

**PHY 712 Electrodynamics**  
**9-9:50 AM MWF Olin 105**

**Plan for Lecture 7:**

**Start reading Chapter 3**

**Solution of Poisson/Laplace equation for special geometries –**

**A. Cylindrical**

**B. Spherical**

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**Course schedule for Spring 2019**  
(Preliminary schedule -- subject to frequent adjustment.)

Lecture date	JDJ Reading	Topic	HW	Due date
1 Mon: 01/14/2019	Chap. 1 & Appen.	Introduction, units and Poisson equation	#1	01/23/2019
2 Wed: 01/16/2019	Chap. 1	Electrostatic energy calculations	#2	01/23/2019
3 Fri: 01/18/2019	Chap. 1	Electrostatic potentials and fields	#3	01/23/2019
Mon: 01/21/2019	No class	Martin Luther King Holiday		
4 Wed: 01/23/2019	Chap. 1 - 3	Poisson's equation in 2 and 3 dimensions		
5 Fri: 01/25/2019	Chap. 1 - 3	Brief introduction to numerical methods	#4	01/28/2019
6 Mon: 01/28/2019	Chap. 2 & 3	Image charge constructions	#5	01/30/2019
7 Wed: 01/30/2019	Chap. 2 & 3	Cylindrical and spherical geometries		
8 Fri: 02/01/2019	Chap. 3 & 4	Spherical geometry and multipole moments	#6	02/04/2019
9 Mon: 02/04/2019				
10 Wed: 02/06/2019				
11 Fri: 02/08/2019				
12 Mon: 02/11/2019				
13 Wed: 02/13/2019				
14 Fri: 02/15/2019				
15 Mon: 02/18/2019				
16 Wed: 02/20/2019				

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Solution of the Poisson/Laplace equation in various geometries -- cylindrical geometry with no z-dependence (infinitely long wire, for example):



Corresponding orthogonal functions from solution of Laplace equation :  $\nabla^2 \Phi = 0$

$$\frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial \Phi}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 \Phi}{\partial \phi^2} = 0$$

$$\Phi(\rho, \phi) = \Phi(\rho, \phi + m2\pi)$$

⇒ General solution of the Laplace equation in these coordinates :

$$\Phi(\rho, \phi) = A_0 + B_0 \ln(\rho) + \sum_{m=1}^{\infty} (A_m \rho^m + B_m \rho^{-m}) \sin(m\phi + \alpha_m)$$

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Cylindrical coordinates with trivial z-dependence – some details:  $\Phi(\rho, \phi) = f(\rho)g(\phi)$

Suppose  $\frac{d^2 g(\phi)}{d\phi^2} = -m^2 g(\phi)$


$g(\phi) = \sin(m\phi + \alpha_m)$

$\frac{1}{\rho} \frac{d}{d\rho} \left( \rho \frac{df_m(\rho)}{d\rho} \right) - \frac{m^2}{\rho^2} f_m(\rho) = 0$

$f_0(\rho) = \begin{cases} 1 & f_{m>0} = \rho^{\pm m} \end{cases}$

⇒ General solution of the Laplace equation in these coordinates:

$\Phi(\rho, \phi) = A_0 + B_0 \ln(\rho) + \sum_{m=1}^{\infty} (A_m \rho^m + B_m \rho^{-m}) \sin(m\phi + \alpha_m)$



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Solution of the Poisson/Laplace equation in various geometries -- cylindrical geometry with no z-dependence (infinitely long wire, for example):

Green's function appropriate for this geometry with boundary conditions at  $\rho = 0$  and  $\rho = \infty$ :

$\left( \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2}{\partial \phi^2} \right) G(\rho, \rho', \phi, \phi') = -4\pi \frac{\delta(\rho - \rho')}{\rho} \delta(\phi - \phi')$

