PHY 712 Electrodynamics

12-12:50 AM MWF via video link:
https://wakeforest-university.zoom.us/my/natalie.holzwarth

Plan for Lecture 26:

Finish Chap. 11 and begin Chap. 14

A. Electromagnetic field transformations & corresponding analysis of Liénard-Wiechert potentials for constant velocity sources

B. Radiation by moving charged particles

In this lecture we will continue to discuss the electromagnetic fields produced by a moving charged particle using the Lienard-Wiechert potentials. First we need to make sure that we obtain consistent results with Lecture 25. Then we will start to discuss the results from more general trajectories.

21	Mon: 03/23/2020	Chap. 9	Radiation from localized oscillating sources	#17	03/25/2020
22	Wed: 03/25/2020	Chap. 9	Radiation from oscillating sources	#18	03/27/2020
23	Fri: 03/27/2020	Chap. 9 and 10	Radiation from oscillating sources	#19	03/30/2020
24	Mon: 03/30/2020	Chap. 11	Special Theory of Relativity	#20	04/03/2020
25	Wed: 04/01/2020	Chap. 11	Special Theory of Relativity		
26	Fri: 04/03/2020	Chap. 11	Special Theory of Relativity	#21	04/06/2020
27	Mon: 04/06/2020	Chap. 14	Radiation from accelerating charged particles		
28	Wed: 04/08/2020	Chap. 14	Synchrotron radiation		
	Fri: 04/10/2020	No class	*Good Friday*		
29	Mon: 04/13/2020	Chap. 14	Synchrotron radiation		
30	Wed: 04/15/2020	Chap. 15	Radiation from collisions of charged particles		
31	Fri: 04/17/2020	Chap. 13	Cherenkov radiation		
32	Mon: 04/20/2020		Special topic: E & M aspects of superconductivity		
33	Wed: 04/22/2020		Special topic: Aspects of optical properties of materials		
34	Fri: 04/24/2020				
35	Mon: 04/27/2020				
36	Wed: 04/29/2020		Review		

The homework from today's lecture involves deriving some of the details of today's lecture.

Inverse transformation of field strength tensor

$$F'^{\alpha\beta} = \mathcal{L}_v^{-1\alpha\gamma} F^{\gamma\delta} \mathcal{L}_v^{-1\delta\beta} \qquad \mathcal{L}_v^{-1} = \begin{pmatrix} \gamma_v & -\gamma_v\beta_v & 0 & 0 \\ -\gamma_v\beta_v & \gamma_v & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$F'^{\alpha\beta} = \begin{pmatrix} 0 & -E_x & -\gamma_v\left(E_y - \beta_v B_z\right) & -\gamma_v\left(E_z + \beta_v B_y\right) \\ E_x & 0 & -\gamma_v\left(B_z - \beta_v E_y\right) & \gamma_v\left(B_y + \beta_v E_z\right) \\ \gamma_v\left(E_y - \beta_v B_z\right) & \gamma_v\left(B_z - \beta_v E_y\right) & 0 & -B_x \\ \gamma_v\left(E_z + \beta_v B_y\right) & -\gamma_v\left(B_y + \beta_v E_z\right) & B_x & 0 \end{pmatrix}$$

Summary of results:

$$E'_x = E_x \qquad B'_x = B_x$$

$$E'_y = \gamma_v\left(E_y - \beta_v B_z\right) \qquad B'_y = \gamma_v\left(B_y + \beta_v E_z\right)$$

$$E'_z = \gamma_v\left(E_z + \beta_v B_y\right) \qquad B'_z = \gamma_v\left(B_z - \beta_v E_y\right)$$

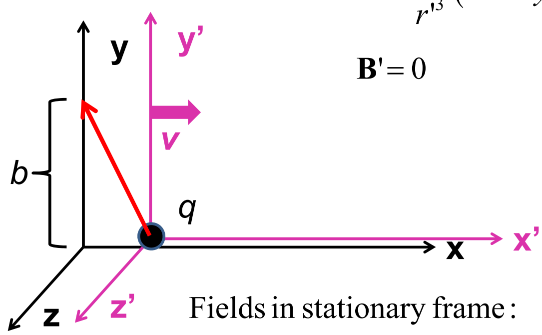
Review of the Lorentz transformation for the field strength tensor --

