PHY 712 Electrodynamics
12-12:50 AM Olin 103

Plan for Lecture 20:

Review of Chap. 1-8

1. Plan for next week
2. Comment on exam
3. Review

1

13	Wed: 02/12/2020	Chap. 5	Magnetic dipoles and dipolar fields	#12	02/17/2020
14	Fri: 02/14/2020	Chap. 6	Maxwell's Equations	#13	02/19/2020
15	Mon: 02/17/2020	Chap. 6	Electromagnetic energy and forces	#14	02/21/2020
16	Wed: 02/19/2020	Chap. 7	Electromagnetic plane waves	#15	02/24/2020
17	Fri: 02/21/2020	Chap. 7	Electromagnetic plane waves	#16	02/26/2020
18	Mon: 02/24/2020	Chap. 7	Optical effects of refractive indices		
19	Wed: 02/26/2020	Chap. 8	EM waves in wave guides		
20	Fri: 02/28/2020	Chap. 1-8	Review		
	Mon: 03/02/2020	No class	*APS March Meeting*	Take Home Exam	
	Wed: 03/04/2020	No class	*APS March Meeting*	Take Home Exam	
	Fri: 03/06/2020	No class	*APS March Meeting*	Take Home Exam	
	Mon: 03/09/2020	No class	*Spring Break*		
	Wed: 03/11/2020	No class	*Spring Break*		
	Fri: 03/13/2020	No class	*Spring Break*		
21	Mon: 03/16/2020	Chap. 9	Radiation from localized oscillating sources		

2

Next week --
Colloquium: "Changes in Blood Clot Structure and Mechanics in Cardiovascular and Thromboembolic Diseases"

Dr. Stephen Baker, Teacher Scholar Postdoctoral Fellow
WFU Physics

George P. Williams, Jr. Lecture Hall, (Olin 101)
Wednesday, March 4, 2020 at 3:00 PM

There will be a reception in the Olin Lounge at approximately 4 PM following the colloquium. All interested persons are cordially invited to attend.

ABSTRACT Studies in recent years have shown blood clot structure and mechanical properties to be a novel risk factor for cardiovascular diseases, the leading cause of morbidity and mortality worldwide. As a result, we need to better understand how the structural and mechanical properties of blood clots from patients with cardiovascular disease are different from those of healthy individuals. To study these properties, we need to determine how they change at different length scales. On the nano- and microscale, an atomic force microscope is an extremely versatile piece of equipment that can be used for nanometer to micrometer scale imaging, normal force unfolding of single molecules, or even novel lateral force techniques.

3

Comment on exam

Three multipart questions

Completed exam accepted until Monday, Mar. 9, 2020

Email your questions to natalie@wfu.edu

Will be present in Friday March 6, 2020

4

Review of mathematical relationships

Some useful identities for vectors and vector operators

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) = \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})$$
$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$$
$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c})$$
$$\nabla\times\nabla\psi = 0$$
$$\nabla\cdot(\nabla\times\mathbf{a}) = 0$$
$$\nabla\times(\nabla\times\mathbf{a}) = \nabla(\nabla\cdot\mathbf{a}) - \nabla^2\mathbf{a}$$
$$\nabla\cdot(\psi\mathbf{a}) = \mathbf{a}\cdot\nabla\psi + \psi\nabla\cdot\mathbf{a}$$
$$\nabla\times(\psi\mathbf{a}) = \nabla\psi\times\mathbf{a} + \psi\nabla\times\mathbf{a}$$
$$\nabla(\mathbf{a}\cdot\mathbf{b}) = (\mathbf{a}\cdot\nabla)\mathbf{b} + (\mathbf{b}\cdot\nabla)\mathbf{a} + \mathbf{a}\times(\nabla\times\mathbf{b}) + \mathbf{b}\times(\nabla\times\mathbf{a})$$
$$\nabla\cdot(\mathbf{a}\times\mathbf{b}) = \mathbf{b}\cdot(\nabla\times\mathbf{a}) - \mathbf{a}\cdot(\nabla\times\mathbf{b})$$
$$\nabla\times(\mathbf{a}\times\mathbf{b}) = \mathbf{a}(\nabla\cdot\mathbf{b}) - \mathbf{b}(\nabla\cdot\mathbf{a}) + (\mathbf{b}\cdot\nabla)\mathbf{a} - (\mathbf{a}\cdot\nabla)\mathbf{b}$$

