf{r}', t') = \frac{1}{|\mathbf{r} - \mathbf{r}'|} \delta\left(t' - \left(t - \frac{|\mathbf{r} - \mathbf{r}'|}{c}\right)\right)$$

Solution for field $\Psi(\mathbf{r}, t)$:

$$\Psi(\mathbf{r}, t) = \Psi_{f=0}(\mathbf{r}, t) +$$

$$\int d^3r' \int dt' \frac{1}{|\mathbf{r} - \mathbf{r}'|} \delta\left(t' - \left(t - \frac{1}{c}|\mathbf{r} - \mathbf{r}'|\right)\right) f(\mathbf{r}', t')$$

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Electromagnetic waves from time harmonic sources

Charge density : $\rho(\mathbf{r}, t) = \Re(\tilde{\rho}(\mathbf{r}, \omega)e^{-i\omega t})$

Current density : $\mathbf{J}(\mathbf{r}, t) = \Re(\tilde{\mathbf{J}}(\mathbf{r}, \omega)e^{-i\omega t})$

Note that the continuity condition :

$$\frac{\partial \rho(\mathbf{r}, t)}{\partial t} + \nabla \cdot \mathbf{J}(\mathbf{r}, t) = 0 \Rightarrow -i\omega \tilde{\rho}(\mathbf{r}, \omega) + \nabla \cdot \tilde{\mathbf{J}}(\mathbf{r}, \omega) = 0$$

General source : $f(\mathbf{r}, t) = \Re(\tilde{f}(\mathbf{r}, \omega)e^{-i\omega t})$

For $\tilde{f}(\mathbf{r}, \omega) = \frac{1}{4\pi\epsilon_0} \tilde{\rho}(\mathbf{r}, \omega)$

or $\tilde{f}(\mathbf{r}, \omega) = \frac{\mu_0}{4\pi} \tilde{\mathbf{J}}_i(\mathbf{r}, \omega)$

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Electromagnetic waves from time harmonic sources -- continued:

$$\Psi(\mathbf{r}, t) = \Psi_{f=0}(\mathbf{r}, t) +$$

$$\int d^3r' \int dt' \frac{1}{|\mathbf{r} - \mathbf{r}'|} \delta\left(t' - \left(t - \frac{1}{c}|\mathbf{r} - \mathbf{r}'|\right)\right) f(\mathbf{r}', t')$$

$$\tilde{\Psi}(\mathbf{r}, \omega)e^{-i\omega t} = \tilde{\Psi}_{f=0}(\mathbf{r}, \omega)e^{-i\omega t} +$$

$$\int d^3r' \int dt' \frac{1}{|\mathbf{r} - \mathbf{r}'|} \delta\left(t' - \left(t - \frac{1}{c}|\mathbf{r} - \mathbf{r}'|\right)\right) \tilde{f}(\mathbf{r}', \omega)e^{-i\omega t'}$$

$$= \tilde{\Psi}_{f=0}(\mathbf{r}, \omega)e^{-i\omega t} + \int d^3r' \frac{e^{i\frac{\omega}{c}|\mathbf{r} - \mathbf{r}'|}}{|\mathbf{r} - \mathbf{r}'|} \tilde{f}(\mathbf{r}', \omega)e^{-i\omega t}$$

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Electromagnetic waves from time harmonic sources – continued:

For scalar potential (Lorentz gauge, $k \equiv \frac{\omega}{c}$)

$$\tilde{\Phi}(\mathbf{r}, \omega) = \tilde{\Phi}_0(\mathbf{r}, \omega) + \frac{1}{4\pi\epsilon_0} \int d^3r' \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \tilde{\rho}(\mathbf{r}', \omega),$$

where $\left(\nabla^2 + \frac{\omega^2}{c^2}\right)\tilde{\Phi}_0(\mathbf{r}, \omega) = 0$

For vector potential (Lorentz gauge, $k \equiv \frac{\omega}{c}$)

$$\tilde{\mathbf{A}}(\mathbf{r}, \omega) = \tilde{\mathbf{A}}_0(\mathbf{r}, \omega) + \frac{\mu_0}{4\pi} \int d^3r' \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \tilde{\mathbf{J}}(\mathbf{r}', \omega),$$

where $\left(\nabla^2 + \frac{\omega^2}{c^2}\right)\tilde{\mathbf{A}}_0(\mathbf{r}, \omega) = 0$

