PHY 712 Electrodynamics
9-9:50 AM MWF Olin 105

Plan for Lecture 6:

Continue reading Chapters 2 & 3

1. **Methods of images -- planes, spheres**
2. **Solution of Poisson equation in for other geometries -- cylindrical**

Course schedule for Spring 2019

(Preliminary schedule -- subject to frequent adjustment.)

	Lecture date	JDJ Reading	Topic	HW	Due date
1	Mon: 01/14/2019	Chap. 1 & Appen.	Introduction, units and Poisson equation	#1	01/23/2019
2	Wed: 01/16/2019	Chap. 1	Electrostatic energy calculations	#2	01/23/2019
3	Fri: 01/18/2019	Chap. 1	Electrostatic potentials and fields	#3	01/23/2019
	Mon: 01/21/2019	No class	Martin Luther King Holiday		
4	Wed: 01/23/2019	Chap. 1 - 3	Poisson's equation in 2 and 3 dimensions		
5	Fri: 01/25/2019	Chap. 1 - 3	Brief introduction to numerical methods	#4	01/28/2019
6	Mon: 01/28/2019	Chap. 2 & 3	Image charge constructions	#5	01/30/2019
7	Wed: 01/30/2019				
8	Fri: 02/01/2019				
9	Mon: 02/04/2019				
10	Wed: 02/06/2019				
11	Fri: 02/08/2019				
12	Mon: 02/11/2019				
13	Wed: 02/13/2019				
14	Fri: 02/15/2019				

Survey of mathematical techniques for analyzing electrostatics – the Poisson equation

$$\nabla^2\Phi(\mathbf{r}) = -\frac{\rho(\mathbf{r})}{\varepsilon_0}$$

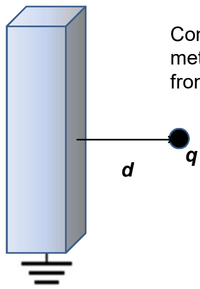
1. Direct solution of differential equation
2. Solution by means of an integral equation; Green's function techniques
3. Orthogonal function expansions
4. Numerical methods (finite differences and finite element methods)
5. Method of images

Method of images

Clever trick for specialized geometries:

- Flat plane (surface)
- Sphere

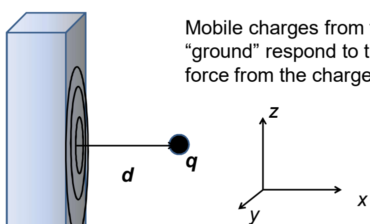
Planar case:



Consider a grounded metal sheet, a distance ***d*** from a point charge ***q***.

A grounded metal sheet, a distance ***d*** from a point charge ***q***.

Mobile charges from the "ground" respond to the force from the charge ***q***.



$$\Phi(x=0,y,z)=0$$

A grounded metal sheet, a distance ***d*** from a point charge ***q***.

$$\nabla^2\Phi = -\frac{q}{\varepsilon_0}\delta^3(\mathbf{r}-d\hat{\mathbf{x}})$$

$$\Phi(x=0,y,z)=0$$

Trick for $x \geq 0$:

$$\Phi(x\geq 0,y,z)=\frac{1}{4\pi\varepsilon_0}\left(\frac{q}{|\mathbf{r}-d\hat{\mathbf{x}}|}-\frac{q}{|\mathbf{r}+d\hat{\mathbf{x}}|}\right)$$

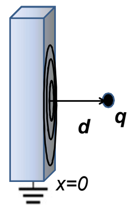
Surface charge density:

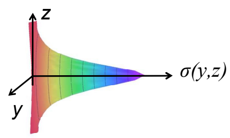
$$\sigma(y,z)=\varepsilon_0 E(0,y,z)=-\varepsilon_0\frac{d\Phi(0,y,z)}{dx}=-\frac{q}{4\pi}\left(\frac{2d}{\left(d^2+y^2+z^2\right)^{3/2}}\right)$$

A grounded metal sheet, a distance ***d*** from a point charge ***q***.

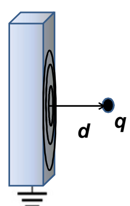
$$\text{Surface charge density:} \quad \sigma(y,z) = -\frac{q}{4\pi}\left(\frac{2d}{\left(d^2+y^2+z^2\right)^{3/2}}\right)$$

$$\text{Note:} \iint dydz \;\; \sigma(y,z) = -\frac{q2d}{4\pi} 2\pi \int_0^{\infty} \frac{udu}{\left(d^2+u^2\right)^{3/2}} = -q$$





A grounded metal sheet, a distance ***d*** from a point charge ***q***.



