PHY 712 Electrodynamics
12-12:50 AM MWF Olin 103

Plan for Lecture 15:

Finish reading Chapter 6

1. **Some details of Liénard-Wiechert results**
2. **Energy density and flux associated with electromagnetic fields**
3. **Time harmonic fields**

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11	Fri: 02/07/2020	Chap. 5	Magnetostatics	#10	02/12/2020
12	Mon: 02/10/2020	Chap. 5	Magnetic dipoles and hyperfine interaction	#11	02/14/2020
13	Wed: 02/12/2020	Chap. 5	Magnetic dipoles and dipolar fields	#12	02/17/2020
14	Fri: 02/14/2020	Chap. 6	Maxwell's Equations	#13	02/19/2020
15	Mon: 02/17/2020	Chap. 6	Electromagnetic energy and forces	#14	02/21/2020
16	Wed: 02/19/2020	Chap. 7	Electromagnetic plane waves		
17	Fri: 02/21/2020	Chap. 7	Electromagnetic plane waves		
18	Mon: 02/24/2020	Chap. 7	Refractive index		
19	Wed: 02/26/2020	Chap. 8	EM waves in wave guides		
20	Fri: 02/28/2020	Chap. 1-8	Review		
	Mon: 03/02/2020	No class	*APS March Meeting*	Take Home Exam	
	Wed: 03/04/2020	No class	*APS March Meeting*	Take Home Exam	
	Fri: 03/06/2020	No class	*APS March Meeting*	Take Home Exam	
	Mon: 03/09/2020	No class	*Spring Break*		
	Wed: 03/11/2020	No class	*Spring Break*		
	Fri: 03/13/2020	No class	*Spring Break*		
21	Mon: 03/16/2020	Chap. 9	Radiation from localized oscillating sources		
22	Wed: 03/18/2020	Chap. 9	Radiation from oscillating sources		

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Solution of Maxwell's equations in the Lorentz gauge -- continued

Liénard-Wiechert potentials and fields --
Determination of the scalar and vector potentials for a moving point particle (also see Landau and Lifshitz ***The Classical Theory of Fields***, Chapter 8.)

Consider the fields produced by the following source: a point charge q moving on a trajectory $R_q(t)$.

Charge density: $\rho(\mathbf{r},t) = q\delta^3(\mathbf{r}-\mathbf{R}_q(t))$

Current density: $\mathbf{J}(\mathbf{r},t) = q\dot{\mathbf{R}}_q(t)\delta^3(\mathbf{r}-\mathbf{R}_q(t)),$ where $\dot{\mathbf{R}}_q(t) \equiv \frac{d\mathbf{R}_q(t)}{dt}.$

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Solution of Maxwell's equations in the Lorentz gauge -- continued

$$\Phi(\mathbf{r},t)=\frac{1}{4\pi\epsilon_0}\int\int d^3r'dt'\frac{\rho(\mathbf{r}',t')}{|\mathbf{r}-\mathbf{r}']|}\delta\left(t'-(t-|\mathbf{r}-\mathbf{r}'|/c)\right)$$

$$\mathbf{A}(\mathbf{r},t)=\frac{1}{4\pi\epsilon_0 c^2}\int\int d^3r'dt'\frac{\mathbf{J}(\mathbf{r}',\mathbf{t}')}{|\mathbf{r}-\mathbf{r}'|}\delta\left(t'-(t-|\mathbf{r}-\mathbf{r}'|/c)\right).$$

We performing the integrations over first *d³r'* and then *dt'* making use of the fact that for any function of *t'*,

$$\int_{-\infty}^{\infty} dt'\, f(t')\delta\left(t'-(t-|\mathbf{r}-\mathbf{R}_q(t')|/c)\right)=\frac{f(t_r)}{1-\frac{\dot{\mathbf{R}}_q(t_r)\cdot(\mathbf{r}-\mathbf{R}_q(t_r))}{c|\mathbf{r}-\mathbf{R}_q(t_r)|}},$$

where the ``retarded time'' is defined to be

$$t_r\equiv t-\frac{|\mathbf{r}-\mathbf{R}_q(t_r)|}{c}.$$

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Solution of Maxwell's equations in the Lorentz gauge -- continued