$G(\rho, \rho', \phi, \phi') = -\ln(\rho_>^2) + 2 \sum_{m=1}^{\infty} \frac{1}{m} \left( \frac{\rho_<}{\rho_>} \right)^m \cos(m(\phi - \phi'))$



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Comments and details


Change notation  $\rho \Rightarrow r$

$G(r, r', \phi, \phi') = -\ln(r_>^2) + 2 \sum_{m=1}^{\infty} \frac{1}{m} \left( \frac{r_<}{r_>} \right)^m \cos(m(\phi - \phi'))$

$\Phi(r, \phi) = \frac{1}{4\pi\epsilon_0} \int_0^{2\pi} d\phi' \int_0^{\infty} r' dr' G(r, r', \phi, \phi') \rho(r', \phi')$

Note that: For this extended charge distribution, Coulomb's law in its original form diverges:

$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int d^3 r' \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}$



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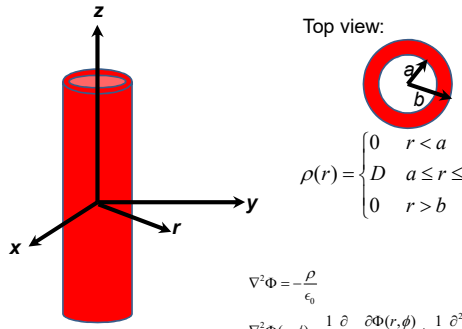
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Example:



Top view:

$$\rho(r) = \begin{cases} 0 & r < a \\ D & a \leq r \leq b \\ 0 & r > b \end{cases}$$

$$\nabla^2 \Phi = -\frac{\rho}{\epsilon_0}$$

$$\nabla^2 \Phi(r, \phi) = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \Phi(r, \phi)}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \Phi(r, \phi)}{\partial \phi^2}$$

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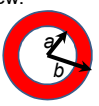
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Example continued --

Top view:



$$\rho(r) = \begin{cases} 0 & 0 < r < a \\ D & a \leq r \leq b \\ 0 & r > b \end{cases}$$

$$\Phi(r) = \begin{cases} \frac{D}{4\epsilon_0} (b^2 - a^2 - 2b^2 \ln(b) + 2a^2 \ln(a)) & 0 < r < a \\ \frac{D}{4\epsilon_0} (b^2 - r^2 - 2b^2 \ln(b) + 2a^2 \ln(r)) & a \leq r \leq b \\ -\frac{D}{2\epsilon_0} (b^2 - a^2) \ln(r) & r > b \end{cases}$$

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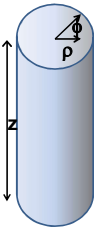
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Solution of the Poisson/Laplace equation in various geometries -- cylindrical geometry with z-dependence



Laplace equation:  $\nabla^2 \Phi = 0$

$$\frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial \Phi}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 \Phi}{\partial \phi^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0$$

$$\Phi(\rho, \phi, z) = R(\rho)Q(\phi)Z(z)$$

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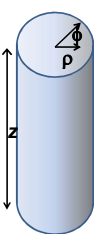
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Cylindrical geometry continued:



Laplace equation :  $\nabla^2\Phi = 0$   
 $\Phi(\rho, \phi, z) = R(\rho)Q(\phi)Z(z)$   
 One possibility :

$$\frac{d^2Z}{dz^2} - k^2Z = 0 \quad \Rightarrow Z(z) = \sinh(kz), \cosh(kz), e^{\pm kz}$$

$$\frac{d^2Q}{d\phi^2} + m^2Q = 0 \quad \Rightarrow Q(\phi) = e^{\pm im\phi}$$

$$\frac{d^2R}{d\rho^2} + \frac{1}{\rho} \frac{dR}{d\rho} + \left(k^2 - \frac{m^2}{\rho^2}\right)R = 0 \quad \Rightarrow J_m(k\rho), N_m(k\rho)$$

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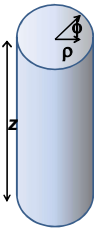
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Cylindrical geometry continued:



Laplace equation :  $\nabla^2\Phi = 0$   
 $\Phi(\rho, \phi, z) = R(\rho)Q(\phi)Z(z)$   
 Another possibility :

$$\frac{d^2Z}{dz^2} + k^2Z = 0 \quad \Rightarrow Z(z) = \sin(kz), \cos(kz), e^{\pm ikz}$$

$$\frac{d^2Q}{d\phi^2} + m^2Q = 0 \quad \Rightarrow Q(\phi) = e^{\pm im\phi}$$

$$\frac{d^2R}{d\rho^2} + \frac{1}{\rho} \frac{dR}{d\rho} + \left(-k^2 - \frac{m^2}{\rho^2}\right)R = 0 \quad \Rightarrow I_m(k\rho), K_m(k\rho)$$

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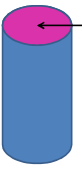
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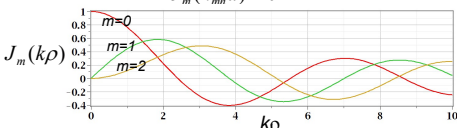
Solutions of Laplace equation inside cylindrical shape  
 Example with non-trivial boundary value at  $z=L$



$\Phi(\rho, \phi, z=L) = V(\rho, \phi)$   
 $\Phi(\rho, \phi, z) = 0$  on all other boundaries

$$\Phi(\rho, \phi, z) = \sum_{n,m} A_{nm} J_m(k_{nm}\rho) \sinh(k_{nm}z) \sin(m\phi + \alpha_{nm})$$

where  $J_m(k_{nm}\alpha) = 0$



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Solutions of Laplace equation inside cylindrical shape  
 Example with non-trivial boundary value at  $z=L$



$\Phi(\rho, \varphi, z=L) = V(\rho, \varphi)$   
 $\Phi(\rho, \varphi, z) = 0$  on all other boundaries  
 $\Phi(\rho, \varphi, z) = \sum_{n,m} A_{mn} J_m(k_{mn} \rho) \sinh(k_{mn} z) \sin(m\varphi + \alpha_{mn})$   
 If  $V(\rho, \varphi)$  is an even function of  $\varphi$  so that  $\alpha_{mn} = \pi/2$ :

$$A_{mn} = \frac{\int_0^{2\pi} d\varphi \cos(m\varphi) \int_0^a \rho d\rho J_m(k_{mn} \rho) V(\rho, \varphi)}{\sinh(k_{mn} L) \int_0^{2\pi} d\varphi \cos^2(m\varphi) \int_0^a \rho d\rho J_m^2(k_{mn} \rho)}$$

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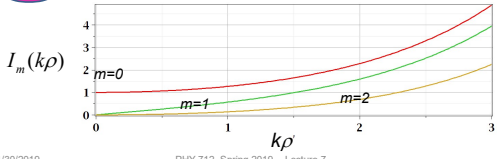
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Solutions of Laplace equation inside cylindrical shape  
 Example with non-trivial boundary value at  $\rho=a$



$\Phi(\rho = a, \phi, z) = V(\phi, z)$   
 $\Phi(\rho, \phi, z) = 0$  on all other boundaries  
 $\Phi(\rho, \phi, z) = \sum_{n,m} A_{mn} I_m\left(\frac{n\pi\rho}{L}\right) \sin\left(\frac{n\pi z}{L}\right) \sin(m\phi + \alpha_{mn})$



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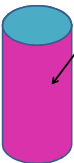
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Solutions of Laplace equation inside cylindrical shape  
 Example with non-trivial boundary value at  $\rho=a$