Example:

Fields in moving frame:

$$\mathbf{E}' = \frac{q}{r'^3}\left(x'\hat{\mathbf{x}} + y'\hat{\mathbf{y}}\right) = \frac{q\left(-vt'\hat{\mathbf{x}} + b\hat{\mathbf{y}}\right)}{\left(\left(-vt'\right)^2 + b^2\right)^{3/2}}$$

$$\mathbf{B}' = 0$$



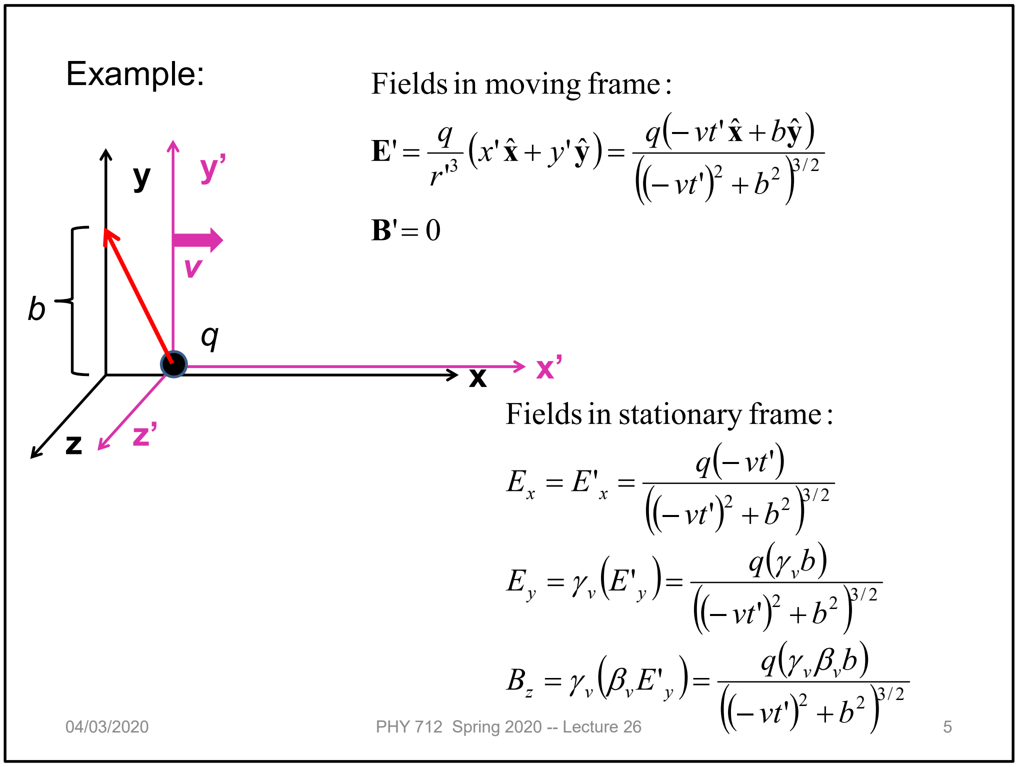
Fields in stationary frame:

$$E_x = E'_x \qquad B_x = B'_x$$

$$E_y = \gamma_v\left(E'_y + \beta_v B'_z\right) \qquad B_y = \gamma_v\left(B'_y - \beta_v E'_z\right)$$

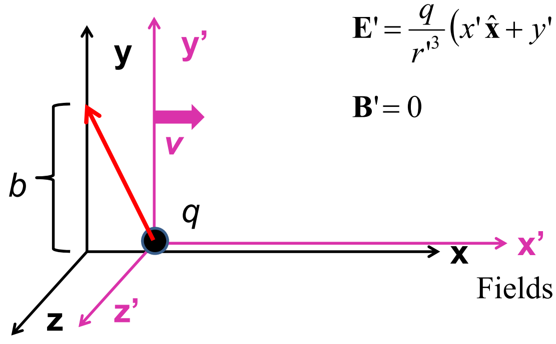
$$E_z = \gamma_v\left(E'_z - \beta_v B'_y\right) \qquad B_z = \gamma_v\left(B'_z + \beta_v E'_y\right)$$

This is the example that we have been studying from Lecture 25.



