PHY 712 Electrodynamics
12-12:50 AM MWF Olin 103

Plan for Lecture 9:

Continue reading Chapter 4

Dipolar fields and dielectrics

A. Electric field due to a dipole

B. Electric polarization P

C. Electric displacement D and dielectric functions

1

Course schedule for Spring 2020

(Preliminary schedule -- subject to frequent adjustment.)

	Lecture date	JDJ Reading	Topic	HW	Due date
1	Mon: 01/13/2020	Chap. 1 & Appen.	Introduction, units and Poisson equation	#1	01/17/2020
2	Wed: 01/15/2020	Chap. 1	Electrostatic energy calculations	#2	01/22/2020
3	Fri: 01/17/2020	Chap. 1	Electrostatic potentials and fields	#3	01/24/2020
	Mon: 01/20/2020	No class	Martin Luther King Holiday		
4	Wed: 01/22/2020	Chap. 1 - 3	Poisson's equation in 2 and 3 dimensions	#4	01/27/2020
5	Fri: 01/24/2020	Chap. 1 - 3	Brief introduction to numerical methods	#5	01/31/2020
6	Mon: 01/27/2020	Chap. 2 & 3	Image charge constructions	#6	02/03/2020
7	Wed: 01/29/2020	Chap. 2 & 3	Cylindrical and spherical geometries	#7	02/05/2020
8	Fri: 01/31/2020	Chap. 3 & 4	Spherical geometry and multipole moments	#8	02/07/2020
9	Mon: 02/03/2020	Chap. 4	Dipoles and Dielectrics	#9	02/09/2020
10	Wed: 02/05/2020	Chap. 4	Polarization and Dielectrics		
11	Fri: 02/07/2020	Chap. 5	Magnetostatics		
12	Mon: 02/10/2020	Chap. 5	Magnetic dipoles and hyperfine interaction		
13	Wed: 02/12/2020	Chap. 5	Magnetic dipoles and dipolar fields		

2

Review: General results for a multipole analysis of the electrostatic potential due to an isolated charge distribution:

General form of electrostatic potential with boundary value $\Phi(r \to \infty) = 0$ for confined charge density $\rho(\mathbf{r})$:

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int d^3r' \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$$

$$= \frac{1}{4\pi\varepsilon_0}\int d^3r' \rho(\mathbf{r}')\left(\sum_{lm}\frac{4\pi}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y_{lm}(\theta,\varphi)Y_{lm}^*(\theta',\varphi')\right)$$

Suppose that $\rho(\mathbf{r}) = \sum_{lm}\rho_{lm}(r)Y_{lm}(\theta,\varphi)$

$$\Rightarrow \Phi(\mathbf{r}) = \frac{1}{\varepsilon_0}\sum_{lm}\frac{1}{2l+1}Y_{lm}(\theta,\varphi)\left(\frac{1}{r^{l+1}}\int_0^r r'^{2+l}dr'\rho_{lm}(r') + r^l\int_r^\infty r'^{1-l}dr'\rho_{lm}(r')\right)$$

For $r \to \infty$: $\Phi(\mathbf{r}) = \frac{1}{\varepsilon_0}\sum_{lm}\frac{1}{2l+1}Y_{lm}(\theta,\varphi)\frac{1}{r^{l+1}}\underbrace{\int_0^\infty r'^{2+l}dr'\rho_{lm}(r')}_{q_{lm}}$

3

Notion of multipole moment:

In the spherical harmonic representation --

define the moment q_{lm} of the (confined) charge distribution $\rho(\mathbf{r})$:

$$q_{lm} \equiv \int d^3r' \, r'^l \, Y^*_{lm}(\theta', \phi') \rho(\mathbf{r}')$$

In the Cartesian representation --

define the monopole moment q:

$$q \equiv \int d^3r' \, \rho(\mathbf{r}')$$

define the dipole moment $\mathbf{p}$:

$$\mathbf{p} \equiv \int d^3r' \, \mathbf{r}' \rho(\mathbf{r}')$$

define the quadrupole moment components Q_{ij} $(i,j \to x,y,z)$:

$$Q_{ij} \equiv \int d^3r' \left(3r'_i r'_j - r'^2 \delta_{ij}\right) \rho(\mathbf{r}')$$

02/03/2020 PHY 712 Spring 2020 -- Lecture 9 4

4

General form of electrostatic potential in terms of multipole moments:

For r outside the extent of $\rho(\mathbf{r})$:

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \sum_{lm} \frac{4\pi}{2l+1} \frac{Y_{lm}(\theta,\varphi)}{r^{l+1}} \left(\int d^3r' \, r'^l \, Y_{lm}^*(\theta',\varphi') \rho(\mathbf{r}') \right)$$

$$= \frac{1}{4\pi\varepsilon_0} \sum_{lm} \frac{4\pi \, q_{lm}}{2l+1} \frac{Y_{lm}(\theta,\varphi)}{r^{l+1}}$$

In terms of Cartesian expansion:

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \left(\frac{q}{r} + \frac{\mathbf{p}\cdot\mathbf{r}}{r^3} + \frac{1}{2} \sum_{i,j} Q_{ij} \frac{r_i r_j}{r^5} \cdots \right)$$

02/03/2020 PHY 712 Spring 2020 -- Lecture 9 5

5

Focus on dipolar contributions:

For r outside the extent of $\rho(\mathbf{r})$:

Electrostatic potential:

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \left(\frac{\mathbf{p}\cdot\mathbf{r}}{r^3} \right)$$

Electrostatic field:

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \left(\frac{3\mathbf{r}\,(\mathbf{p}\cdot\mathbf{r}) - r^2\mathbf{p}}{r^5} - \frac{4\pi}{3} \mathbf{p} \delta^3(\mathbf{r}) \right)$$

Poorly defined for $r \to 0$ Correct value for $r \to 0$

02/03/2020 PHY 712 Spring 2020 -- Lecture 9 6

6

"Justification" of surprizing δ-function term in dipole electric field -- Assuming dipole is located at *r=0*, we need to need to evaluate the electrostatic field near *r=0*:

We will use the approximation:

$$\mathbf{E}(\mathbf{r}\approx\mathbf{0}) \approx \left(\int_{\text{sphere}} \mathbf{E}(\mathbf{r}) d^3r\right)\delta^3(\mathbf{r}).$$

First we note that:

$$\int_{r\le R} \mathbf{E}(\mathbf{r}) d^3r = -R^2 \int_{r=R} \Phi(\mathbf{r})\hat{\mathbf{r}} d\boldsymbol{\Omega}.$$

7

Some details:

$$\int_{r\le R} \mathbf{E}(\mathbf{r}) d^3r = -R^2 \int_{r=R} \Phi(\mathbf{r})\hat{\mathbf{r}} d\boldsymbol{\Omega}.$$

This result follows from the divergence theorem:

$$\int_{\text{vol}} \nabla\cdot\mathcal{V} d^3r = \int_{\text{surface}} \mathcal{V}\cdot d\mathbf{A}.$$

In our case, this theorem can be used for each cartesian coordinate if we choose $\mathcal{V} \equiv \hat{\mathbf{x}}\Phi(\mathbf{r})$ for the x component, etc.

$$\int_{r\le R} \nabla\Phi(\mathbf{r}) d^3r = \hat{\mathbf{x}}\int_{r\le R} \nabla\cdot(\hat{\mathbf{x}}\Phi) d^3r + \hat{\mathbf{y}}\int_{r\le R} \nabla\cdot(\hat{\mathbf{y}}\Phi) d^3r + \hat{\mathbf{z}}\int_{r\le R} \nabla\cdot(\hat{\mathbf{z}}\Phi) d^3r,$$

which is equal to:

$$\int_{r=R} \Phi(\mathbf{r}) R^2 d\Omega\left((\hat{\mathbf{x}}\cdot\hat{\mathbf{r}})\hat{\mathbf{x}} + (\hat{\mathbf{y}}\cdot\hat{\mathbf{r}})\hat{\mathbf{y}} + (\hat{\mathbf{z}}\cdot\hat{\mathbf{r}})\hat{\mathbf{z}}\right) = \int_{r=R} \Phi(\mathbf{r}) R^2 d\Omega\hat{\mathbf{r}}.$$

Therefore --

$$\int_{r\le R} \mathbf{E}(\mathbf{r}) d^3r = -\int_{r\le R} \nabla\Phi(\mathbf{r}) d^3r = -R^2 \int_{r=R} \Phi(\mathbf{r})\hat{\mathbf{r}} d\boldsymbol{\Omega}.$$

8

More details

$$\int_{r\le R} \mathbf{E}(\mathbf{r}) d^3r = -R^2 \int_{r=R} \Phi(\mathbf{r})\hat{\mathbf{r}} d\boldsymbol{\Omega}.$$

Now, we notice that the electrostatic potential can be determined from the charge density $\rho(\mathbf{r})$ according to:

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int d^3r' \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} = \frac{1}{4\pi\epsilon_0}\sum_{lm}\frac{4\pi}{2l+1}\int d^3r' \rho(\mathbf{r}')\frac{r_<^l}{r_>^{l+1}} Y_{lm}^*(\hat{\mathbf{r}}) Y_{lm}(\hat{\mathbf{r}}').$$