5

Vector relations for spherical polar coordinates

$$\nabla\psi = \hat{\mathbf{r}}\frac{\partial\psi}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial\psi}{\partial\theta} + \hat{\boldsymbol{\phi}}\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}$$
$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}$$
$$\nabla\cdot\mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2A_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta A_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$$
$$\nabla\times\mathbf{A} = \hat{\mathbf{r}}\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta A_\phi\right) - \frac{\partial A_\theta}{\partial\phi}\right]$$
$$+\hat{\boldsymbol{\theta}}\left[\frac{1}{r\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{1}{r}\frac{\partial}{\partial r}\left(rA_\phi\right)\right] + \hat{\boldsymbol{\phi}}\frac{1}{r}\left[\frac{\partial}{\partial r}\left(rA_\theta\right) - \frac{\partial A_r}{\partial\theta}\right]$$
$$\hat{\mathbf{x}} = \hat{\mathbf{r}}\sin\theta\cos\phi + \hat{\boldsymbol{\theta}}\cos\theta\cos\phi - \hat{\boldsymbol{\phi}}\sin\phi$$
$$\hat{\mathbf{y}} = \hat{\mathbf{r}}\sin\theta\sin\phi + \hat{\boldsymbol{\theta}}\cos\theta\sin\phi + \hat{\boldsymbol{\phi}}\cos\phi$$
$$\hat{\mathbf{z}} = \hat{\mathbf{r}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta$$
$$\frac{\partial}{\partial x} = \sin\theta\cos\phi\frac{\partial}{\partial r} + \cos\theta\cos\phi\frac{1}{r}\frac{\partial}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial\phi}$$
$$\frac{\partial}{\partial y} = \sin\theta\sin\phi\frac{\partial}{\partial r} + \cos\theta\sin\phi\frac{1}{r}\frac{\partial}{\partial\theta} + \frac{\cos\phi}{r\sin\theta}\frac{\partial}{\partial\phi}$$
$$\frac{\partial}{\partial z} = \cos\theta\frac{\partial}{\partial r} - \sin\theta\frac{\partial}{\partial\theta}$$

6

Some properties of a delta function

In one-dimension:

Note that for any function $F(x)$:

$$\int_{-\infty}^{\infty} F(x)\delta(x-x_0)dx = F(x_0)$$

Now consider a function $p(x)$, for which $p(x_i)=0$ for $i=1,2,\cdots$

$$\int_{-\infty}^{\infty} F(x)\delta(p(x))dx = \int_{-\infty}^{\infty} F(x)\left(\sum_i \delta\left((x-x_i)\left.\frac{dp}{dx}\right|_{x_i}\right)\right)dx$$

$$= \sum_i \frac{F(x_i)}{\left|\frac{dp}{dx}\right|_{x_i}}$$

In three-dimensions:

$$\delta^3(\mathbf{r}-\mathbf{r}_0) \equiv \delta(x-x_0)\delta(y-y_0)\delta(z-z_0) \quad \text{Cartesian}$$

$$= \frac{1}{r^2}\delta(r-r_0)\delta(\phi-\phi_0)\delta(\cos\theta-\cos\theta_0) \quad \text{Spherical}$$

$$= \frac{1}{r}\delta(r-r_0)\delta(\phi-\phi_0)\delta(z-z_0) \quad \text{Cylindrical}$$

02/28/2020 PHY 712 Spring 2020 -- Lecture 20 7

7

Orthogonal functions useful for angular representations

Legendre polynomials for $-1 \le x \le 1$:

$P_0(x) = 1 \qquad P_1(x) = x$

$P_2(x) = \frac{1}{2}(3x^2-1) \qquad P_3(x) = \frac{1}{2}(5x^3-3x)$

Spherical harmonic functions

$l=0: \quad Y_{00}(\hat{\mathbf{r}}) = \frac{1}{\sqrt{4\pi}}$

$l=1: \quad Y_{1(\pm1)}(\hat{\mathbf{r}}) = \mp\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{\pm i\varphi} \qquad Y_{10}(\hat{\mathbf{r}}) = \sqrt{\frac{3}{4\pi}}\cos\theta$

$l=2: \quad Y_{2(\pm2)}(\hat{\mathbf{r}}) = \sqrt{\frac{15}{32\pi}}\sin^2\theta\, e^{\pm 2i\varphi} \qquad Y_{2(\pm1)}(\hat{\mathbf{r}}) = \mp\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{\pm i\varphi}$