Using the fields from the moving frame, we can write the expressions for the fields in the stationary frame.

Example:



Fields in moving frame:

$$\mathbf{E}' = \frac{q}{r'^3}\left(x'\hat{\mathbf{x}} + y'\hat{\mathbf{y}}\right) = \frac{q\left(-vt'\hat{\mathbf{x}} + b\hat{\mathbf{y}}\right)}{\left(\left(-vt'\right)^2 + b^2\right)^{3/2}}$$

$$\mathbf{B}' = 0$$

Fields in stationary frame:

$$E_x = E'_x = \frac{q\left(-v\gamma_v t\right)}{\left(\left(-v\gamma_v t\right)^2 + b^2\right)^{3/2}}$$

$$E_y = \gamma_v\left(E'_y\right) = \frac{q\left(\gamma_v b\right)}{\left(\left(-v\gamma_v t\right)^2 + b^2\right)^{3/2}}$$

$$B_z = \gamma_v\left(\beta_v E'_y\right) = \frac{q\left(\gamma_v \beta_v b\right)}{\left(\left(-v\gamma_v t\right)^2 + b^2\right)^{3/2}}$$

Expression in terms of consistent coordinates

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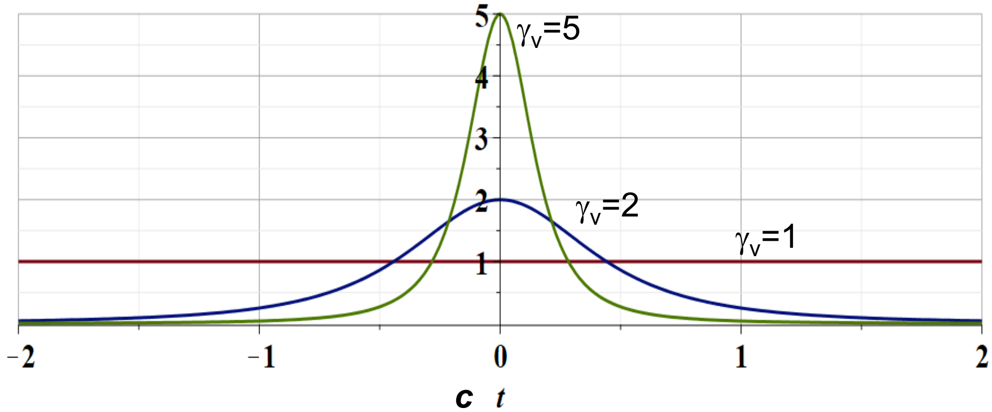
Here the fields measured in the stationary frame are expressed in terms of the time *t* measured in the stationary frame.

$$E_y = \frac{q(\gamma_v b)}{\left((-v\gamma_v t)^2 + b^2\right)^{3/2}} = \frac{q(\gamma_v b)}{\left((\gamma_v^2 - 1)c^2t^2 + b^2\right)^{3/2}}$$



This is a plot shown in Lecture 25 of E_y as a function of time.