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Electromagnetic waves from time harmonic sources – continued:

Useful expansion :

$$\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} = ik \sum_{lm} j_l(kr_<) h_l(kr_>) Y_{lm}(\hat{\mathbf{r}}) Y_{lm}^*(\hat{\mathbf{r}}')$$

Spherical Bessel function : $j_l(kr)$

Spherical Hankel function : $h_l(kr) = j_l(kr) + in_l(kr)$

$$\tilde{\Phi}(\mathbf{r}, \omega) = \tilde{\Phi}_0(\mathbf{r}, \omega) + \sum_{lm} \tilde{\phi}_{lm}(r, \omega) Y_{lm}(\hat{\mathbf{r}})$$

$$\tilde{\phi}_{lm}(r, \omega) = \frac{ik}{\epsilon_0} \int d^3r' \tilde{\rho}(\mathbf{r}', \omega) j_l(kr_<) h_l(kr_>) Y_{lm}^*(\hat{\mathbf{r}}')$$

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Electromagnetic waves from time harmonic sources – continued:

Useful expansion :

$$\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} = ik \sum_{lm} j_l(kr_<) h_l(kr_>) Y_{lm}(\hat{\mathbf{r}}) Y_{lm}^*(\hat{\mathbf{r}}')$$

Spherical Bessel function : $j_l(kr)$

Spherical Hankel function : $h_l(kr) = j_l(kr) + in_l(kr)$

$$\tilde{\mathbf{A}}(\mathbf{r}, \omega) = \tilde{\mathbf{A}}_0(\mathbf{r}, \omega) + \sum_{lm} \tilde{\mathbf{a}}_{lm}(r, \omega) Y_{lm}(\hat{\mathbf{r}})$$

$$\tilde{\mathbf{a}}_{lm}(r, \omega) = ik\mu_0 \int d^3r' \tilde{\mathbf{J}}(\mathbf{r}', \omega) j_l(kr_<) h_l(kr_>) Y_{lm}^*(\hat{\mathbf{r}}')$$

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Forms of spherical Bessel and Hankel functions:

$$j_0(x) = \frac{\sin(x)}{x} \qquad h_0(x) = \frac{e^{ix}}{ix}$$

$$j_1(x) = \frac{\sin(x)}{x^2} - \frac{\cos(x)}{x} \qquad h_1(x) = -\left(1 + \frac{i}{x}\right) \frac{e^{ix}}{x}$$

$$j_2(x) = \left(\frac{3}{x^3} - \frac{1}{x}\right) \sin(x) - \frac{3\cos(x)}{x^2} \qquad h_2(x) = i\left(1 + \frac{3i}{x} - \frac{3}{x^2}\right) \frac{e^{ix}}{x}$$

Asymptotic behavior:

$$x \ll 1 \quad \Rightarrow \quad j_l(x) \approx \frac{(x)^l}{(2l+1)!!}$$

$$x \gg 1 \quad \Rightarrow \quad h_l(x) \approx (-i)^{l+1} \frac{e^{ix}}{x}$$

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Digression on spherical Bessel functions --
 Consider the homogeneous wave equation

$$\left(\nabla^2 + \frac{\omega^2}{c^2}\right) \tilde{\Phi}_0(\mathbf{r}, \omega) = 0$$

Suppose $\tilde{\Phi}_0(\mathbf{r}, \omega) = \psi_{lm}(r) Y_{lm}(\hat{\mathbf{r}})$
 $\Rightarrow \psi_{lm}(r)$ must satisfy the following for $k = \omega / c$:

$$\left(\frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} - \frac{l(l+1)}{r^2} + k^2\right) \psi_{lm}(r) = 0$$

General Bessel function equation:

$$\left(\frac{d^2}{dx^2} + \frac{2}{x} \frac{d}{dx} - \frac{l(l+1)}{x^2} + 1\right) w_l(x) = 0$$