$Y_{20}(\hat{\mathbf{r}}) = \sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$

Note that $Y_{l0}(\hat{\mathbf{r}}) = \sqrt{\frac{2l+1}{4\pi}}P_l(\cos\theta)$

02/28/2020 PHY 712 Spring 2020 -- Lecture 20 8

8

Useful identities related to Coulomb kernel:

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \frac{1}{\sqrt{r^2+r'^2-2\mathbf{r}\cdot\mathbf{r}'}} = \sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}P_l(\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}')$$

$$P_l(\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}') = \frac{4\pi}{2l+1}\sum_{m=-l}^{l}Y_{lm}(\hat{\mathbf{r}})Y_{lm}^*(\hat{\mathbf{r}}')$$

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \sum_{lm}\frac{4\pi}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y_{lm}(\theta,\varphi)Y_{lm}^*(\theta',\varphi')$$

Also note that: $\nabla^2\frac{1}{|\mathbf{r}-\mathbf{r}'|} = -4\pi\delta^3(\mathbf{r}-\mathbf{r}')$

02/28/2020 PHY 712 Spring 2020 -- Lecture 20 9

9

Green's theorem for electrostatics

Poisson equation: $\nabla^2\Phi(\mathbf{r}) = -\frac{\rho(\mathbf{r})}{\varepsilon_0}$

Green's relation: $\nabla'^2 G(\mathbf{r},\mathbf{r}') = -4\pi\delta^3(\mathbf{r}-\mathbf{r}')$.

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_V d^3r'\,\rho(\mathbf{r}')G(\mathbf{r},\mathbf{r}') +$$

volume integral over source

$$\frac{1}{4\pi}\int_S d^2r'\left[G(\mathbf{r},\mathbf{r}')\nabla'\Phi(\mathbf{r}') - \Phi(\mathbf{r}')\nabla'G(\mathbf{r},\mathbf{r}')\right]\cdot\hat{\mathbf{r}}'.$$

surface integrals incorporating boundary values

10

Review

Maxwell's equations

Coulomb's law: $\nabla\cdot\mathbf{D} = \rho_{free}$

Ampere-Maxwell's law: $\nabla\times\mathbf{H} - \frac{\partial\mathbf{D}}{\partial t} = \mathbf{J}_{free}$

Faraday's law: $\nabla\times\mathbf{E} + \frac{\partial\mathbf{B}}{\partial t} = 0$

No magnetic monopoles: $\nabla\cdot\mathbf{B} = 0$

For linear isotropic media and no sources: $\mathbf{D} = \varepsilon\mathbf{E}$; $\mathbf{B} = \mu\mathbf{H}$

11

Review -- continued

Maxwell's equations

Microscopic or vacuum form $(\mathbf{P} = 0;\ \mathbf{M} = 0)$:

Coulomb's law: $\nabla\cdot\mathbf{E} = \rho/\varepsilon_0$

Ampere-Maxwell's law: $\nabla\times\mathbf{B} - \frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t} = \mu_0\mathbf{J}$

Faraday's law: $\nabla\times\mathbf{E} + \frac{\partial\mathbf{B}}{\partial t} = 0$

No magnetic monopoles: $\nabla\cdot\mathbf{B} = 0$

$$\Rightarrow c^2 = \frac{1}{\varepsilon_0\mu_0}$$

12

Review -- continued

$$\nabla \cdot \mathbf{B} = 0 \quad \Rightarrow \mathbf{B} = \nabla \times \mathbf{A}$$

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \quad \Rightarrow \nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} \right) = 0$$

$$\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\nabla \Phi$$

$$\text{or} \quad \mathbf{E} = -\nabla \Phi - \frac{\partial \mathbf{A}}{\partial t}$$

02/28/2020 PHY 712 Spring 2020 -- Lecture 20 13

13

Review -- continued

Analysis of the scalar and vector potential equations:

$$-\nabla^2 \Phi - \frac{\partial (\nabla \cdot \mathbf{A})}{\partial t} = \rho / \varepsilon_0$$

$$\nabla \times (\nabla \times \mathbf{A}) + \frac{1}{c^2} \left(\frac{\partial (\nabla \Phi)}{\partial t} + \frac{\partial^2 \mathbf{A}}{\partial t^2} \right) = \mu_0 \mathbf{J}$$

Lorentz gauge form -- require $\nabla \cdot \mathbf{A}_L + \frac{1}{c^2} \frac{\partial \Phi_L}{\partial t} = 0$

$$-\nabla^2 \Phi_L + \frac{1}{c^2} \frac{\partial^2 \Phi_L}{\partial t^2} = \rho / \varepsilon_0$$

$$-\nabla^2 \mathbf{A}_L + \frac{1}{c^2} \frac{\partial^2 \mathbf{A}_L}{\partial t^2} = \mu_0 \mathbf{J}$$

02/28/2020 PHY 712 Spring 2020 -- Lecture 20 14

14

Review – continued – focusing on statics --

When to solve equations using integral form versus differential form?

