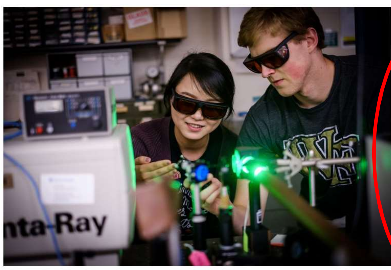
PHY 712 Electrodynamics
9-9:50 AM MWF Olin 105
Plan for Lecture 4:

Reading: Chapter 1 - 3 in JDJ

Electrostatic potentials

1. One, two, and three dimensions (Cartesian coordinates)
2. Mean value theorem for the electrostatic potential



Events

Colloquium: "Gravitational Wave Astrophysics: A New Era of Discovery." – January 23, 2019, at 4:00 PM
Jessica McIver, PhD, Senior Postdoctoral Scholar in Physics, Caltech George P. Williams, Jr. Lecture Hall, (Olin 101) Wednesday, January 23, 2018, at 4:00 PM There will be a reception with ...

Colloquium: "Wearable Smart Interfaces: A Pervasive Power Interaction with the Environment" Thursday, January 24, 2019, at 2:00 PM
Jun Chen, PhD, Postdoctoral Research Fellow, Department of Materials Science and Engineering, Stanford University George P. Williams, Jr. Lecture Hall, (Olin 101) Wednesday, January 24, 2018, at 2:00 PM There ...

Course schedule for Spring 2018

(Preliminary schedule -- subject to frequent adjustment.)

	Lecture date	JDJ Reading	Topic	HW	Due date
1	Mon: 01/14/2019	Chap. 1 & Appen.	Introduction, units and Poisson equation	#1	01/23/2019
2	Wed: 01/16/2019	Chap. 1	Electrostatic energy calculations	#2	01/23/2019
3	Fri: 01/18/2019	Chap. 1	Electrostatic potentials and fields	#3	01/23/2019
	Mon: 01/21/2019	No class	Martin Luther King Holiday		
4	Wed: 01/23/2019	Chap. 1 - 3	Poisson's equation in 2 and 3 dimensions		
5	Fri: 01/25/2019	Chap. 1 - 3	Brief introduction to numberical methods	#4	01/28/2019
6	Mon: 01/28/2019				
7	Wed: 01/30/2019				
8	Fri: 02/01/2019				
9	Mon: 02/04/2019				
10	Wed: 02/06/2019				
11	Fri: 02/08/2019				
12	Mon: 02/11/2019				
13	Wed: 02/13/2019				
14	Fri: 02/15/2019				

Poisson Equation

$$\nabla^2 \Phi_P(\mathbf{r}) = -\frac{\rho(\mathbf{r})}{\varepsilon_0}$$

Solution to Poisson equation using Green's function $G(\mathbf{r},\mathbf{r}')$:

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_V d^3r'\,\rho(\mathbf{r}')G(\mathbf{r},\mathbf{r}') +$$
$$\frac{1}{4\pi}\int_S d^2r'\left[G(\mathbf{r},\mathbf{r}')\nabla'\Phi(\mathbf{r}') - \Phi(\mathbf{r}')\nabla' G(\mathbf{r},\mathbf{r}')\right]\cdot\hat{\mathbf{r}}'.$$

Poisson equation for one-dimensional system

$$\frac{d^2\Phi_P(x)}{dx^2} = -\frac{\rho(x)}{\varepsilon_0}$$

Example solution:

$$\Phi_P(x) = \frac{1}{4\pi\varepsilon_0}\int_{-\infty}^{\infty} G(x,x')\rho(x')dx' + C_1 + C_2 x$$

where $G(x,x') = 4\pi x_<$ where $x_<$ is the smaller of x and x';
C_1 and C_2 are constants.

Check:

$$\Phi_P(x) = \frac{1}{\varepsilon_0}\left\{\int_{-\infty}^{x} x'\rho(x')dx' + x\int_x^{\infty}\rho(x')dx'\right\} + C_1 + C_2 x$$

$$\frac{d\Phi_P(x)}{dx} = \frac{1}{\varepsilon_0}\int_x^{\infty}\rho(x')dx' + C_2 \quad \Rightarrow \frac{d^2\Phi_P(x)}{dx^2} = -\frac{\rho(x)}{\varepsilon_0}$$

General procedure for constructing Green's function for one-dimensional system using 2 independent solutions of the homogeneous equations

Consider two independent solutions to the homogeneous equation

$$\nabla^2\phi_i(x) = 0$$

where $i = 1$ or 2. Let

$$G(x,x') = \frac{4\pi}{W}\phi_1(x_<)\phi_2(x_>).$$

This notation means that $x_<$ should be taken as the smaller of x and x' and $x_>$ should be taken as the larger.