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Electromagnetic waves from time harmonic sources –
 continued:

$$\tilde{\Phi}(\mathbf{r}, \omega) = \tilde{\Phi}_0(\mathbf{r}, \omega) + \sum_{lm} \tilde{\phi}_{lm}(r, \omega) Y_{lm}(\hat{\mathbf{r}})$$

$$\tilde{\phi}_{lm}(r, \omega) = \frac{ik}{\epsilon_0} \int d^3r' \tilde{\rho}(\mathbf{r}', \omega) j_l(kr_<) h_l(kr_>) Y_{lm}^*(\hat{\mathbf{r}}')$$

$$\tilde{\mathbf{A}}(\mathbf{r}, \omega) = \tilde{\mathbf{A}}_0(\mathbf{r}, \omega) + \sum_{lm} \tilde{\mathbf{a}}_{lm}(r, \omega) Y_{lm}(\hat{\mathbf{r}})$$

$$\tilde{\mathbf{a}}_{lm}(r, \omega) = ik\mu_0 \int d^3r' \tilde{\mathbf{J}}(\mathbf{r}', \omega) j_l(kr_<) h_l(kr_>) Y_{lm}^*(\hat{\mathbf{r}}')$$

For $r \gg$ (extent of source)

$$\tilde{\phi}_{lm}(r, \omega) \approx \frac{ik}{\epsilon_0} h_l(kr) \int d^3r' \tilde{\rho}(\mathbf{r}', \omega) j_l(kr') Y_{lm}^*(\hat{\mathbf{r}}')$$

$$\tilde{\mathbf{a}}_{lm}(r, \omega) \approx ik\mu_0 h_l(kr) \int d^3r' \tilde{\mathbf{J}}(\mathbf{r}', \omega) j_l(kr') Y_{lm}^*(\hat{\mathbf{r}}')$$

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Some details:

$$\tilde{\Phi}(\mathbf{r}, \omega) = \tilde{\Phi}_0(\mathbf{r}, \omega) + \sum_{lm} \tilde{\phi}_{lm}(r, \omega) Y_{lm}^*(\hat{\mathbf{r}})$$

$$\tilde{\phi}_{lm}(r, \omega) = \frac{ik}{\epsilon_0} \int d^3 r' \tilde{\rho}(\mathbf{r}', \omega) j_l(kr_\zeta) h_l(kr_\rho) Y_{lm}^*(\hat{\mathbf{r}}')$$

$$= \frac{ik}{\epsilon_0} \int d\Omega Y_{lm}^*(\hat{\mathbf{r}}') \left(h_l(kr) \int_0^r r'^2 dr' j_l(kr') \tilde{\rho}(\mathbf{r}', \omega) + j_l(kr) \int_r^\infty r'^2 dr' h_l(kr') \tilde{\rho}(\mathbf{r}', \omega) \right)$$

For $r \gg$ (extent of source)

$$\tilde{\phi}_{lm}(r, \omega) \approx \frac{ik}{\epsilon_0} h_l(kr) \int d^3 r' \tilde{\rho}(\mathbf{r}', \omega) j_l(kr') Y_{lm}^*(\hat{\mathbf{r}}')$$

$$\tilde{\mathbf{a}}_{lm}(r, \omega) \approx ik\mu_0 h_l(kr) \int d^3 r' \tilde{\mathbf{J}}(\mathbf{r}', \omega) j_l(kr') Y_{lm}^*(\hat{\mathbf{r}}')$$

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Electromagnetic waves from time harmonic sources – continued -- some details:

$$\tilde{\phi}_{lm}(r, \omega) = \frac{ik}{\epsilon_0} \int d^3 r' \tilde{\rho}(\mathbf{r}', \omega) j_l(kr_\zeta) h_l(kr_\rho) Y_{lm}^*(\hat{\mathbf{r}}')$$