Examples from electrostatic and magnetostatic cases:

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \int d^3 r' \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}$$

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int d^3 r' \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}$$

Useful for spatially confined sources.

02/28/2020 PHY 712 Spring 2020 -- Lecture 20 15

15

Review -- continued

General form of electrostatic potential with boundary value $r \to \infty$, for isolated charge density $\rho(\mathbf{r})$:

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int d^3r' \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$$

$$= \frac{1}{4\pi\varepsilon_0}\int d^3r' \rho(\mathbf{r}')\left(\sum_{lm}\frac{4\pi}{2l+1}\frac{r_<^{\,l}}{r_>^{\,l+1}} Y_{lm}(\theta,\varphi)Y_{lm}{}^*(\theta',\varphi')\right)$$

Suppose that $\rho(\mathbf{r}) = \sum_{lm}\rho_{lm}(r)Y_{lm}(\theta,\varphi)$

$$\Rightarrow \Phi(\mathbf{r}) = \frac{1}{\varepsilon_0}\sum_{lm}\frac{1}{2l+1}Y_{lm}(\theta,\varphi)\left(\frac{1}{r^{l+1}}\int_0^r r'^{2+l}\,dr'\,\rho_{lm}(r') + r^l\int_r^\infty r'^{1-l}\,dr'\rho_{lm}(r')\right)$$

02/28/2020 PHY 712 Spring 2020 -- Lecture 20 16

16

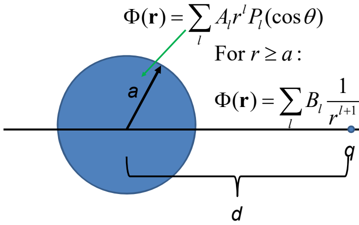
Example of use of spherical harmonic expansion: Problem #4.9 in Jackson. A point charge *q* is located in free space a distance d from the center of a dielectric sphere of radius *a* (*a*<*d*) and dielectric constant $\varepsilon/\varepsilon_0$. Find the electrostatic potential.

For $r \le a$:

$$\Phi(\mathbf{r}) = \sum_l A_l r^l P_l(\cos\theta)$$

For $r \ge a$:

$$\Phi(\mathbf{r}) = \sum_l B_l \frac{1}{r^{l+1}} P_l(\cos\theta) + \frac{q}{4\pi\epsilon_0}\frac{1}{|\mathbf{r}-d\hat{\mathbf{z}}|}$$



02/28/2020 PHY 712 Spring 2020 -- Lecture 20 17

17

Review of HW -- continued

For $r \le a$:

$$\Phi(\mathbf{r}) = \sum_l A_l r^l P_l(\cos\theta)$$

For $r \ge a$:

$$\Phi(\mathbf{r}) = \sum_l B_l \frac{1}{r^{l+1}} P_l(\cos\theta) + \frac{q}{4\pi\epsilon_0}\frac{1}{|\mathbf{r}-d\hat{\mathbf{z}}|}$$

In order to match BC's at $r = a$:

$$\frac{1}{|\mathbf{r}-d\hat{\mathbf{z}}|} = \sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}P_l(\cos\theta) = \sum_{l=0}^{\infty}\frac{a^l}{d^{l+1}}P_l(\cos\theta)$$

02/28/2020 PHY 712 Spring 2020 -- Lecture 20 18

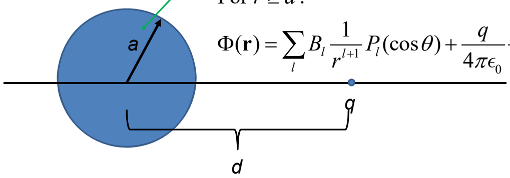
18

For $r \le a$:

$$\Phi(\mathbf{r}) = \sum_l A_l r^l P_l(\cos\theta)$$

For $r \ge a$:

$$\Phi(\mathbf{r}) = \sum_l B_l \frac{1}{r^{l+1}} P_l(\cos\theta) + \frac{q}{4\pi\epsilon_0}\frac{1}{|\mathbf{r} - d\hat{\mathbf{z}}|}$$