"Wronskian": $W \equiv \frac{d\phi_1(x)}{dx}\phi_2(x) - \phi_1(x)\frac{d\phi_2(x)}{dx}.$

Beautiful method; but only works in one dimension.

Orthogonal function expansions and Green's functions

Suppose we have a "complete" set of orthogonal functions $\{u_n(x)\}$ defined in the interval $x_1 \leq x \leq x_2$ such that

$$\int_{x_1}^{x_2} u_n(x) u_m(x)\, dx = \delta_{nm}.$$

We can show that the completeness of this functions implies that

$$\sum_{n=1}^{\infty} u_n(x) u_n(x') = \delta(x - x').$$

This relation allows us to use these functions to represent a Green's function for our system. For the 1-dimensional Poisson equation, the Green's function satisfies

$$\frac{\partial^2}{\partial x^2} G(x, x') = -4\pi\delta(x - x').$$

Orthogonal function expansion -- continued

Suppose the orthogonal functions satisfy an eigenvalue equation:

$$\frac{d^2}{dx^2} u_n(x) = -\alpha_n u_n(x)$$

where the functions $u_n(x)$ also satisfy the appropriate boundary conditions, then we can construct the Green's function:

$$G(x, x') = 4\pi \sum_n \frac{u_n(x) u_n(x')}{\alpha_n}.$$

Check:

$$\frac{d^2}{dx^2} G(x, x') = 4\pi \sum_n \frac{\left(-\alpha_n u_n(x)\right) u_n(x')}{\alpha_n} = -4\pi \sum_n u_n(x) u_n(x')$$

$$= -4\pi\delta(x - x')$$

Example

For example, consider the previous example in the interval $-a \leq x \leq a$:

$$\rho(x) = \begin{cases} 0 & \text{for } x < -a \\ -\rho_0 & \text{for } -a < x < 0 \\ +\rho_0 & \text{for } 0 < x < a \\ 0 & \text{for } x > a \end{cases}$$

We want to solve the Poisson equation with boundary condition $d\Phi(-a)/dx = 0$ and $d\Phi(a)/dx = 0$. We may choose

$u_n(x) = \sqrt{\frac{1}{a}} \sin\left(\frac{[2n+1]\pi x}{2a}\right)$ and the corresponding Green's function

$$G(x, x') = \frac{4\pi}{a} \sum_{n=0}^{\infty} \frac{\sin\left(\frac{[2n+1]\pi x}{2a}\right) \sin\left(\frac{[2n+1]\pi x'}{2a}\right)}{\left(\frac{[2n+1]\pi}{2a}\right)^2}.$$

Example -- continued

This form of the one-dimensional Green's function only allows us to find a solution to the Poisson equation within the interval $-a \le x \le a$ from

$$\Phi(x) = \frac{1}{4\pi\varepsilon_0}\int_{-a}^{a} dx'\, G(x,x')\rho(x') + C_1$$

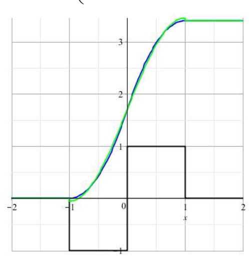
$$\Rightarrow \Phi(x) = \frac{\rho_0 a^2}{\epsilon_0}\left(16\sum_{n=0}^{\infty}\frac{\sin\left(\frac{[2n+1]\pi x}{2a}\right)}{([2n+1]\pi)^3} + \frac{1}{2}\right),$$

choosing C_1 so that $\Phi(-a) = 0$.

$$\text{Exact result:}\quad \Phi(x) = \begin{cases} 0 & \text{for } x < -a \\ \frac{\rho_0}{2\varepsilon_0}(x+a)^2 & \text{for } -a < x < 0 \\ -\frac{\rho_0}{2\varepsilon_0}(x-a)^2 + \frac{\rho_0 a^2}{\varepsilon_0} & \text{for } 0 < x < a \\ \frac{\rho_0}{\varepsilon_0}a^2 & \text{for } x > a \end{cases}$$

Example -- continued

$$\Phi(x) = \frac{\rho_0 a^2}{\epsilon_0}\left(16\sum_{n=0}^{\infty}\frac{\sin\left(\frac{[2n+1]\pi x}{2a}\right)}{([2n+1]\pi)^3} + \frac{1}{2}\right)$$