Resulting scalar and vector potentials:

$$\Phi(\mathbf{r},t)=\frac{q}{4\pi\epsilon_0}\frac{1}{R-\frac{\mathbf{v}\cdot\mathbf{R}}{c}},$$

$$\mathbf{A}(\mathbf{r},t)=\frac{q}{4\pi\epsilon_0 c^2}\frac{\mathbf{v}}{R-\frac{\mathbf{v}\cdot\mathbf{R}}{c}},$$

Notation: $\mathbf{R}\equiv \mathbf{r}-\mathbf{R}_q(t_r)$ $\quad t_r\equiv t-\frac{|\mathbf{r}-\mathbf{R}_q(t_r)|}{c}.$

$\mathbf{v}\equiv\dot{\mathbf{R}}_q(t_r),$

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Solution of Maxwell's equations in the Lorentz gauge -- continued

In order to find the electric and magnetic fields, we need to evaluate

$$\mathbf{E}(\mathbf{r},t)=-\nabla\Phi(\mathbf{r},t)-\frac{\partial\mathbf{A}(\mathbf{r},t)}{\partial t}$$

$$\mathbf{B}(\mathbf{r},t)=\nabla\times\mathbf{A}(\mathbf{r},t)$$

The trick of evaluating these derivatives is that the retarded time t_r depends on position **r** and on itself. We can show the following results using the shorthand notation:

$$\nabla t_r=-\frac{\mathbf{R}}{c\left(R-\frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)} \quad \text{and} \quad \frac{\partial t_r}{\partial t}=\frac{R}{\left(R-\frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)}.$$

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Solution of Maxwell's equations in the Lorentz gauge -- continued

$$-\nabla\Phi(\mathbf{r},t)=\frac{q}{4\pi\epsilon_0}\frac{1}{\left(R-\frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left[\mathbf{R}\left(1-\frac{v^2}{c^2}\right)-\frac{v}{c}\left(R-\frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)+\mathbf{R}\frac{\dot{\mathbf{v}}\cdot\mathbf{R}}{c^2}\right],$$

$$-\frac{\partial\mathbf{A}(\mathbf{r},t)}{\partial t}=\frac{q}{4\pi\epsilon_0}\frac{1}{\left(R-\frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left[\frac{\mathbf{v}R}{c}\left(\frac{v^2}{c^2}-\frac{\mathbf{v}\cdot\mathbf{R}}{Rc}-\frac{\dot{\mathbf{v}}\cdot R}{c^2}\right)-\frac{\dot{\mathbf{v}}R}{c^2}\left(R-\frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)\right].$$

$$\mathbf{E}(\mathbf{r},t)=\frac{q}{4\pi\epsilon_0}\frac{1}{\left(R-\frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left[\left(\mathbf{R}-\frac{\mathbf{v}R}{c}\right)\left(1-\frac{v^2}{c^2}\right)+\left(\mathbf{R}\times\left\{\left(\mathbf{R}-\frac{\mathbf{v}R}{c}\right)\times\frac{\dot{\mathbf{v}}}{c^2}\right\}\right)\right].$$

$$\mathbf{B}(\mathbf{r},t)=\frac{q}{4\pi\epsilon_0 c^2}\left[\frac{-\mathbf{R}\times\mathbf{v}}{\left(R-\frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left(1-\frac{v^2}{c^2}+\frac{\dot{\mathbf{v}}\cdot\mathbf{R}}{c^2}\right)-\frac{\mathbf{R}\times\dot{\mathbf{v}}/c}{\left(R-\frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^2}\right]=\frac{\mathbf{R}\times\mathbf{E}(\mathbf{r},t)}{cR}$$

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Maxwell's equations

Coulomb's law: $\nabla\cdot\mathbf{D}=\rho_{free}$

Ampere-Maxwell's law: $\nabla\times\mathbf{H}-\frac{\partial\mathbf{D}}{\partial t}=\mathbf{J}_{free}$