We also note that the unit vector can be written in terms of spherical harmonic functions:

$$\hat{\mathbf{r}} = \begin{cases} \sin(\theta)\cos(\phi)\hat{\mathbf{x}} + \sin(\theta)\sin(\phi)\hat{\mathbf{y}} + \cos(\theta)\hat{\mathbf{z}} \\ \sqrt{\frac{4\pi}{3}}\left(Y_{1-1}(\hat{\mathbf{r}})\frac{\hat{\mathbf{x}}+i\hat{\mathbf{y}}}{\sqrt{2}} + Y_{11}(\hat{\mathbf{r}})\frac{-\hat{\mathbf{x}}+i\hat{\mathbf{y}}}{\sqrt{2}} + Y_{10}(\hat{\mathbf{r}})\hat{\mathbf{z}}\right) \end{cases}$$

$$\int_{r=R} \Phi(\mathbf{r})\hat{\mathbf{r}} d\boldsymbol{\Omega} = \frac{1}{3\varepsilon_0}\int d^3r' \rho(\mathbf{r}')\frac{r_<}{r_>^2}\sqrt{\frac{4\pi}{3}}\left(Y_{1-1}(\hat{\mathbf{r}}')\frac{\hat{\mathbf{x}}+i\hat{\mathbf{y}}}{\sqrt{2}} + Y_{11}(\hat{\mathbf{r}}')\frac{-\hat{\mathbf{x}}+i\hat{\mathbf{y}}}{\sqrt{2}} + Y_{10}(\hat{\mathbf{r}}')\hat{\mathbf{z}}\right)$$

$$= \frac{1}{3\varepsilon_0}\int d^3r' \rho(\mathbf{r}')\frac{r_<}{r_>^2}\hat{\mathbf{r}}'$$

9

More details continued --

When we evaluate the integral over solid angle $d\Omega$, only the $l = 1$ terms contribute, and the result of the integration reduces to:

$$-R^2\int_{r=R}\Phi(\mathbf{r})\hat{\mathbf{r}}d\mathbf{\Omega} = -\frac{1}{4\pi\epsilon_0}\frac{4\pi R^2}{3}\int d^3r'\rho(\mathbf{r}')\frac{r_<}{r_>^2}\hat{\mathbf{r}}'.$$

The choice of $r_<$ and $r_>$ is a choice between the integration variables r' and the sphere radius R. If the sphere encloses the charge distribution, $\rho(\mathbf{r}')$, then $r_< = r'$ and $r_> = R$ so that the result is:

$$-R^2\int_{r=R}\Phi(\mathbf{r})\hat{\mathbf{r}}d\mathbf{\Omega} = -\frac{1}{4\pi\epsilon_0}\frac{4\pi R^2}{3}\frac{1}{R^2}\int d^3r'\rho(\mathbf{r}')r'\hat{\mathbf{r}}' \equiv -\frac{\mathbf{p}}{3\varepsilon_0}.$$

Otherwise, if the charge distribution $\rho(\mathbf{r}')$ lies outside of the sphere, then $r_< = R$ and $r_> = r'$ and the result is:

$$-R^2\int_{r=R}\Phi(\mathbf{r})\hat{\mathbf{r}}d\mathbf{\Omega} = -\frac{1}{4\pi\epsilon_0}\frac{4\pi R^2}{3}R\int d^3r'\frac{\rho(\mathbf{r}')}{r'^2}\hat{\mathbf{r}}' \equiv \frac{4\pi R^3}{3}\mathbf{E}(0).$$

10

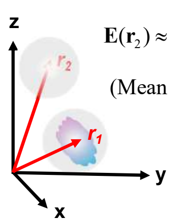
In summary --

Electrostatic dipolar field for dipole moment $\mathbf{p}$ at $\mathbf{r}$=0:

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\left(\frac{3\mathbf{r}\ (\mathbf{p}\cdot\mathbf{r}) - r^2\mathbf{p}}{r^5} - \frac{4\pi}{3}\mathbf{p}\delta^3(\mathbf{r})\right)$$

11

Summary of key argument:



$$\mathbf{E}(\mathbf{r}_2) \approx \frac{3}{4\pi R^3}\int_{r\le R} d^3r\ \mathbf{E}(\mathbf{r}_2+\mathbf{r}) = \mathbf{E}(\mathbf{r}_2)$$