Surface charge density:

$$\sigma(y,z) = -\frac{q}{4\pi}\left(\frac{2d}{\left(d^2+y^2+z^2\right)^{3/2}}\right)$$

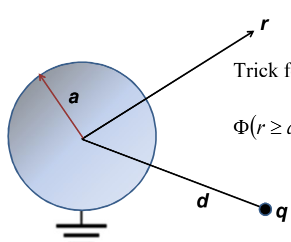
Force between charge and sheet:

$$\mathbf{F} = \frac{-q^2\hat{\mathbf{x}}}{4\pi\varepsilon_0(2d)^2}$$

Image potential between charge and sheet at distance x:

$$V(x) = \frac{-q^2}{4\pi\varepsilon_0(4x)}$$

A grounded metal sphere of radius *a*, in the presence of a point charge ***q*** at a distance *d* from its center.



Trick for $r \geq a$:

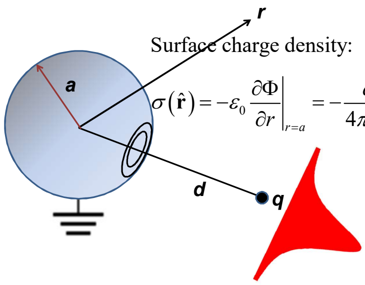
$$\Phi(r \geq a) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{|\mathbf{r}-\mathbf{d}|} - \frac{q}{\frac{d}{a}\left|\mathbf{r}-\mathbf{d}\frac{a^2}{d^2}\right|}\right)$$

Interpreted as

Image charge of $q' = -q\frac{a}{d}$

Located along $\hat{\mathbf{d}}$ at $\hat{\mathbf{d}}a\frac{a}{d}$

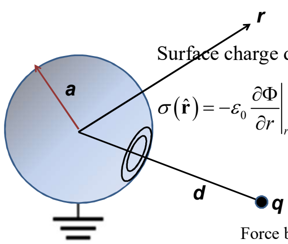
A grounded metal sphere of radius *a*, in the presence of a point charge ***q*** at a distance *d* from its center.



Surface charge density:

$$\sigma(\hat{\mathbf{r}}) = -\varepsilon_0 \left.\frac{\partial \Phi}{\partial r}\right|_{r=a} = -\frac{q}{4\pi a^2}\frac{a}{d}\frac{\left(1-\frac{a^2}{d^2}\right)}{\left(1+\frac{a^2}{d^2}-2\frac{a}{d}\hat{\mathbf{r}}\cdot\hat{\mathbf{d}}\right)^{3/2}}$$

A grounded metal sphere of radius *a*, in the presence of a point charge ***q*** at a distance *d* from its center.



Surface charge density:

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Force between q and sphere

$$|\mathbf{F}| = \frac{1}{4\pi\epsilon_0}\frac{q^2(a/d)}{\left(d-a^2/d\right)^2} = \frac{q^2}{4\pi\epsilon_0}\frac{ad}{\left(d^2-a^2\right)^2}$$

Use of image charge formalism to construct Green's function

Example:

Suppose we have a Dirichlet boundary value problem on a sphere of radius a:

$$\nabla^2\Phi = -\frac{\rho(\mathbf{r})}{\varepsilon_0} \qquad \Phi(r=a)=0$$

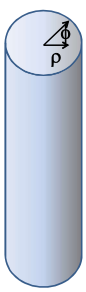
$$\nabla^2 G(\mathbf{r},\mathbf{r}') = -4\pi\delta^3(\mathbf{r}-\mathbf{r}')$$

$$\Rightarrow \text{For} \quad r,r' > a: \qquad G(\mathbf{r},\mathbf{r}') = \frac{1}{|\mathbf{r}-\mathbf{r}'|} - \frac{1}{\frac{r'}{a}\left|\mathbf{r}-\frac{a^2}{r'^2}\mathbf{r}'\right|}$$

Analysis of Poisson/Laplace equation in various regular geometries

1. Rectangular geometries
2. Cylindrical geometries
3. Spherical geometries

Solution of the Poisson/Laplace equation in various geometries -- cylindrical geometry with no *z*-dependence (infinitely long wire, for example):