Faraday's law: $\nabla\times\mathbf{E}+\frac{\partial\mathbf{B}}{\partial t}=0$

No magnetic monopoles: $\nabla\cdot\mathbf{B}=0$

Energy analysis of electromagnetic fields and sources

Rate of work done on source $\mathbf{J}(\mathbf{r},t)$ by electromagnetic field:

$$\frac{dW_{mech}}{dt}\equiv\frac{dE_{mech}}{dt}=\int d^3r\ \mathbf{E}\cdot\mathbf{J}_{free}$$

Expressing source current in terms of fields it produces:

$$\frac{dW_{mech}}{dt}=\int d^3r\ \mathbf{E}\cdot\left(\nabla\times\mathbf{H}-\frac{\partial\mathbf{D}}{\partial t}\right)$$

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Energy analysis of electromagnetic fields and sources - continued

$$\frac{dW_{mech}}{dt}=\int d^3r\ \mathbf{E}\cdot\mathbf{J}_{free}=\int d^3r\ \mathbf{E}\cdot\left(\nabla\times\mathbf{H}-\frac{\partial\mathbf{D}}{\partial t}\right)$$

$$=-\int d^3r\ \left(\nabla\cdot(\mathbf{E}\times\mathbf{H})+\mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial t}+\mathbf{H}\cdot\frac{\partial\mathbf{B}}{\partial t}\right)$$

Let $\mathbf{S}\equiv\mathbf{E}\times\mathbf{H}$ "Poynting vector"

$$u\equiv\frac{1}{2}(\mathbf{E}\cdot\mathbf{D}+\mathbf{H}\cdot\mathbf{B})\quad\text{energy density}$$

$$\Rightarrow\frac{\partial u}{\partial t}+\nabla\cdot\mathbf{S}=-\mathbf{E}\cdot\mathbf{J}_{free}$$

Assuming that $\mathbf{D}=\epsilon\mathbf{E}$ and that $\mathbf{B}=\mu\mathbf{H}$

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Energy analysis of electromagnetic fields and sources - continued

$$\frac{dE_{mech}}{dt} \equiv \int d^3r \;\; \mathbf{E}\cdot\mathbf{J}_{free}$$

Electromagnetic energy density: $\quad u \equiv \frac{1}{2}\left(\mathbf{E}\cdot\mathbf{D}+\mathbf{H}\cdot\mathbf{B}\right)$

$$E_{field} \equiv \int d^3r \;\; u(\mathbf{r},t)$$

Poynting vector: $\quad \mathbf{S} \equiv \mathbf{E}\times\mathbf{H}$

From the previous energy analysis: $\frac{\partial u}{\partial t} + \nabla\cdot\mathbf{S} = -\mathbf{E}\cdot\mathbf{J}_{free}$

$$\Rightarrow \frac{dE_{mech}}{dt} + \frac{dE_{field}}{dt} = -\int d^3r \;\; \nabla\cdot\mathbf{S}(\mathbf{r},t) = -\oint d^2r \;\hat{\mathbf{r}}\cdot\mathbf{S}(\mathbf{r},t)$$

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Momentum analysis of electromagnetic fields and sources

$$\frac{d\mathbf{P}_{mech}}{dt} \equiv \int d^3r \;\; \left(\rho\mathbf{E}+\mathbf{J}\times\mathbf{B}\right)$$

Follows by analogy with Lorentz force:

$$\mathbf{F} = q\left(\mathbf{E}+\mathbf{v}\times\mathbf{B}\right)$$

$$\mathbf{P}_{field} = \varepsilon_0 \int d^3r \;\; \left(\mathbf{E}\times\mathbf{B}\right)$$

Expression for vacuum fields:

$$\left(\frac{d\mathbf{P}_{mech}}{dt} + \frac{d\mathbf{P}_{field}}{dt}\right)_i = \sum_j \int d^3r \,\frac{\partial T_{ij}}{\partial r_j}$$

Maxwell stress tensor:

$$T_{ij} \equiv \varepsilon_0 \left(E_iE_j + c^2B_iB_j - \delta_{ij}\frac{1}{2}\left(\mathbf{E}\cdot\mathbf{E}+c^2\mathbf{B}\cdot\mathbf{B}\right)\right)$$