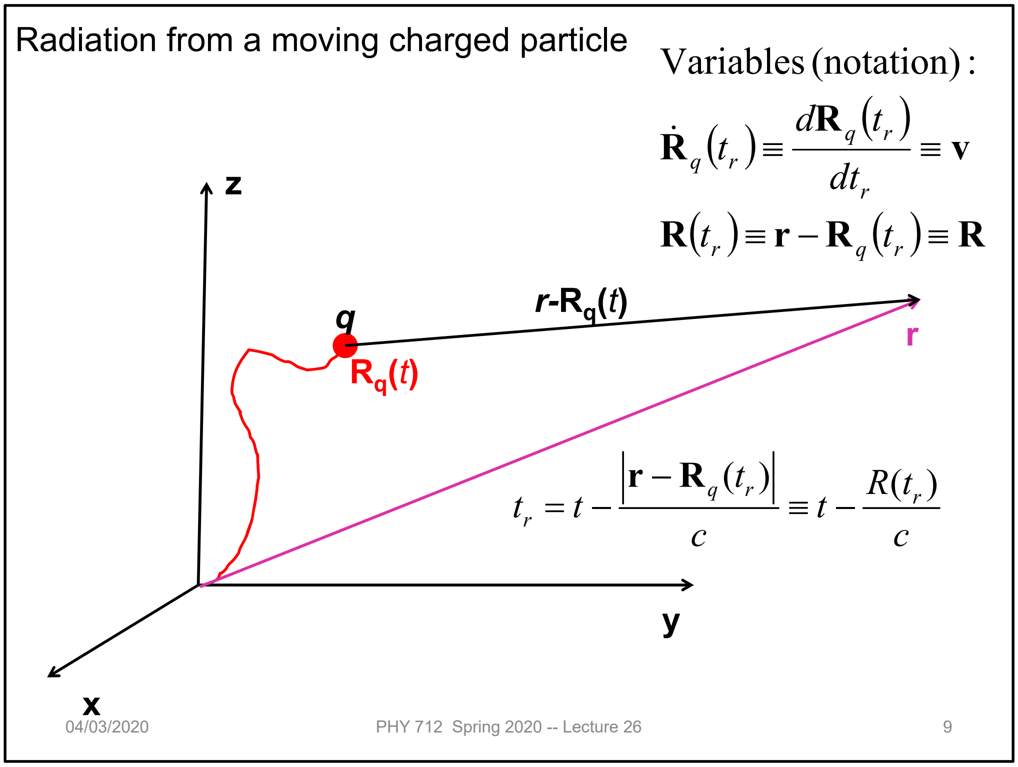
Examination of this system from the viewpoint of the the Liénard-Wiechert potentials –(Gaussian units)

$$\mathbf{E}(\mathbf{r},t) = \frac{q}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left[\left(\mathbf{R} - \frac{\mathbf{v}R}{c}\right)\left(1 - \frac{v^2}{c^2}\right) + \left(\mathbf{R} \times \left\{\left(\mathbf{R} - \frac{\mathbf{v}R}{c}\right) \times \frac{\dot{\mathbf{v}}}{c^2}\right\}\right)\right]$$

$$\mathbf{B}(\mathbf{r},t) = \frac{q}{c}\left[\frac{-\mathbf{R}\times\mathbf{v}}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left(1 - \frac{v^2}{c^2} + \frac{\dot{\mathbf{v}}\cdot\mathbf{R}}{c^2}\right) - \frac{\mathbf{R}\times\dot{\mathbf{v}}/c}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^2}\right]$$

$$\mathbf{B}(\mathbf{r},t) = \frac{\mathbf{R}\times\mathbf{E}(\mathbf{r},t)}{R}.$$

Now we consider how we may arrive at the same result without changing reference frames by analyzing the EM fields produced by a moving charge using the Lienard-Wiechert analysis.



Here we consider a charged particle (charge q) moving along the red trajectory. The vector **r** indicates the point at which we will evaluate the fields. The retarded time t_r is defined here.

Examination of this system from the viewpoint of the the Liènard-Wiechert potentials –(Gaussian units)

$$\mathbf{E}(\mathbf{r},t) = \frac{q}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left[\left(\mathbf{R} - \frac{\mathbf{v}R}{c}\right)\left(1 - \frac{v^2}{c^2}\right)\right]$$

$$\mathbf{B}(\mathbf{r},t) = \frac{q}{c}\left[\frac{-\mathbf{R}\times\mathbf{v}}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left(1 - \frac{v^2}{c^2}\right)\right]$$

For our example:

$\mathbf{R}_q(t_r) = vt_r\hat{\mathbf{x}}$ $\quad$ $\mathbf{r} = b\hat{\mathbf{y}}$