(Mean value theorem for Laplace equation)

$$\mathbf{E}(\mathbf{r}_1) \approx \frac{3}{4\pi R^3}\int_{r\le R} d^3r\ \mathbf{E}(\mathbf{r}_1+\mathbf{r})$$

$$\approx \frac{3}{4\pi R^3}\left(-\frac{\mathbf{p}}{3\epsilon_0}\right) \Rightarrow -\frac{\mathbf{p}}{3\epsilon_0}\delta^3(\mathbf{r}-\mathbf{r}_1)$$

Summary:

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\left(\frac{3\mathbf{r}\ (\mathbf{p}\cdot\mathbf{r}) - r^2\mathbf{p}}{r^5} - \frac{4\pi}{3}\mathbf{p}\delta^3(\mathbf{r})\right)$$

12

Coarse grain representation of macroscopic distribution of dipoles:

Electric polarization $\mathbf{P}(\mathbf{r})$ due to collection of dipoles:

$$\mathbf{P}(\mathbf{r}) \equiv \sum_i \mathbf{p}_i \delta^3(\mathbf{r}-\mathbf{r}_i)$$

Monopole electric charge density $\rho_{\text{mono}}(\mathbf{r})$:

$$\rho_{\text{mono}}(\mathbf{r}) \equiv \sum_i q_i \delta^3(\mathbf{r}-\mathbf{r}_i)$$

Electrostatic potential for a single monopole charge q and a single dipole $\mathbf{p}$:

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r} + \frac{\mathbf{p}\cdot\mathbf{r}}{r^3}\right)$$

02/03/2020 PHY 712 Spring 2020 -- Lecture 9 13

13

Coarse grain representation of macroscopic distribution of dipoles -- continued:

Electrostatic potential for a single monopole charge q and a single dipole $\mathbf{p}$:

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r} + \frac{\mathbf{p}\cdot\mathbf{r}}{r^3}\right)$$

Electrostatic potential for collections of monopole charges q_i and dipoles $\mathbf{p}_i$:

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\left(\int d^3r' \frac{\rho_{mono}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} + \int d^3r' \frac{\mathbf{P}(\mathbf{r}')\cdot(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\right)$$

Note: $\int d^3r' \frac{\mathbf{P}(\mathbf{r}')\cdot(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} = \int d^3r' \, \mathbf{P}(\mathbf{r}')\cdot\nabla' \frac{1}{|\mathbf{r}-\mathbf{r}'|} = -\int d^3r' \frac{\nabla'\cdot\mathbf{P}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$

02/03/2020 PHY 712 Spring 2020 -- Lecture 9 14

14

Coarse grain representation of macroscopic distribution of dipoles -- continued:

Electrostatic potential for collections of monopole charges q_i and dipoles $\mathbf{p}_i$:

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\left(\int d^3r' \frac{\rho_{mono}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} - \int d^3r' \frac{\nabla'\cdot\mathbf{P}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\right)$$

$$-\nabla^2\Phi(\mathbf{r}) = \nabla\cdot\mathbf{E}(\mathbf{r}) = \frac{1}{\varepsilon_0}\left(\rho_{mono}(\mathbf{r}) - \nabla\cdot\mathbf{P}(\mathbf{r})\right)$$

$$\Rightarrow \nabla\cdot\left(\varepsilon_0\mathbf{E}(\mathbf{r}) + \mathbf{P}(\mathbf{r})\right) = \rho_{mono}(\mathbf{r})$$

Define Displacement field: $\mathbf{D}(\mathbf{r}) \equiv \varepsilon_0\mathbf{E}(\mathbf{r}) + \mathbf{P}(\mathbf{r})$

Macroscopic form of Gauss's law: $\nabla\cdot\mathbf{D}(\mathbf{r}) = \rho_{mono}(\mathbf{r})$

02/03/2020 PHY 712 Spring 2020 -- Lecture 9 15

15

Coarse grain representation of macroscopic distribution of dipoles -- continued:

Many materials are polarizable and produce a polarization field in the presence of an electric field with a proportionality constant χ_e :

$$\mathbf{P}(\mathbf{r}) = \varepsilon_0 \chi_e \mathbf{E}(\mathbf{r})$$

$$\mathbf{D}(\mathbf{r}) \equiv \varepsilon_0 \mathbf{E}(\mathbf{r}) + \mathbf{P}(\mathbf{r}) = \varepsilon_0 (1+\chi_e) \mathbf{E}(\mathbf{r}) \equiv \varepsilon \mathbf{E}(\mathbf{r})$$