Boundary conditions:

$\mathbf{D}\cdot\hat{\mathbf{r}}\big|_{r=a}$ = continuous $\qquad$ $\mathbf{E}\cdot\hat{\boldsymbol{\theta}}\big|_{r=a}$ = continuous

$$\varepsilon \frac{\partial\Phi_{in}(r)}{\partial r}\bigg|_{r=a} = \varepsilon_0 \frac{\partial\Phi_{out}(r)}{\partial r}\bigg|_{r=a} \qquad \frac{\partial\Phi_{in}(r)}{\partial\theta}\bigg|_{r=a} = \frac{\partial\Phi_{out}(r)}{\partial\theta}\bigg|_{r=a}$$

19

Boundary conditions:

$\mathbf{D}\cdot\hat{\mathbf{r}}\big|_{r=a}$ = continuous $\qquad$ $\mathbf{E}\cdot\hat{\boldsymbol{\theta}}\big|_{r=a}$ = continuous

$$\varepsilon \frac{\partial\Phi_{in}(r)}{\partial r}\bigg|_{r=a} = \varepsilon_0 \frac{\partial\Phi_{out}(r)}{\partial r}\bigg|_{r=a} \qquad \frac{\partial\Phi_{in}(r)}{\partial\theta}\bigg|_{r=a} = \frac{\partial\Phi_{out}(r)}{\partial\theta}\bigg|_{r=a}$$

Equality for each l:

$$\varepsilon l a^{l-1} A_l = -\frac{\varepsilon_0 (l+1)}{a^{l+2}} B_l + \frac{q}{4\pi}\frac{l a^{l-1}}{d^{l+1}}$$

$$a^l A_l = +\frac{1}{a^{l+1}} B_l + \frac{q}{4\pi}\frac{a^l}{d^{l+1}}$$

2 equations and 2 unknowns for each l

20

Review -- continued

Hyperfine interaction energy:

$$E_{int} \equiv H_{HF} = -\mu_e \cdot \mathbf{B}_{\mu_N} - \mu_N \cdot \mathbf{B}_{\mathbf{J}_e}(0)$$

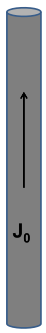
Putting all of the terms together:

$$H_{\mathrm{HF}} = -\frac{\mu_0}{4\pi}\left(\left\langle \frac{3(\boldsymbol{\mu}_N\cdot\hat{\mathbf{r}})(\boldsymbol{\mu}_e\cdot\hat{\mathbf{r}}) - \boldsymbol{\mu}_N\cdot\boldsymbol{\mu}_e}{r^3} + \frac{8\pi}{3}\boldsymbol{\mu}_N\cdot\boldsymbol{\mu}_e\delta^3(\mathbf{r})\right\rangle + \frac{e}{m_e}\left\langle\frac{\mathbf{L}\cdot\boldsymbol{\mu}_N}{r^3}\right\rangle\right).$$

In this expression the brackets $\langle\ \rangle$ indicate evaluating the expectation value relative to the electronic state.

21

Comment on HW problem

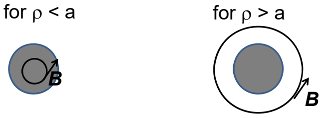


1. Consider an infinitely long wire with radius a, oriented along the **z** axis. There is a steady uniform current inside the wire. Specifically the current is along the z-axis with the magnitude of J_0 for $\rho \leq a$ and zero for $\rho > a$, where ρ denotes the radial parameter of the natural cylindrical coordinates of the system.
   a. Find the vector potential (**A**) for all ρ.
   b. Find the magnetic flux field (**B**) for all ρ.

Solution to problem using PHY 114 ideas

In this case, it is convenient to solve part b first.