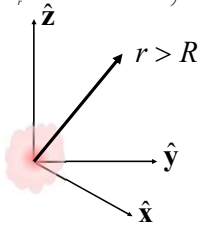
$$= \frac{ik}{\epsilon_0} \left(h_l(kr) \int_0^r r'^2 dr' \rho_{lm}(\mathbf{r}', \omega) j_l(kr') + j_l(kr) \int_r^\infty r'^2 dr' \rho_{lm}(\mathbf{r}', \omega) h_l(kr') \right)$$

where $\rho_{lm}(\mathbf{r}', \omega) \equiv \int d\Omega Y_{lm}(\hat{\mathbf{r}}') \rho(\mathbf{r}', \omega)$

note that for $r > R$, where $\rho_{lm}(\mathbf{r}, \omega) \approx 0$,

$$\tilde{\phi}_{lm}(r, \omega) \approx \frac{ik}{\epsilon_0} h_l(kr) \int_0^\infty r'^2 dr' \rho_{lm}(\mathbf{r}', \omega) j_l(kr')$$

Similar relationships can be written for $\tilde{\mathbf{a}}_{lm}(r, \omega)$ and $\tilde{\mathbf{J}}(\mathbf{r}', \omega)$.



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Electromagnetic waves from time harmonic sources – continued:

For $r \gg$ (extent of source)

$$\tilde{\phi}_{lm}(r, \omega) \approx \frac{ik}{\epsilon_0} h_l(kr) \int d^3 r' \tilde{\rho}(\mathbf{r}', \omega) j_l(kr') Y_{lm}^*(\hat{\mathbf{r}}')$$

$$\tilde{\mathbf{a}}_{lm}(r, \omega) \approx ik\mu_0 h_l(kr) \int d^3 r' \tilde{\mathbf{J}}(\mathbf{r}', \omega) j_l(kr') Y_{lm}^*(\hat{\mathbf{r}}')$$

Note that $\tilde{\rho}(\mathbf{r}', \omega)$ and $\tilde{\mathbf{J}}(\mathbf{r}', \omega)$ are connected via the continuity condition: $-i\omega \tilde{\rho}(\mathbf{r}, \omega) + \nabla \cdot \tilde{\mathbf{J}}(\mathbf{r}, \omega) = 0$

$$\tilde{\phi}_{lm}(r, \omega) \approx \frac{ik}{\epsilon_0} h_l(kr) \int d^3 r' \tilde{\rho}(\mathbf{r}', \omega) j_l(kr') Y_{lm}^*(\hat{\mathbf{r}}')$$

$$= -\frac{k}{\omega\epsilon_0} h_l(kr) \int d^3 r' \tilde{\mathbf{J}}(\mathbf{r}', \omega) \cdot \nabla' (j_l(kr') Y_{lm}^*(\hat{\mathbf{r}}'))$$

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Electromagnetic waves from time harmonic sources – continued:

Various approximations:

$$kr \gg 1 \Rightarrow h_l(kr) \approx (-i)^{l+1} \frac{e^{ikr}}{kr}$$

$$kr \ll 1 \Rightarrow j_l(kr) \approx \frac{(kr)^l}{(2l+1)!!}$$

Lowest (non-trivial) contributions in l expansions:

$$\tilde{\varphi}_{1m}(r, \omega) \approx \frac{ik}{\epsilon_0} h_1(kr) \int d^3r' \tilde{\rho}(\mathbf{r}', \omega) \frac{kr'}{3} Y_{1m}^*(\hat{\mathbf{r}}')$$

$$\tilde{\mathbf{a}}_{00}(r, \omega) \approx ik\mu_0 h_0(kr) \int d^3r' \tilde{\mathbf{J}}(\mathbf{r}', \omega) Y_{00}^*(\hat{\mathbf{r}}')$$

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Some details:

Lowest (non-trivial) contributions in l expansions:

$$\tilde{\varphi}_{1m}(r, \omega) \approx \frac{ik}{\epsilon_0} h_1(kr) \int d^3r' \tilde{\rho}(\mathbf{r}', \omega) \frac{kr'}{3} Y_{1m}^*(\hat{\mathbf{r}}')$$

$$\tilde{\mathbf{a}}_{00}(r, \omega) \approx ik\mu_0 h_0(kr) \int d^3r' \tilde{\mathbf{J}}(\mathbf{r}', \omega) Y_{00}^*(\hat{\mathbf{r}}')$$