$\Phi(\rho = a, \varphi, z) = V(\varphi, z)$   
 $\Phi(\rho, \varphi, z) = 0$  on all other boundaries  
 $\Phi(\rho, \varphi, z) = \sum_{n,m} A_{mn} I_m\left(\frac{n\pi\rho}{L}\right) \sin\left(\frac{n\pi z}{L}\right) \sin(m\varphi + \alpha_{mn})$   
 If  $V(z, \varphi)$  is an even function of  $\varphi$  so that  $\alpha_{mn} = \pi/2$ :

$$A_{mn} = \frac{\int_0^{2\pi} d\varphi \cos(m\varphi) \int_0^L dz \sin\left(\frac{n\pi z}{L}\right) V(z, \varphi)}{I_m\left(\frac{n\pi a}{L}\right) \int_0^{2\pi} d\varphi \cos^2(m\varphi) \int_0^L dz \sin^2\left(\frac{n\pi z}{L}\right)}$$

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Green's function for Dirchelet boundary value inside cylinder:



$\Phi(\rho, \phi, z=L) = V(\rho, \phi)$   
 $\Phi(\rho=a, \phi, z) = 0, \Phi(\rho, \phi, z=0) = 0$   
 Expansion in terms of Bessel function zeros:  $J_m(k_{mn}a) = 0$   
 $G(\rho, \rho', \phi, \phi', z, z') = \frac{8\pi}{\pi a^2} \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} \frac{e^{im(\phi-\phi')} J_m(k_{mn}\rho) J_m(k_{mn}\rho') \sinh(k_{mn}z) \sinh(k_{mn}(L-z))}{k_{mn} (J_{m+1}(k_{mn}a))^2 \sinh(k_{mn}L)}$   
 $\Phi(\rho, \phi, z) = \frac{1}{4\pi\epsilon_0} \int_V d\phi' \rho' d\rho' dz' G(\rho, \rho', \phi, \phi', z, z') \rho(\rho', \phi', z')$   
 $+ \frac{1}{4\pi} \int_{S, z=L} d\phi' \rho' d\rho' \frac{\partial G(\rho, \rho', \phi, \phi', z, z')}{\partial z'} \Big|_{z=L} V(\rho', \phi')$

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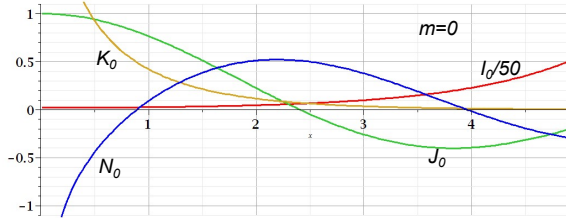
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Comments on cylindrical Bessel functions

$$\left( \frac{d^2}{du^2} + \frac{1}{u} \frac{d}{du} + \left( \pm 1 - \frac{m^2}{u^2} \right) \right) F_m^{\pm}(u) = 0$$

$F_m^+(u) = J_m(u), N_m(u), H_m(u) \equiv J_m(u) \pm iN_m(u)$   
 $F_m^-(u) = I_m(u), K_m(u)$



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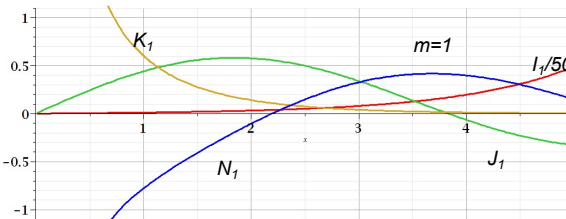
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Comments on cylindrical Bessel functions

$$\left( \frac{d^2}{du^2} + \frac{1}{u} \frac{d}{du} + \left( \pm 1 - \frac{m^2}{u^2} \right) \right) F_m^{\pm}(u) = 0$$

$F_m^+(u) = J_m(u), N_m(u), H_m(u) \equiv J_m(u) \pm iN_m(u)$   
 $F_m^-(u) = I_m(u), K_m(u)$



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Some useful identities involving cylindrical Bessel functions

$$\left( \frac{d^2}{du^2} + \frac{1}{u} \frac{d}{du} + \left( 1 - \frac{m^2}{u^2} \right) \right) J_m(u) = 0 \quad \text{for integer } m$$