Orthogonal function expansions in 2 and 3 dimensions

$$\nabla^2\Phi(\mathbf{r}) \equiv \frac{\partial^2\Phi(\mathbf{r})}{\partial x^2} + \frac{\partial^2\Phi(\mathbf{r})}{\partial y^2} + \frac{\partial^2\Phi(\mathbf{r})}{\partial z^2} = -\rho(\mathbf{r})/\epsilon_0.$$

Let $\{u_n(x)\}$, $\{v_n(y)\}$, $\{w_n(z)\}$ denote complete orthogonal function sets in the x, y, and z dimensions, respectively. The Green's function construction becomes:

$$G(x,x',y,y',z,z') = 4\pi\sum_{lmn}\frac{u_l(x)u_l(x')v_m(y)v_m(y')w_n(z)w_n(z')}{\alpha_l + \beta_m + \gamma_n},$$

where

$$\frac{d^2}{dx^2}u_l(x) = -\alpha_l u_l(x),\ \frac{d^2}{dy^2}v_m(x) = -\beta_m v_m(y),\ \text{and}\ \frac{d^2}{dz^2}w_n(z) = -\gamma_n w_n(z).$$

(See Eq. 3.167 in Jackson for example.)

Combined orthogonal function expansion and homogeneous solution construction of Green's function in 2 and 3 dimensions.

An alternative method of finding Green's functions for a second order ordinary differential equations (in 1 dimension) is based on a product of two independent solutions of the homogeneous equation, $\phi_1(x)$ and $\phi_2(x)$:

$$G(x,x') = K\phi_1(x_<)\phi_2(x_>), \text{ where } K \equiv \frac{4\pi}{\frac{d\phi_1}{dx}\phi_2 - \phi_1\frac{d\phi_2}{dx}},$$

where $x_<$ denotes the smaller of x and x'.

For the two and three dimensional cases, we can use this technique in one of the dimensions in order to reduce the number of summation terms. These ideas are discussed in Section 3.11 of Jackson.

Green's function construction -- continued

For the two dimensional case, for example, we can assume that the Green's function can be written in the form:

$$G(x,x',y,y') = \sum_n u_n(x)u_n(x')g_n(y,y').$$

The y dependence of this equation will have the required behavior, if we choose: $\left[-\alpha_n + \frac{\partial^2}{\partial y^2}\right]g_n(y,y') = -4\pi\delta(y-y'),$

which in turn can be expressed in terms of the two independent solutions $v_{n_1}(y)$ and $v_{n_2}(y)$ of the homogeneous equation:

$$\left[\frac{d^2}{dy^2} - \alpha_n\right]v_{n_i}(y) = 0,$$

and the Wronskian constant: $K_n \equiv \frac{dv_{n_1}}{dy}v_{n_2} - v_{n_1}\frac{dv_{n_2}}{dy}$

$$\left[-\alpha_n + \frac{\partial^2}{\partial y^2}\right]g_n(y,y') = -4\pi\delta(y-y'),$$

$$g_n(y,y') = \frac{4\pi}{K_n}v_{n_1}(y_<)v_{n_2}(y_>)$$

where: $\left[\frac{d^2}{dy^2} - \alpha_n\right]v_{n_i}(y) = 0,$

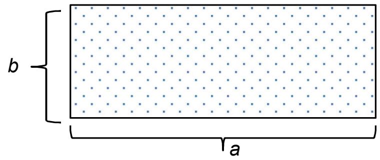
and $K_n \equiv \frac{dv_{n_1}}{dy}v_{n_2} - v_{n_1}\frac{dv_{n_2}}{dy}$

For example, choose $v_{n_1}(y) = \sinh(\sqrt{\alpha_n}y)$ and $v_{n_2}(y) = \sinh(\sqrt{\alpha_n}(b-y))$

where $K_n = \sqrt{\alpha_n}\sinh(\sqrt{\alpha_n}b)$

using the identity: $\cosh(r)\sinh(s) + \sinh(r)\cosh(s) = \sinh(r+s)$

Example:



Two dimensional box with sides a and b with boundary conditions: *Φ(0,y)=Φ(a,y)=Φ(x,0)=Φ(x,b)=0*

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_V d^3r'\,\rho(\mathbf{r}')G(\mathbf{r},\mathbf{r}') +$$