$\mathbf{R} = b\hat{\mathbf{y}} - vt_r\hat{\mathbf{x}}$ $\quad$ $R = \sqrt{v^2t_r^2 + b^2}$

$\mathbf{v} = v\hat{\mathbf{x}}$ $\quad$ $t_r = t - \frac{R}{c}$

This should be equivalent to the result given in Jackson (11.152):

$$\mathbf{E}(x,y,z,t) = \mathbf{E}(0,b,0,t) = q\frac{-v\gamma t\hat{\mathbf{x}} + \gamma b\hat{\mathbf{y}}}{\left(b^2 + (v\gamma t)^2\right)^{3/2}}$$

$$\mathbf{B}(x,y,z,t) = \mathbf{B}(0,b,0,t) = q\frac{\gamma\beta b\hat{\mathbf{z}}}{\left(b^2 + (v\gamma t)^2\right)^{3/2}}$$

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In our case, the trajectory of the moving particle is described as constant velocity along the x-axis while the fields are measured at the fixed point b along the y axis.

Some details

$$\mathbf{E}(\mathbf{r},t)=\frac{q}{\left(R-\frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left[\left(\mathbf{R}-\frac{\mathbf{v}R}{c}\right)\left(1-\frac{v^2}{c^2}\right)\right]$$

$$\mathbf{B}(\mathbf{r},t)=\frac{q}{c}\left[\frac{-\mathbf{R}\times\mathbf{v}}{\left(R-\frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left(1-\frac{v^2}{c^2}\right)\right]$$

For our example:

$\mathbf{R}_q(t_r)=vt_r\hat{\mathbf{x}}$ $\quad \mathbf{r}=b\hat{\mathbf{y}}$

$\mathbf{R}=b\hat{\mathbf{y}}-vt_r\hat{\mathbf{x}}$ $\quad R=\sqrt{v^2t_r^2+b^2}$

$\mathbf{v}=v\hat{\mathbf{x}}$ $\quad t_r=t-\frac{R}{c}$

t_r must be a solution to a quadradic equation:

$$t_r-t=-\frac{R}{c} \qquad \Rightarrow \qquad t_r^2-2\gamma^2tt_r+\gamma^2t^2-\gamma^2b^2/c^2=0$$

with the physical solution:

$$t_r=\gamma\left(\gamma t-\frac{\sqrt{(v\gamma t)^2+b^2}}{c}\right)$$

For your homework for this lecture, you are asked to review the evaluations here.

Some details continued:

Now we can express R as:

$$R = \gamma\left(-\beta v\gamma t + \sqrt{(v\gamma t)^2 + b^2}\right)$$

and the related quantities:

$$\mathbf{R} - \mathbf{v}R/c = -vt\hat{\mathbf{x}} + b\hat{\mathbf{y}}$$

$$R - \mathbf{v}\cdot\mathbf{R}/c = \frac{\sqrt{(v\gamma t)^2 + b^2}}{\gamma}$$

$$\mathbf{E}(\mathbf{r},t) = \frac{q}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left[\left(\mathbf{R} - \frac{\mathbf{v}R}{c}\right)\left(1 - \frac{v^2}{c^2}\right)\right] = q\frac{-v\gamma t\hat{\mathbf{x}} + \gamma b\hat{\mathbf{y}}}{\left(b^2 + (v\gamma t)^2\right)^{3/2}}$$

$$\mathbf{B}(\mathbf{r},t) = \frac{q}{c}\left[\frac{-\mathbf{R}\times\mathbf{v}}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left(1 - \frac{v^2}{c^2}\right)\right] = q\frac{\gamma\beta b\hat{\mathbf{z}}}{\left(b^2 + (v\gamma t)^2\right)^{3/2}}$$