Top view for $\rho < a$ — Top view for $\rho > a$



22

Comment on HW -- continued

Top view for $\rho < a$ — Top view for $\rho > a$

For $\rho < a$:

$$\oint \mathbf{B}\cdot d\boldsymbol{\ell} = \mu_0 \int \mathbf{J}\cdot d\mathbf{A}$$

$$2\pi\rho B = \mu_0 J_0 \pi\rho^2$$

$$B = \frac{\mu_0 J_0 \rho}{2}$$

$$\mathbf{B} = \frac{\mu_0 J_0 \rho}{2}\hat{\boldsymbol{\varphi}} = \nabla\times\mathbf{A}$$

$$\mathbf{A} = -\frac{\mu_0 J_0\left(\rho^2 - a^2\right)}{4}\hat{\mathbf{z}}$$

For $\rho > a$:

$$\oint \mathbf{B}\cdot d\boldsymbol{\ell} = \mu_0 \int \mathbf{J}\cdot d\mathbf{A}$$

$$2\pi\rho B = \mu_0 J_0 \pi a^2$$

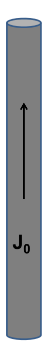
$$B = \frac{\mu_0 J_0 a^2}{2\rho}$$

$$\mathbf{B} = \frac{\mu_0 J_0 a^2}{2\rho}\hat{\boldsymbol{\varphi}} = \nabla\times\mathbf{A}$$

$$\mathbf{A} = -\frac{\mu_0 J_0 a^2 \ln\left(\rho/a\right)}{2}\hat{\mathbf{z}}$$

23

Comment on HW -- continued



Alternative treatment using differential equations:

$$-\nabla^2\mathbf{A} = \begin{cases} \mu_0 J_0 \hat{\mathbf{z}} & \text{for } \rho \leq a \\ 0 & \text{for } \rho > a \end{cases}$$

$$-\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial A_z(\rho)}{\partial\rho} = \begin{cases} \mu_0 J_0 & \text{for } \rho \leq a \\ 0 & \text{for } \rho > a \end{cases}$$

$$A_z(\rho) = \begin{cases} -\frac{\mu_0 J_0 \rho^2}{4} + C_1 & \text{for } \rho \leq a \\ C_2 + C_3 \ln(\rho) & \text{for } \rho > a \end{cases}$$

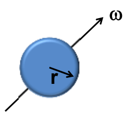
Choosing constants from continuity requirements:

$$A_z(\rho) = \begin{cases} -\frac{\mu_0 J_0 \rho^2}{4} + \frac{\mu_0 J_0 a^2}{4} & \text{for } \rho \leq a \\ -\frac{\mu_0 J_0 a^2}{2}\ln(\rho/a) & \text{for } \rho > a \end{cases}$$

$$\mathbf{B} = -\frac{\partial A_z(\rho)}{\partial\rho}\hat{\boldsymbol{\varphi}}$$

24

Comment on magnetic problem



A sphere of radius a carries a uniform surface charge distribution σ. The sphere is rotated about a diameter with constant angular velocity ω. Find the vector potential **A** and magnetic field **B** both inside and outside the sphere.

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int d^3r' \, \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}.$$

$$\mathbf{J}(\mathbf{r}') = \begin{cases} \sigma \; \delta(r'-a) \; \boldsymbol{\omega}\times\mathbf{r}' & \text{for } r' \leq a \\ 0 & \text{otherwise} \end{cases}$$

Note that: $$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \sum_{lm} \frac{4\pi}{2l+1} \frac{r_<^l}{r_>^{l+1}} Y_{lm}(\hat{\mathbf{r}}) Y_{lm}^*(\hat{\mathbf{r}}')$$

and: $$\int d\Omega' \sum_m Y_{lm}(\hat{\mathbf{r}}) Y_{lm}^*(\hat{\mathbf{r}}') \, \mathbf{r}' = \frac{r'}{r}\mathbf{r} \, \delta_{l1}.$$

25

Comment on magnetic problem -- continued

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int d^3r' \, \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} = \frac{\mu_0\sigma}{4\pi} \frac{\boldsymbol{\omega}\times\mathbf{r}}{r} \frac{4\pi}{3} \int_0^a r'^3 dr' \, \delta(r'-a) \frac{r_<}{r_>^2}$$

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0\sigma}{3} \boldsymbol{\omega}\times\mathbf{r} \begin{cases} a & \text{for } r \leq a \\ \dfrac{a^4}{r^3} & \text{for } r > a \end{cases}$$

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0\sigma}{3} \begin{cases} 2\boldsymbol{\omega} a & \text{for } r \leq a \\ \dfrac{a^4}{r^3}\left(3\left(\hat{\mathbf{r}}\cdot\boldsymbol{\omega}\right)\hat{\mathbf{r}} - \boldsymbol{\omega}\right) & \text{for } r > a \end{cases}$$

26