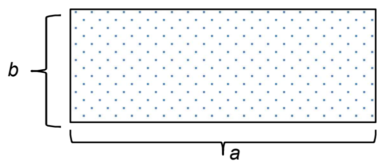
Don't know this term

Know this term

$$\frac{1}{4\pi}\int_S d^2r'\left[G(\mathbf{r},\mathbf{r}')\nabla'\Phi(\mathbf{r}') - \Phi(\mathbf{r}')\nabla' G(\mathbf{r},\mathbf{r}')\right]\cdot\hat{\mathbf{r}}'.$$

$$G(x,x',y,y') = \sum_n u_n(x)u_n(x')\frac{4\pi}{K_n}v_{n_1}(y_<)v_{n_2}(y_>).$$

Example:



Two dimensional box with sides a and b with boundary conditions: *Φ(0,y)=Φ(a,y)=Φ(x,0)=Φ(x,b)=0*

For this type of problem, it is prudent to construct $G(x,x',y,y')$ so that it vanishes on the boundary:

$G(x,x',y,0) = G(x,x',y,b) = G(x,0,y,y') = G(x,a,y,b') = 0$

$$G(x,x',y,y') = \sum_n u_n(x)u_n(x')\frac{4\pi}{K_n}v_{n_1}(y_<)v_{n_2}(y_>).$$

$$\frac{d^2}{dx^2}u_n(x) = -\alpha_n u_n(x) \quad \text{where} \quad u_n(0) = u_n(a) = 0$$

$$\Rightarrow u_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right) \qquad \alpha_n = \left(\frac{n\pi}{a}\right)^2$$

$$\left[\frac{d^2}{dy^2} - \left(\frac{n\pi}{a}\right)^2\right]v_{n_i}(y) = 0$$

$$v_{n_1}(y) = \sinh\left(\frac{n\pi}{a}y\right) \qquad v_{n_2}(y) = \sinh\left(\frac{n\pi}{a}(b-y)\right)$$

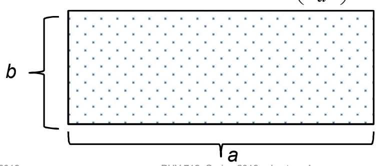
$$K_n = \frac{n\pi}{a}\sinh\left(\frac{n\pi b}{a}\right)$$

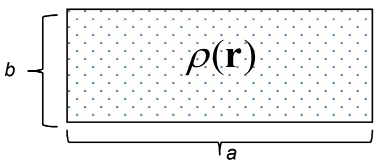
Green's function construction -- continued

$$G(x,x',y,y') = \sum_n u_n(x)u_n(x')K_n v_{n_1}(y_<)v_{n_2}(y_>).$$

For example, a Green's function for a two-dimensional rectangular system with $0 \le x \le a$ and $0 \le y \le b$, which vanishes on the rectangular boundaries:

$$G(x,x',y,y') = 8\sum_{n=1}^{\infty} \frac{\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{n\pi x'}{a}\right)\sinh\left(\frac{n\pi y_<}{a}\right)\sinh\left(\frac{n\pi}{a}(b-y_>)\right)}{n\sinh\left(\frac{n\pi b}{a}\right)}.$$





$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_V d^3r'\,\rho(\mathbf{r}')G(\mathbf{r},\mathbf{r}') +$$

$$\frac{1}{4\pi}\int_S d^2r'\left[G(\mathbf{r},\mathbf{r}')\nabla'\Phi(\mathbf{r}') - \Phi(\mathbf{r}')\nabla' G(\mathbf{r},\mathbf{r}')\right]\cdot\hat{\mathbf{r}}'.$$

$$G(x,x',y,y') = 8\sum_{n=1}^{\infty} \frac{\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{n\pi x'}{a}\right)\sinh\left(\frac{n\pi y_<}{a}\right)\sinh\left(\frac{n\pi}{a}(b-y_>)\right)}{n\sinh\left(\frac{n\pi b}{a}\right)}.$$

$$G(x,x',y,y') = 8\sum_{n=1}^{\infty} \frac{\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{n\pi x'}{a}\right)\sinh\left(\frac{n\pi y_<}{a}\right)\sinh\left(\frac{n\pi}{a}(b-y_>)\right)}{n\sinh\left(\frac{n\pi b}{a}\right)}.$$

Example: $\rho(x,y) = \rho_0 \sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi y}{b}\right)$

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_V d^3r'\,\rho(\mathbf{r}')G(\mathbf{r},\mathbf{r}')$$