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Comment on treatment of time-harmonic fields

Fourier transformation in time domain :

$$\mathbf{E}(\mathbf{r},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \,\tilde{\mathbf{E}}(\mathbf{r},\omega)\, e^{-i\omega t}$$

$$\tilde{\mathbf{E}}(\mathbf{r},\omega) = \int_{-\infty}^{\infty} dt \,\mathbf{E}(\mathbf{r},t)\, e^{i\omega t}$$

Note that $\mathbf{E}(\mathbf{r},t)$ is real $\Rightarrow \tilde{\mathbf{E}}(\mathbf{r},\omega) = \tilde{\mathbf{E}}^*(\mathbf{r},-\omega)$

These relations and the notion of the superposition principle, lead to the common treatment:

$$\mathbf{E}(\mathbf{r},t) = \Re\left(\tilde{\mathbf{E}}(\mathbf{r},\omega)e^{-i\omega t}\right) \equiv \frac{1}{2}\left(\tilde{\mathbf{E}}(\mathbf{r},\omega)e^{-i\omega t} + \tilde{\mathbf{E}}^*(\mathbf{r},\omega)e^{i\omega t}\right)$$

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Comment on treatment of time-harmonic fields -- continued

Equations for time harmonic fields:

$$\mathbf{E}(\mathbf{r},t) = \Re\left(\tilde{\mathbf{E}}(\mathbf{r},\omega)e^{-i\omega t}\right) \equiv \frac{1}{2}\left(\tilde{\mathbf{E}}(\mathbf{r},\omega)e^{-i\omega t} + \tilde{\mathbf{E}}^*(\mathbf{r},\omega)e^{i\omega t}\right)$$

Equations	in time domain	in frequency domain
Coulomb's law:	$\nabla\cdot\mathbf{D} = \rho_{free}$	$\nabla\cdot\tilde{\mathbf{D}} = \tilde{\rho}_{free}$
Ampere-Maxwell's law:	$\nabla\times\mathbf{H} - \frac{\partial\mathbf{D}}{\partial t} = \mathbf{J}_{free}$	$\nabla\times\tilde{\mathbf{H}} + i\omega\tilde{\mathbf{D}} = \tilde{\mathbf{J}}_{free}$
Faraday's law:	$\nabla\times\mathbf{E} + \frac{\partial\mathbf{B}}{\partial t} = 0$	$\nabla\times\tilde{\mathbf{E}} - i\omega\tilde{\mathbf{B}} = 0$
No magnetic monopoles:	$\nabla\cdot\mathbf{B} = 0$	$\nabla\cdot\tilde{\mathbf{B}} = 0$

Note -- in all of these, the real part is taken at the end of the calculation.

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Comment on treatment of time-harmonic fields -- continued

Equations for time harmonic fields:

$$\mathbf{E}(\mathbf{r},t) = \Re\left(\tilde{\mathbf{E}}(\mathbf{r},\omega)e^{-i\omega t}\right) \equiv \frac{1}{2}\left(\tilde{\mathbf{E}}(\mathbf{r},\omega)e^{-i\omega t} + \tilde{\mathbf{E}}^*(\mathbf{r},\omega)e^{i\omega t}\right)$$

Poynting vector: $\mathbf{S}(\mathbf{r},t) = \mathbf{E}(\mathbf{r},t)\times\mathbf{H}(\mathbf{r},t)$

$$\mathbf{S}(\mathbf{r},t) = \frac{1}{4}\left(\tilde{\mathbf{E}}(\mathbf{r},\omega)e^{-i\omega t} + \tilde{\mathbf{E}}^*(\mathbf{r},\omega)e^{i\omega t}\right)\times\left(\tilde{\mathbf{H}}(\mathbf{r},\omega)e^{-i\omega t} + \tilde{\mathbf{H}}^*(\mathbf{r},\omega)e^{i\omega t}\right)$$