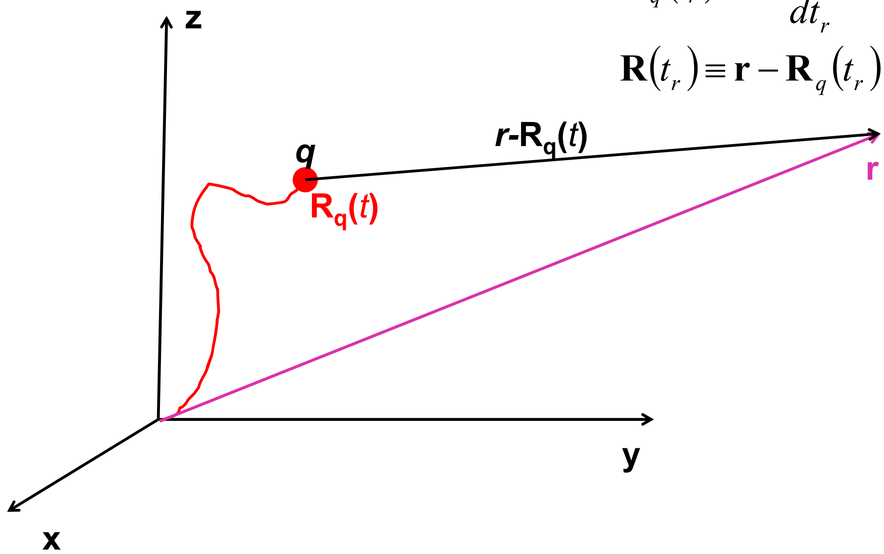
When the dust clears, we do verify the E and B fields obtained using the Lorentz transformation.

Radiation from a moving charged particle

Variables (notation):

$$\dot{\mathbf{R}}_q(t_r) \equiv \frac{d\mathbf{R}_q(t_r)}{dt_r} \equiv \mathbf{v}$$

$$\mathbf{R}(t_r) \equiv \mathbf{r} - \mathbf{R}_q(t_r) \equiv \mathbf{R}$$



With this success, we are motivated to apply this approach to more general particle trajectories.

Liénard-Wiechert fields (cgs Gaussian units):

$$\mathbf{E}(\mathbf{r},t) = \frac{q}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left[\left(\mathbf{R} - \frac{\mathbf{v}R}{c}\right)\left(1 - \frac{v^2}{c^2}\right) + \left(\mathbf{R} \times \left\{\left(\mathbf{R} - \frac{\mathbf{v}R}{c}\right) \times \frac{\dot{\mathbf{v}}}{c^2}\right\}\right)\right]. \tag{19}$$

$$\mathbf{B}(\mathbf{r},t) = \frac{q}{c}\left[\frac{-\mathbf{R} \times \mathbf{v}}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left(1 - \frac{v^2}{c^2} + \frac{\dot{\mathbf{v}}\cdot\mathbf{R}}{c^2}\right) - \frac{\mathbf{R} \times \dot{\mathbf{v}}/c}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^2}\right]. \tag{20}$$

In this case, the electric and magnetic fields are related according to

$$\mathbf{B}(\mathbf{r},t) = \frac{\mathbf{R} \times \mathbf{E}(\mathbf{r},t)}{R}. \tag{21}$$

$$\dot{\mathbf{R}}_q(t_r) \equiv \frac{d\mathbf{R}_q(t_r)}{dt_r} \equiv \mathbf{v} \qquad \mathbf{R}(t_r) \equiv \mathbf{r} - \mathbf{R}_q(t_r) \equiv \mathbf{R} \qquad \dot{\mathbf{v}} \equiv \frac{d^2\mathbf{R}_q(t_r)}{dt_r^{\ 2}}$$

Here we review the equations from the Lienard-Wiechert analysis. We particularly notice that for the fields very far from the particle positions, the dominant terms are those which involve the acceleration of the particle.