In this example, only n=1 contributes because

$$\int_0^a dx'\sin\left(\frac{\pi x'}{a}\right)\sin\left(\frac{n\pi x'}{a}\right) = \frac{a}{2}\delta_{1n}$$

$$\Phi(x,y) = \frac{8\rho_0}{4\pi\epsilon_0}\frac{a}{2\sinh(\pi a/b)}\sin\left(\frac{\pi x}{a}\right)\times$$

$$\left(\sinh\left(\frac{\pi(b-y)}{a}\right)\int_0^y dy'\sin\left(\frac{\pi y'}{b}\right)\sinh\left(\frac{\pi y'}{a}\right) + \sinh\left(\frac{\pi y}{a}\right)\int_y^b dy'\sin\left(\frac{\pi y'}{b}\right)\sinh\left(\frac{\pi(b-y')}{a}\right)\right)$$

When the dust clears: $\Phi(x,y) = \frac{\rho_0}{\epsilon_0}\frac{a^2b^2}{\pi^2(a^2+b^2)}\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi y}{b}\right)$

A useful theorem for electrostatics
The mean value theorem (Problem 1.10 in Jackson)

The "mean value theorem" value theorem (problem 1.10 of your textbook) states that the value of $\Phi(\mathbf{r})$ at the arbitrary (charge-free) point $\mathbf{r}$ is equal to the average of $\Phi(\mathbf{r}')$ over the surface of any sphere centered on the point $\mathbf{r}$ (see Jackson problem #1.10). One way to prove this theorem is the following. Consider a point $\mathbf{r}' = \mathbf{r} + \mathbf{u}$, where $\mathbf{u}$ will describe a sphere of radius R about the fixed point $\mathbf{r}$. We can make a Taylor series expansion of the electrostatic potential $\Phi(\mathbf{r}')$ about the fixed point $\mathbf{r}$:

$$\Phi(\mathbf{r}+\mathbf{u}) = \Phi(\mathbf{r}) + \mathbf{u}\cdot\nabla\Phi(\mathbf{r}) + \frac{1}{2!}(\mathbf{u}\cdot\nabla)^2\Phi(\mathbf{r}) + \frac{1}{3!}(\mathbf{u}\cdot\nabla)^3\Phi(\mathbf{r}) + \frac{1}{4!}(\mathbf{u}\cdot\nabla)^4\Phi(\mathbf{r}) + \cdots . \tag{1}$$

According to the premise of the theorem, we want to integrate both sides of the equation 1 over a sphere of radius R in the variable $\mathbf{u}$:

$$\int_{\text{sphere}} dS_u = R^2 \int_0^{2\pi} d\phi_u \int_{-1}^{+1} d\cos(\theta_u). \tag{2}$$

Mean value theorem – continued

We note that

$$R^2 \int_0^{2\pi} d\phi_u \int_{-1}^{+1} d\cos(\theta_u) 1 = 4\pi R^2,$$

$$R^2 \int_0^{2\pi} d\phi_u \int_{-1}^{+1} d\cos(\theta_u)\mathbf{u}\cdot\nabla = 0,$$

$$R^2 \int_0^{2\pi} d\phi_u \int_{-1}^{+1} d\cos(\theta_u)(\mathbf{u}\cdot\nabla)^2 = \frac{4\pi R^4}{3}\nabla^2,$$

$$R^2 \int_0^{2\pi} d\phi_u \int_{-1}^{+1} d\cos(\theta_u)(\mathbf{u}\cdot\nabla)^3 = 0,$$

and

$$R^2 \int_0^{2\pi} d\phi_u \int_{-1}^{+1} d\cos(\theta_u)(\mathbf{u}\cdot\nabla)^4 = \frac{4\pi R^6}{5}\nabla^4.$$

Since $\nabla^2\Phi(\mathbf{r}) = 0$, the only non-zero term of the average is thus the first term:

$$R^2 \int_0^{2\pi} d\phi_u \int_{-1}^{+1} d\cos(\theta_u)\Phi(\mathbf{r}+\mathbf{u}) = 4\pi R^2\Phi(\mathbf{r}),$$

or

$$\Phi(\mathbf{r}) = \frac{1}{4\pi R^2}\; R^2 \int_0^{2\pi} d\phi_u \int_{-1}^{+1} d\cos(\theta_u)\Phi(\mathbf{r}+\mathbf{u}) \equiv \frac{1}{4\pi R^2}\int_{\text{sphere}} dS_u \Phi(\mathbf{r}+\mathbf{u}).$$

Since this result is independent of the radius R, we see that we have the theorem.

Summary: Mean value theorem

$$\Phi(\mathbf{r}) = \frac{1}{4\pi R^2}\int R^2 d\Omega\; \Phi(\mathbf{r}+\mathbf{u})$$