In analogy to the electrostatic case --

electric dipole moment: $\mathbf{p}(\omega) \equiv \int d^3r' \mathbf{r}' \tilde{\rho}(\mathbf{r}', \omega)$

Note that $\mathbf{p}(\omega) \cdot \hat{\mathbf{r}} = \frac{4\pi}{3} \sum_{m=-1}^1 \left(\int d^3r' r' \tilde{\rho}(\mathbf{r}', \omega) Y_{1m}^*(\hat{\mathbf{r}}') \right) Y_{1m}(\hat{\mathbf{r}})$

$$\Rightarrow \sum_{m=-1}^1 \tilde{\varphi}_{1m}(r, \omega) Y_{1m}(\hat{\mathbf{r}}) \approx \frac{ik^2}{4\pi\epsilon_0} h_1(kr) \mathbf{p}(\omega) \cdot \hat{\mathbf{r}}$$

Similarly: $\tilde{\mathbf{a}}_{00}(r, \omega) Y_{00}(\hat{\mathbf{r}}) \approx \frac{\mu_0}{4\pi} ik h_0(kr) \int d^3r' \tilde{\mathbf{J}}(\mathbf{r}', \omega)$

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Some details -- continued: (assuming confined source)

Recall continuity condition: $-i\omega \tilde{\rho}(\mathbf{r}, \omega) + \nabla \cdot \tilde{\mathbf{J}}(\mathbf{r}, \omega) = 0$

$$-i\omega \tilde{\rho}(\mathbf{r}, \omega) + \mathbf{r} \cdot \nabla \tilde{\rho}(\mathbf{r}, \omega)$$

$$\int d^3r \mathbf{r} \tilde{\rho}(\mathbf{r}, \omega) = \frac{1}{i\omega} \int d^3r \mathbf{r} \nabla \cdot \tilde{\mathbf{J}}(\mathbf{r}, \omega)$$

$$= -\frac{1}{i\omega} \int d^3r \tilde{\mathbf{J}}(\mathbf{r}, \omega) = \mathbf{p}(\omega)$$

Here we have used the identity:

$$\nabla \cdot (\psi \mathbf{V}) = \nabla \psi \cdot \mathbf{V} + \psi (\nabla \cdot \mathbf{V})$$

We have also assumed that

$$\lim_{r \rightarrow \infty} (r \tilde{\mathbf{J}}(\mathbf{r}, \omega)) = 0$$

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Electromagnetic waves from time harmonic sources – continued:

Lowest order contribution; dipole radiation:

Define dipole moment at frequency ω :

$$\mathbf{p}(\omega) \equiv \int d^3r \mathbf{r} \tilde{\rho}(\mathbf{r}, \omega) = -\frac{1}{i\omega} \int d^3r \tilde{\mathbf{J}}(\mathbf{r}, \omega)$$

$$\tilde{\mathbf{A}}(\mathbf{r}, \omega) = -\frac{i\mu_0\omega}{4\pi} \mathbf{p}(\omega) \frac{e^{ikr}}{r}$$

$$\tilde{\Phi}(\mathbf{r}, \omega) = -\frac{ik}{4\pi\epsilon_0} \mathbf{p}(\omega) \cdot \hat{\mathbf{r}} \left(1 + \frac{i}{kr}\right) \frac{e^{ikr}}{r}$$

Note: in this case we have assumed a restricted extent of the source such that $kr \ll 1$.

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Electromagnetic waves from time harmonic sources – continued:

$$\tilde{\mathbf{E}}(\mathbf{r}, \omega) = -\nabla \tilde{\Phi}(\mathbf{r}, \omega) + i\omega \tilde{\mathbf{A}}(\mathbf{r}, \omega)$$

$$= \frac{1}{4\pi\epsilon_0} \frac{e^{ikr}}{r} \left(k^2 ((\hat{\mathbf{r}} \times \mathbf{p}(\omega)) \times \hat{\mathbf{r}}) + \left(\frac{3\hat{\mathbf{r}}(\hat{\mathbf{r}} \cdot \mathbf{p}(\omega)) - \mathbf{p}(\omega)}{r^2} \right) (1 - ikr) \right)$$

$$\tilde{\mathbf{B}}(\mathbf{r}, \omega) = \nabla \times \tilde{\mathbf{A}}(\mathbf{r}, \omega)$$

$$= \frac{1}{4\pi\epsilon_0 c^2} \frac{e^{ikr}}{r} k^2 (\hat{\mathbf{r}} \times \mathbf{p}(\omega)) \left(1 - \frac{1}{ikr}\right)$$