$$= \frac{1}{4}\left(\tilde{\mathbf{E}}(\mathbf{r},\omega)\times\tilde{\mathbf{H}}^*(\mathbf{r},\omega) + \tilde{\mathbf{E}}^*(\mathbf{r},\omega)\times\tilde{\mathbf{H}}(\mathbf{r},\omega)\right)$$

$$+\frac{1}{4}\left(\tilde{\mathbf{E}}(\mathbf{r},\omega)\times\tilde{\mathbf{H}}(\mathbf{r},\omega)e^{-2i\omega t} + \tilde{\mathbf{E}}^*(\mathbf{r},\omega)\times\tilde{\mathbf{H}}^*(\mathbf{r},\omega)e^{2i\omega t}\right)$$

$$\langle\mathbf{S}(\mathbf{r},t)\rangle_{t\,\text{avg}} = \Re\left(\frac{1}{2}\left(\tilde{\mathbf{E}}(\mathbf{r},\omega)\times\tilde{\mathbf{H}}^*(\mathbf{r},\omega)\right)\right)$$

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Summary and review

Maxwell's equations

Coulomb's law: $\nabla\cdot\mathbf{D} = \rho_{free}$

Ampere-Maxwell's law: $\nabla\times\mathbf{H} - \frac{\partial\mathbf{D}}{\partial t} = \mathbf{J}_{free}$

Faraday's law: $\nabla\times\mathbf{E} + \frac{\partial\mathbf{B}}{\partial t} = 0$

No magnetic monopoles: $\nabla\cdot\mathbf{B} = 0$

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Maxwell's equations

For linear isotropic media -- $\mathbf{D} = \varepsilon\mathbf{E}$; $\mathbf{B} = \mu\mathbf{H}$

and no sources :

Coulomb's law : $\nabla \cdot \mathbf{E} = 0$

Ampere - Maxwell's law : $\nabla \times \mathbf{B} - \mu\varepsilon \frac{\partial \mathbf{E}}{\partial t} = 0$

Faraday's law : $\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$

No magnetic monopoles : $\nabla \cdot \mathbf{B} = 0$

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Analysis of Maxwell's equations without sources -- continued:

Coulomb's law : $\nabla \cdot \mathbf{E} = 0$

Ampere-Maxwell's law : $\nabla \times \mathbf{B} - \mu\varepsilon \frac{\partial \mathbf{E}}{\partial t} = 0$

Faraday's law : $\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$

No magnetic monopoles : $\nabla \cdot \mathbf{B} = 0$

$$\nabla \times \left(\nabla \times \mathbf{B} - \mu\varepsilon \frac{\partial \mathbf{E}}{\partial t} \right) = -\nabla^2 \mathbf{B} - \mu\varepsilon \frac{\partial (\nabla \times \mathbf{E})}{\partial t}$$

$$= -\nabla^2 \mathbf{B} + \mu\varepsilon \frac{\partial^2 \mathbf{B}}{\partial t^2} = 0$$

$$\nabla \times \left(\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} \right) = -\nabla^2 \mathbf{E} + \frac{\partial (\nabla \times \mathbf{B})}{\partial t}$$

$$= -\nabla^2 \mathbf{E} + \mu\varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0$$

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Analysis of Maxwell's equations without sources -- continued:

Both E and B fields are solutions to a wave equation:

$$\nabla^2 \mathbf{B} - \frac{1}{v^2} \frac{\partial^2 \mathbf{B}}{\partial t^2} = 0$$

$$\nabla^2 \mathbf{E} - \frac{1}{v^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0$$

where $v^2 \equiv c^2 \frac{\mu_0 \varepsilon_0}{\mu\varepsilon} \equiv \frac{c^2}{n^2}$

Plane wave solutions to wave equation :

$$\mathbf{B}(\mathbf{r}, t) = \Re\left(\mathbf{B}_0 e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t}\right) \qquad \mathbf{E}(\mathbf{r}, t) = \Re\left(\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t}\right)$$

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Analysis of Maxwell's equations without sources -- continued:

Plane wave solutions to wave equation:

$$\mathbf{B}(\mathbf{r},t) = \Re\left(\mathbf{B}_0 e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\right) \qquad \mathbf{E}(\mathbf{r},t) = \Re\left(\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\right)$$

$$|\mathbf{k}|^2 = \left(\frac{\omega}{v}\right)^2 = \left(\frac{n\omega}{c}\right)^2 \qquad \text{where } n \equiv \sqrt{\frac{\mu\varepsilon}{\mu_0\varepsilon_0}}$$

Note: ε, μ, n, k can all be complex; for the moment we will assume that they are all real (no dissipation).