Electric field far from source:

$$\mathbf{E}(\mathbf{r},t) = \frac{q}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left\{\mathbf{R}\times\left[\left(\mathbf{R} - \frac{\mathbf{v}R}{c}\right)\times\frac{\dot{\mathbf{v}}}{c^2}\right]\right\}$$

$$\mathbf{B}(\mathbf{r},t) = \frac{\mathbf{R}\times\mathbf{E}(\mathbf{r},t)}{R}$$

Let $\hat{\mathbf{R}} \equiv \frac{\mathbf{R}}{R}$ $\quad \boldsymbol{\beta} \equiv \frac{\mathbf{v}}{c}$ $\quad \dot{\boldsymbol{\beta}} \equiv \frac{\dot{\mathbf{v}}}{c}$

$$\mathbf{E}(\mathbf{r},t) = \frac{q}{cR\left(1 - \boldsymbol{\beta}\cdot\hat{\mathbf{R}}\right)^3}\left\{\hat{\mathbf{R}}\times\left[\left(\hat{\mathbf{R}} - \boldsymbol{\beta}\right)\times\dot{\boldsymbol{\beta}}\right]\right\}$$

$$\mathbf{B}(\mathbf{r},t) = \hat{\mathbf{R}}\times\mathbf{E}(\mathbf{r},t)$$

These acceleration terms are given here. These are the terms that we will focus on. Here we define a unit vector Rhat. Jackson calls this vector **n**. In principle, this unit vector varies in time, but at large enough distances from the source, it is an approximately constant unit vector.

Poynting vector:

$$\mathbf{S}(\mathbf{r},t) = \frac{c}{4\pi}(\mathbf{E}\times\mathbf{B})$$

$$\mathbf{E}(\mathbf{r},t) = \frac{q}{cR\left(1-\boldsymbol{\beta}\cdot\hat{\mathbf{R}}\right)^3}\left\{\hat{\mathbf{R}}\times\left[\left(\hat{\mathbf{R}}-\boldsymbol{\beta}\right)\times\dot{\boldsymbol{\beta}}\right]\right\}$$

$$\mathbf{B}(\mathbf{r},t) = \hat{\mathbf{R}}\times\mathbf{E}(\mathbf{r},t)$$

$$\mathbf{S}(\mathbf{r},t) = \frac{c}{4\pi}\hat{\mathbf{R}}\left|\mathbf{E}(\mathbf{r},t)\right|^2 = \frac{q^2}{4\pi cR^2}\hat{\mathbf{R}}\frac{\left|\hat{\mathbf{R}}\times\left[\left(\hat{\mathbf{R}}-\boldsymbol{\beta}\right)\times\dot{\boldsymbol{\beta}}\right]\right|^2}{\left(1-\boldsymbol{\beta}\cdot\hat{\mathbf{R}}\right)^6}$$

Note: We have used the fact that

$$\hat{\mathbf{R}}\cdot\mathbf{E}(\mathbf{r},t) = 0$$

In addition to calculating the fields themselves, we will be interested in calculating the Poynting vector due to the fields in the radiation zone.