Properties of Bessel functions in terms of zeros:  $x_{mn}$ ;  $J_m(x_{mn}) = 0$

$$\int_0^a \rho d\rho J_m(x_{mn}\rho/a) J_m(x_{m'n'}\rho/a) = \frac{a^2}{2} (J_{m+1}(x_{mn}))^2 \delta_{nn'}$$

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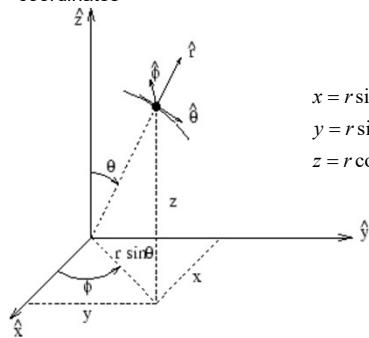
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Poisson and Laplace equation in spherical polar coordinates



$$\begin{aligned} x &= r \sin \theta \cos \phi \\ y &= r \sin \theta \sin \phi \\ z &= r \cos \theta \end{aligned}$$

<http://www.uic.edu/classes/eecs/eecs520/textbook/node32.html>

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Poisson and Laplace equation in spherical polar coordinates -- continued

Laplace equation for electrostatic potential  $\Phi(r, \theta, \phi)$ :

$$\frac{1}{r} \frac{\partial^2}{\partial r^2} (r\Phi) + \frac{1}{r^2} \left( \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right) \Phi = 0$$

$$\Phi(r, \theta, \phi) = \sum_{lm} R_{lm}(r) Y_{lm}(\theta, \phi)$$

Spherical harmonic functions :

$$\left( \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right) Y_{lm}(\theta, \phi) = -l(l+1) Y_{lm}(\theta, \phi)$$

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## Properties of spherical harmonic functions

$$Y_{lm}(\theta, \phi) = (-1)^m Y_{l(-m)}^*(\theta, \phi) \quad (\text{standard Condon - Shortley convention})$$

$$\int d\Omega Y_{lm}(\theta, \phi) Y_{l'm'}^*(\theta, \phi) \equiv \int \sin \theta d\theta d\phi Y_{lm}(\theta, \phi) Y_{l'm'}^*(\theta, \phi) = \delta_{ll'} \delta_{mm'}$$

Completeness :

$$\sum_{lm} Y_{lm}(\theta, \phi) Y_{lm}^*(\theta', \phi') = \delta(\hat{\mathbf{r}} - \hat{\mathbf{r}}') \equiv \delta(\cos \theta - \cos \theta') \delta(\phi - \phi')$$

Relationship to Legendre polynomials :

$$Y_{l0}(\theta, \phi) = \sqrt{\frac{2l+1}{4\pi}} P_l(\cos \theta)$$

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Useful identity:

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \sum_{lm} \frac{4\pi}{2l+1} \frac{r_{<}^l}{r_{>}^{l+1}} Y_{lm}(\theta, \phi) Y_{lm}^*(\theta', \phi')$$

Example for isolated charge density  $\rho(\mathbf{r})$  with electrostatic potential vanishing for  $r \rightarrow \infty$  :

$$\begin{aligned} \Phi(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0} \int d^3r' \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \\ &= \frac{1}{4\pi\epsilon_0} \int d^3r' \rho(\mathbf{r}') \left( \sum_{lm} \frac{4\pi}{2l+1} \frac{r_{<}^l}{r_{>}^{l+1}} Y_{lm}(\theta, \phi) Y_{lm}^*(\theta', \phi') \right) \end{aligned}$$