Corresponding orthogonal functions from solution of

Laplace equation : $\nabla^2 \Phi = 0$

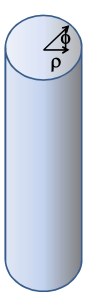
$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial \Phi}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 \Phi}{\partial \phi^2} = 0$$

$$\Phi(\rho,\phi) = \Phi(\rho,\phi + m2\pi)$$

$\Rightarrow$ General solution of the Laplace equation in these coordinates :

$$\Phi(\rho,\phi) = \mathrm{A}_0 + \mathrm{B}_0 \ln(\rho) + \sum_{m=1}^{\infty}\left(A_m \rho^m + B_m \rho^{-m}\right)\sin(m\phi + \alpha_m)$$

Solution of the Poisson/Laplace equation in various geometries -- cylindrical geometry with no *z*-dependence (infinitely long wire, for example):

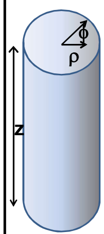


Green's function appropriate for this geometry with boundary conditions at $\rho = 0$ and $\rho = \infty$:

$$\left(\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial \phi^2}\right) G(\rho,\rho',\phi,\phi') = -4\pi \frac{\delta(\rho - \rho')}{\rho}\delta(\phi - \phi')$$

$$G(\rho,\rho',\phi,\phi') = -\ln\left(\rho_>{}^2\right) + 2\sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{\rho_<}{\rho_>}\right)^m \cos\left(m(\phi - \phi')\right)$$

Solution of the Poisson/Laplace equation in various geometries -- cylindrical geometry with *z*-dependence



Corresponding orthogonal functions from solution of Laplace equation : $\nabla^2\Phi = 0$

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\phi^2}+\frac{\partial^2\Phi}{\partial z^2}=0$$

$$\Phi(\rho,\phi,z)=\Phi(\rho,\phi+m2\pi,z)$$

$$\Phi(\rho,\phi,z)=R(\rho)Q(\phi)Z(z)$$

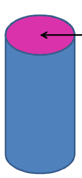
Cylindrical geometry continued:

$$\frac{d^2Z}{dz^2}-k^2Z=0 \qquad \Rightarrow Z(z)=\sinh(kz),\cosh(kz),e^{\pm kz}$$

$$\frac{d^2Q}{d\phi^2}+m^2Q=0 \qquad \Rightarrow Q(\phi)=e^{\pm im\phi}$$

$$\frac{d^2R}{d\rho^2}+\frac{1}{\rho}\frac{dR}{d\rho}+\left(k^2-\frac{m^2}{\rho^2}\right)R=0 \qquad \Rightarrow J_m(k\rho),N_m(k\rho)$$

Cylindrical geometry example:

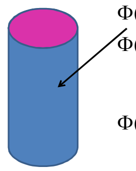


$$\Phi(\rho,\phi,z=L)=V(\rho,\phi)$$

$\Phi(\rho,\phi,z)=0$ on all other boundaries

$$\Phi(\rho,\phi,z)=\sum_{n,m}A_{mn}J_m(k_{mn}\rho)\sinh(k_{mn}z)\sin(m\phi+\alpha_{mn})$$

Cylindrical geometry example:

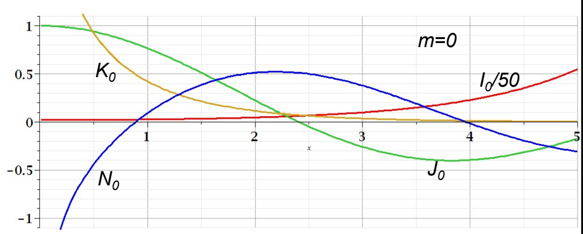


$$\Phi(\rho = a, \phi, z) = V(\phi, z)$$
$$\Phi(\rho, \phi, z) = 0 \quad \text{on all other boundaries}$$

$$\Phi(\rho, \phi, z) = \sum_{n,m} A_{mn} I_m\left(\frac{n\pi\rho}{L}\right) \sin\left(\frac{n\pi z}{L}\right) \sin(m\phi + \alpha_{mn})$$

Comments on cylindrical Bessel functions

$$\left(\frac{d^2}{du^2} + \frac{1}{u}\frac{d}{du} + \left(\pm 1 - \frac{m^2}{u^2}\right)\right) F_m^{\pm}(u) = 0$$
$$F_m^{+}(u) = J_m(u), N_m(u), H_m(u) \equiv J_m(u) \pm i N_m(u)$$
$$F_m^{-}(u) = I_m(u), K_m(u)$$



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