Power radiated

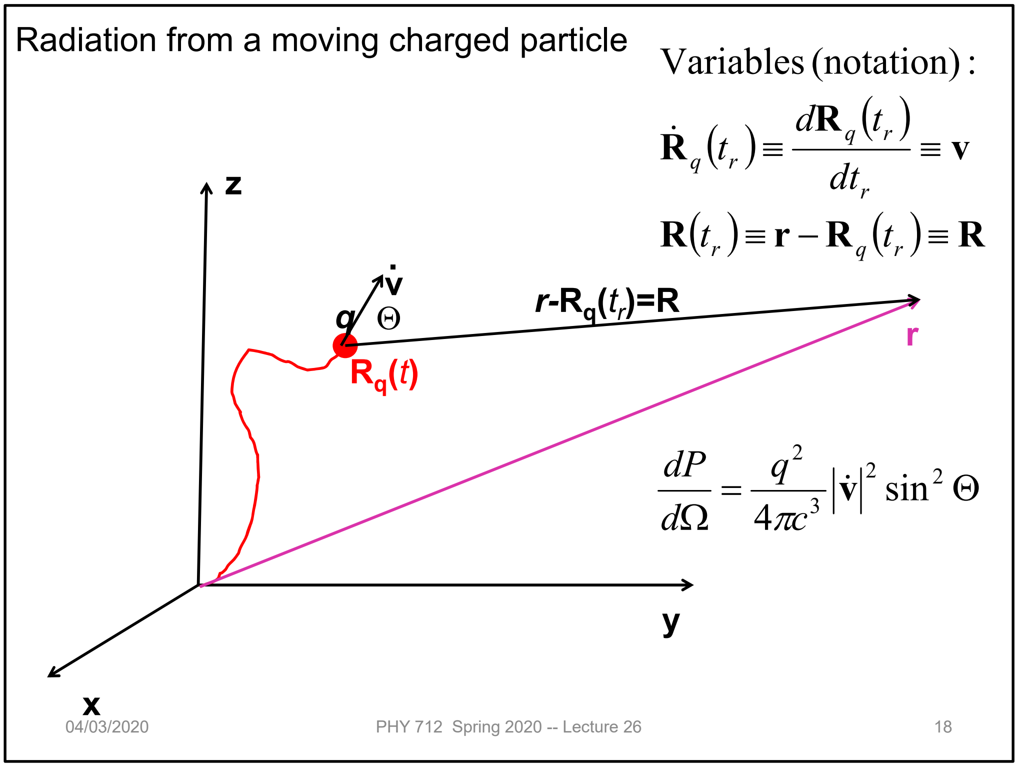
$$\mathbf{S}(\mathbf{r},t) = \frac{c}{4\pi}\hat{\mathbf{R}}\left|\mathbf{E}(\mathbf{r},t)\right|^2 = \frac{q^2}{4\pi c R^2}\hat{\mathbf{R}}\frac{\left|\hat{\mathbf{R}}\times\left[\left(\hat{\mathbf{R}}-\boldsymbol{\beta}\right)\times\dot{\boldsymbol{\beta}}\right]\right|^2}{\left(1-\boldsymbol{\beta}\cdot\hat{\mathbf{R}}\right)^6}$$

$$\frac{dP}{d\Omega} = \mathbf{S}\cdot\hat{\mathbf{R}}R^2 = \frac{q^2}{4\pi c}\frac{\left|\hat{\mathbf{R}}\times\left[\left(\hat{\mathbf{R}}-\boldsymbol{\beta}\right)\times\dot{\boldsymbol{\beta}}\right]\right|^2}{\left(1-\boldsymbol{\beta}\cdot\hat{\mathbf{R}}\right)^6}$$

In the non-relativistic limit: $\beta << 1$

$$\frac{dP}{d\Omega} = \frac{q^2}{4\pi c}\left|\hat{\mathbf{R}}\times\left[\hat{\mathbf{R}}\times\dot{\boldsymbol{\beta}}\right]\right|^2 = \frac{q^2}{4\pi c^3}\left|\dot{\mathbf{v}}\right|^2\sin^2\Theta$$

After some algebra, we arrive at the expression for the power radiated per unit solid angle. We will examine this result more in detail next time, but for now, we will consider the result in the non-relativistic limit when beta is nearly 0.

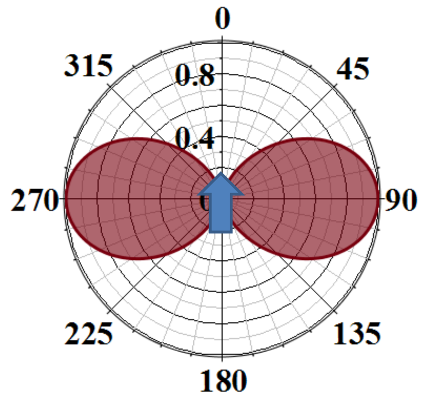


This slide attempts to show the geometry of the trajectory and fields.

Radiation power in non-relativistic case -- continued

$$\frac{dP}{d\Omega} = \frac{q^2}{4\pi c^3}|\dot{\mathbf{v}}|^2 \sin^2\Theta$$

$$P = \int d\Omega \frac{dP}{d\Omega} = \frac{2}{3}\frac{q^2}{c^3}|\dot{\mathbf{v}}|^2$$



Blue arrow indicates the particle acceleration direction

Here we illustrate the non-relativistic power distribution, showing that the radiation intensity is concentrated in the directions perpendicular to the particle acceleration. Next time we will see how relativistic effects change this radiation pattern.