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Example -- continued

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int d^3r' \rho(\mathbf{r}') \left( \sum_{lm} \frac{4\pi}{2l+1} \frac{r_{<}^l}{r_{>}^{l+1}} Y_{lm}(\theta, \phi) Y_{lm}^*(\theta', \phi') \right)$$

$$\text{Suppose: } \rho(\mathbf{r}') = \frac{Q}{a^3 \pi^{3/2}} e^{-r'^2/a^2}$$

$$\int d\Omega Y_{lm}^*(\theta', \phi') = \sqrt{4\pi} \delta_{l0} \delta_{m0}$$

$$\begin{aligned} \Phi(\mathbf{r}) &= \frac{4\pi}{4\pi\epsilon_0} \int_0^\infty r'^2 dr' \int_{r_{>}^{-1}}^{r_{<}^0} \frac{Q}{a^3 \pi^{3/2}} e^{-r'^2/a^2} \\ &= \frac{Q}{4\pi\epsilon_0} \frac{\text{erf}(r/a)}{r} \end{aligned}$$

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Useful identity:

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \sum_{lm} \frac{4\pi}{2l+1} \frac{r_{<}^l}{r_{>}^{l+1}} Y_{lm}(\theta, \varphi) Y_{lm}^*(\theta', \varphi')$$

Elements of "proof":

$$\begin{aligned} \frac{1}{|\mathbf{r} - \mathbf{r}'|} &= \frac{1}{r_{>} \left( 1 + \left(\frac{r_{<}}{r_{>}}\right)^2 - 2\left(\frac{r_{<}}{r_{>}}\right) \hat{\mathbf{r}} \cdot \hat{\mathbf{r}}' \right)^{1/2}} \\ &= \frac{1}{r_{>}} \left( 1 + \left(\frac{r_{<}}{r_{>}}\right) \hat{\mathbf{r}} \cdot \hat{\mathbf{r}}' + \left(\frac{r_{<}}{r_{>}}\right)^2 \left( \frac{3}{2} (\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}')^2 - \frac{1}{2} \right) + \dots \right) \\ &= \frac{1}{r_{>}} \left( \sum_{l=0}^{\infty} \left(\frac{r_{<}}{r_{>}}\right)^l P_l(\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}') \right) \end{aligned}$$

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Useful identity:

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \sum_{lm} \frac{4\pi}{2l+1} \frac{r_{<}^l}{r_{>}^{l+1}} Y_{lm}(\theta, \varphi) Y_{lm}^*(\theta', \varphi')$$

Elements of "proof" -- continued:

Sum rule for spherical harmonics:

$$P_l(\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}') = \frac{4\pi}{2l+1} \sum_{m=-l}^l Y_{lm}(\hat{\mathbf{r}}) Y_{lm}^*(\hat{\mathbf{r}}')$$

Note that for  $\hat{\mathbf{r}} = \hat{\mathbf{r}}'$ ,  $P_l(\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}') = 1$

$$\Rightarrow \frac{4\pi}{2l+1} \sum_{m=-l}^l |Y_{lm}(\hat{\mathbf{r}})|^2 = 1$$

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Some spherical harmonic functions:

$$Y_{00}(\hat{\mathbf{r}}) = \frac{1}{\sqrt{4\pi}}$$

$$Y_{l(\pm 1)}(\hat{\mathbf{r}}) = \mp \sqrt{\frac{3}{8\pi}} \sin \theta e^{\pm i\phi}$$

$$Y_{10}(\hat{\mathbf{r}}) = \sqrt{\frac{3}{4\pi}} \cos \theta$$

$$Y_{2(\pm 2)}(\hat{\mathbf{r}}) = \sqrt{\frac{15}{32\pi}} \sin^2 \theta e^{\pm 2i\phi}$$

$$Y_{2(\pm 1)}(\hat{\mathbf{r}}) = \mp \sqrt{\frac{15}{8\pi}} \sin \theta \cos \theta e^{\pm i\phi}$$

$$Y_{20}(\hat{\mathbf{r}}) = \sqrt{\frac{5}{4\pi}} \left( \frac{3}{2} \cos^2 \theta - \frac{1}{2} \right)$$

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