ε represents the dielectric function of the material

Boundary value problems in dielectric materials

For $\rho_{\text{mono}}(\mathbf{r}) = 0$

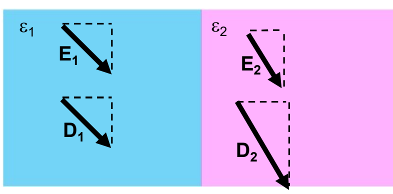
$\nabla \cdot \mathbf{D}(\mathbf{r}) = 0$ and $\nabla \times \mathbf{E}(\mathbf{r}) = 0$

$\Rightarrow$ At a surface between two dielectrics, in terms of surface normal $\hat{\mathbf{r}}$:

$\hat{\mathbf{r}} \cdot \mathbf{D}(\mathbf{r}) =$ continuous $= \hat{\mathbf{r}} \times \mathbf{E}(\mathbf{r})$

16

Boundary value problems in the presence of dielectrics – example:



For $\frac{\varepsilon_2}{\varepsilon_1} = 2$

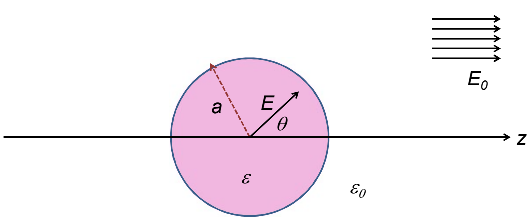
For isotropic dielectrics:

$$D_{1n} = D_{2n} \qquad \varepsilon_1 E_{1n} = \varepsilon_2 E_{2n}$$

$$E_{1t} = E_{2t} \qquad \frac{D_{1t}}{\varepsilon_1} = \frac{D_{2t}}{\varepsilon_2}$$

17

Boundary value problems in the presence of dielectrics – example:



$\nabla \cdot \mathbf{D}(\mathbf{r}) = 0$ and $\nabla \times \mathbf{E}(\mathbf{r}) = 0$

For $r \le a$ $\mathbf{D}(\mathbf{r}) = -\varepsilon \nabla \Phi(\mathbf{r})$

For $r > a$ $\mathbf{D}(\mathbf{r}) = -\varepsilon_0 \nabla \Phi(\mathbf{r})$

At $r = a$:

$$\varepsilon \frac{\partial \Phi_<(\mathbf{r})}{\partial r} = \varepsilon_0 \frac{\partial \Phi_>(\mathbf{r})}{\partial r}$$

$$\frac{\partial \Phi_<(\mathbf{r})}{\partial \theta} = \frac{\partial \Phi_>(\mathbf{r})}{\partial \theta}$$

18

Boundary value problems in the presence of dielectrics – example -- continued:

$$\Phi_<(\mathbf{r}) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$$

$$\Phi_>(\mathbf{r}) = \sum_{l=0}^{\infty} \left(B_l r^l + \frac{C_l}{r^{l+1}} \right) P_l(\cos\theta)$$

At $r = a$: $\varepsilon \frac{\partial \Phi_<(\mathbf{r})}{\partial r} = \varepsilon_0 \frac{\partial \Phi_>(\mathbf{r})}{\partial r}$

$$\frac{\partial \Phi_<(\mathbf{r})}{\partial \theta} = \frac{\partial \Phi_>(\mathbf{r})}{\partial \theta}$$

For $r \to \infty$ $\quad \Phi_>(\mathbf{r}) = -E_0 r \cos\theta$

Solution -- only $l = 1$ contributes

$B_1 = -E_0$

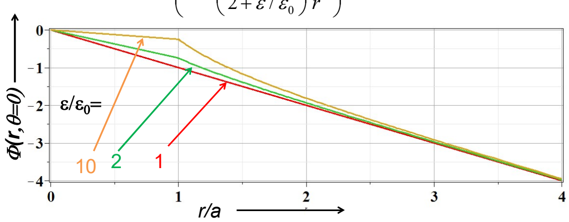
$$A_1 = -\left(\frac{3}{2 + \varepsilon/\varepsilon_0} \right) E_0 \qquad C_1 = \left(\frac{\varepsilon/\varepsilon_0 - 1}{2 + \varepsilon/\varepsilon_0} \right) a^3 E_0$$

19

Boundary value problems in the presence of dielectrics – example -- continued:

$$\Phi_<(\mathbf{r}) = -\left(\frac{3}{2 + \varepsilon/\varepsilon_0} \right) E_0 \, r \cos\theta$$

$$\Phi_>(\mathbf{r}) = -\left(r - \left(\frac{\varepsilon/\varepsilon_0 - 1}{2 + \varepsilon/\varepsilon_0} \right) \frac{a^3}{r^2} \right) E_0 \cos\theta$$



20