Power radiated for $kr \gg 1$:

$$\frac{dP}{d\Omega} = r^2 \hat{\mathbf{r}} \cdot \langle \mathbf{S} \rangle_{\text{avg}} = \frac{r^2 \hat{\mathbf{r}}}{2\mu_0} \hat{\mathbf{r}} \cdot \Re(\tilde{\mathbf{E}}(\mathbf{r}, \omega) \times \tilde{\mathbf{B}}^*(\mathbf{r}, \omega))$$

$$= \frac{c^2 k^4}{32\pi^2} \sqrt{\frac{\mu_0}{\epsilon_0}} |(\hat{\mathbf{r}} \times \mathbf{p}(\omega)) \times \hat{\mathbf{r}}|^2$$

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Example of dipole radiation source

$$\tilde{\mathbf{J}}(\mathbf{r}, \omega) = \hat{\mathbf{z}} J_0 e^{-r/R} \quad \tilde{\rho}(\mathbf{r}, \omega) = \frac{J_0}{-i\omega R} \cos\theta e^{-r/R}$$

$$\tilde{\mathbf{A}}(\mathbf{r}, \omega) = \hat{\mathbf{z}} J_0 (ik\mu_0) \int_0^\infty r'^2 dr' e^{-r'/R} h_0(kr_>) j_0(kr_<)$$

$$\tilde{\Phi}(\mathbf{r}, \omega) = -\frac{J_0 k}{\epsilon_0 \omega R} \cos\theta \int_0^\infty r'^2 dr' e^{-r'/R} h_1(kr_>) j_1(kr_<)$$

Evaluation for $r \gg R$:

$$\tilde{\mathbf{A}}(\mathbf{r}, \omega) = \hat{\mathbf{z}} J_0 \mu_0 \frac{e^{ikr}}{r} \frac{2R^3}{(1+k^2 R^2)^2}$$

$$\tilde{\Phi}(\mathbf{r}, \omega) = \frac{J_0 k}{\epsilon_0 \omega} \cos\theta \frac{e^{ikr}}{r} \left(1 + \frac{i}{kr}\right) \frac{2R^3}{(1+k^2 R^2)^2}$$

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Example of dipole radiation source -- continued
 Evaluation for $r \gg R$:

$$\tilde{\mathbf{A}}(\mathbf{r}, \omega) = \hat{\mathbf{z}} J_0 \mu_0 \frac{e^{ikr}}{r} \frac{2R^3}{(1+k^2R^2)^2}$$

$$\tilde{\Phi}(\mathbf{r}, \omega) = \frac{J_0 k}{\varepsilon_0 \omega} \cos\theta \frac{e^{ikr}}{r} \left(1 + \frac{i}{kr}\right) \frac{2R^3}{(1+k^2R^2)^2}$$

Relationship to pure dipole approximation (exact when $kR \rightarrow 0$)

$$\mathbf{p}(\omega) \equiv \int d^3r \mathbf{r} \tilde{\rho}(\mathbf{r}, \omega) = -\frac{1}{i\omega} \int d^3r \tilde{\mathbf{J}}(\mathbf{r}, \omega) = -\frac{8\pi R^3 J_0}{i\omega} \hat{\mathbf{z}}$$

Corresponding dipole fields: $\tilde{\mathbf{A}}(\mathbf{r}, \omega) = -\frac{i\mu_0 \omega}{4\pi} \mathbf{p}(\omega) \frac{e^{ikr}}{r}$

$$\tilde{\Phi}(\mathbf{r}, \omega) = -\frac{ik}{4\pi\varepsilon_0} \mathbf{p}(\omega) \cdot \hat{\mathbf{r}} \left(1 + \frac{i}{kr}\right) \frac{e^{ikr}}{r}$$

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