Note that $\mathbf{E}_0$ and $\mathbf{B}_0$ are not independent;

from Faraday's law: $\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$

$$\Rightarrow \mathbf{B}_0 = \frac{\mathbf{k}\times\mathbf{E}_0}{\omega} = \frac{n\hat{\mathbf{k}}\times\mathbf{E}_0}{c}$$

also note: $\hat{\mathbf{k}}\cdot\mathbf{E}_0 = 0$ and $\hat{\mathbf{k}}\cdot\mathbf{B}_0 = 0$

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Analysis of Maxwell's equations without sources -- continued:

Summary of plane electromagnetic waves:

$$\mathbf{B}(\mathbf{r},t) = \Re\left(\frac{n\hat{\mathbf{k}}\times\mathbf{E}_0}{c} e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\right) \qquad \mathbf{E}(\mathbf{r},t) = \Re\left(\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\right)$$

$$|\mathbf{k}|^2 = \left(\frac{\omega}{v}\right)^2 = \left(\frac{n\omega}{c}\right)^2 \quad \text{where } n \equiv \sqrt{\frac{\mu\varepsilon}{\mu_0\varepsilon_0}} \quad \text{and } \hat{\mathbf{k}}\cdot\mathbf{E}_0 = 0$$

Poynting vector for plane electromagnetic waves:

$$\langle \mathbf{S}\rangle_{avg} = \frac{1}{2}\Re\left(\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t} \times \frac{1}{\mu}\left(\frac{n\hat{\mathbf{k}}\times\mathbf{E}_0}{c} e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\right)^*\right)$$

$$= \frac{n|\mathbf{E}_0|^2}{2\mu c}\hat{\mathbf{k}} = \frac{1}{2}\sqrt{\frac{\varepsilon}{\mu}}|\mathbf{E}_0|^2\hat{\mathbf{k}}$$

Note that:

$$\mathbf{E}_0 \times \left(\hat{\mathbf{k}}\times\mathbf{E}_0\right) = \hat{\mathbf{k}}\left(\mathbf{E}_0\cdot\mathbf{E}_0\right) - \mathbf{E}_0\left(\hat{\mathbf{k}}\cdot\mathbf{E}_0\right)$$
$$= \hat{\mathbf{k}}|\mathbf{E}_0|^2$$

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Analysis of Maxwell's equations without sources -- continued:

Transverse Electric and Magnetic (TEM) waves

Summary of plane electromagnetic waves:

$$\mathbf{B}(\mathbf{r},t) = \Re\left(\frac{n\hat{\mathbf{k}}\times\mathbf{E}_0}{c} e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\right) \qquad \mathbf{E}(\mathbf{r},t) = \Re\left(\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\right)$$

$$|\mathbf{k}|^2 = \left(\frac{\omega}{v}\right)^2 = \left(\frac{n\omega}{c}\right)^2 \quad \text{where } n \equiv \sqrt{\frac{\mu\varepsilon}{\mu_0\varepsilon_0}} \quad \text{and } \hat{\mathbf{k}}\cdot\mathbf{E}_0 = 0$$

Energy density for plane electromagnetic waves:

$$\langle u\rangle_{avg} = \frac{1}{4}\Re\left(\varepsilon\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\cdot\left(\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\right)^*\right) +$$
$$\frac{1}{4}\Re\left(\frac{1}{\mu}\frac{n\hat{\mathbf{k}}\times\mathbf{E}_0}{c} e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\cdot\left(\frac{n\hat{\mathbf{k}}\times\mathbf{E}_0}{c} e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\right)^*\right)$$
$$= \frac{1}{2}\varepsilon|\mathbf{E}_0|^2$$

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