PHY 742 Quantum Mechanics II
1-1:50 AM MWF Olin 103
Plan for Lecture 14

The Dirac equation –
Relativistic treatment of spin ½ particles
Ref: Chapter 16

1. **Some simple concepts of special theory of relativity**
2. **Energy and momentum relationships**
3. **Dirac equation for a free particle**

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Topics for Quantum Mechanics II

Single particle analysis
- Single particle interacting with electromagnetic fields – EC Chap. 9
- Scattering of a particle from a spherical potential – EC Chap. 14
- More time independent perturbation methods – EC Chap. 12, 13
- Single electron states of a multi-well potential ➔ molecules and solids – EC Chap. 2,6
- Time dependent perturbation methods – EC Chap. 15
- Relativistic effects and the Dirac Equation – EC Chap. 16
- Path integral formalism (Feynman) – EC Chap. 11.C

Multiple particle analysis
- Quantization of the electromagnetic fields – EC Chap. 17
- Photons and atoms – EC Chap. 18
- Multi particle systems; Bose and Fermi particles – EC Chap. 10
- Multi electron atoms and materials
  - Hartree-Fock approximation
  - Density functional approximation

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Course schedule for Spring 2020

(Preliminary schedule -- subject to frequent adjustment.)

	Lecture date	Reading	Topic	HW	Due date
1	Mon: 01/13/2020	Chap. 9	Quantum mechanics of electromagnetic forces	#1	01/22/2020
2	Wed: 01/15/2020	Chap. 9	Quantum mechanics of particle in electrostatic field	#2	01/24/2020
3	Fri: 01/17/2020	Chap. 9	Quantum mechanics of particle in magnetostatic field	#3	01/27/2020
	Mon: 01/20/2020	No class	Martin Luther King Holiday		
4	Wed: 01/22/2020	Chap. 14	Scattering theory	#4	01/29/2020
5	Fri: 01/24/2020	Chap. 14	Scattering theory	#5	01/31/2020
6	Mon: 01/27/2020	Chap. 14	Scattering theory	#6	02/03/2020
7	Wed: 01/29/2020	Chap. 12	Variational methods	#7	02/05/2020
8	Fri: 01/31/2020	Chap. 12	Variational and other approximation methods	#8	02/07/2020
9	Mon: 02/03/2020	Chap. 2,6	Single particle states of molecules and solids	#9	02/10/2020
10	Wed: 02/05/2020	Chap. 2,6	H_2^+ molecular ion; Born Oppenheimer approximation	#10	02/12/2020
11	Fri: 02/07/2020	Chap. 15	Time-dependent perturbations	#11	02/14/2020
12	Mon: 02/10/2020	Chap. 15	Time-dependent perturbations	#12	02/14/2020
13	Wed: 02/12/2020	Chap. 15	Time-dependent perturbations	#13	02/17/2020
14	Fri: 02/14/2020	Chap. 16	The Dirac equation		
15	Mon: 02/17/2020				
16	Wed: 02/19/2020				
17	Fri: 02/21/2020				
18	Mon: 02/24/2020				
19	Wed: 02/26/2020				
20	Fri: 02/28/2020				

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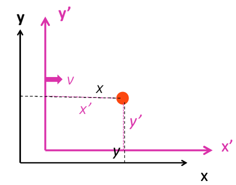
Notions of special relativity

- The basic laws of physics are the same in all frames of reference (at rest or moving at constant velocity).
- The speed of light in vacuum *c* is the same in all frames of reference.



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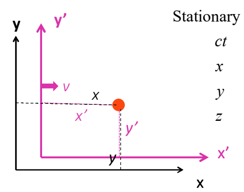
Lorentz transformations

Convenient notation:

$$\beta \equiv \frac{v}{c}$$

$$\gamma \equiv \frac{1}{\sqrt{1-\beta^2}}$$



Stationary frame		Moving frame
ct	$=$	$\gamma(ct'+\beta x')$
x	$=$	$\gamma(x'+\beta ct')$
y	$=$	y'
z	$=$	z'

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Lorentz transformations -- continued $\quad \beta \equiv \frac{v}{c} \quad \gamma \equiv \frac{1}{\sqrt{1-\beta^2}}$

For the moving frame with $\mathbf{v} = v\hat{\mathbf{x}}$:

$$\mathcal{L} = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \mathcal{L}^{-1} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \mathcal{L} \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} \qquad \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \mathcal{L}^{-1} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

Notice:

$$c^2t^2 - x^2 - y^2 - z^2 = c^2t'^2 - x'^2 - y'^2 - z'^2$$

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Lorentz transformation of the velocity -- continued

Stationary frame		Moving frame
cdt	$=$	$\gamma(cdt'+\beta dx')$
dx	$=$	$\gamma(dx'+\beta cdt')$
dy	$=$	dy'
dz	$=$	dz'

$$\text{Define}: \quad u_x \equiv \frac{dx}{dt} \quad u_y \equiv \frac{dy}{dt} \quad u_z \equiv \frac{dz}{dt}$$

$$u'_x \equiv \frac{dx'}{dt'} \quad u'_y \equiv \frac{dy'}{dt'} \quad u'_z \equiv \frac{dz'}{dt'}$$

$$\frac{dx}{dt} = \frac{\gamma(dx'+\beta cdt')}{\gamma(dt'+\beta dx'/c)} = \frac{u'_x+v}{1+vu'_x/c^2} = u_x$$

$$\frac{dy}{dt} = \frac{dy'}{\gamma(dt'+\beta dx'/c)} = \frac{u'_y}{\gamma(1+vu'_x/c^2)} = u_y$$

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Summary of velocity relationships

$$u_x = \frac{u'_x+v}{1+vu'_x/c^2}$$

$$u_y = \frac{u'_y}{\gamma\left(1+vu'_x/c^2\right)} \equiv \frac{u'_y}{\gamma_v\left(1+vu'_x/c^2\right)}$$

$$u_z = \frac{u'_z}{\gamma\left(1+vu'_x/c^2\right)} \equiv \frac{u'_z}{\gamma_v\left(1+vu'_x/c^2\right)}$$

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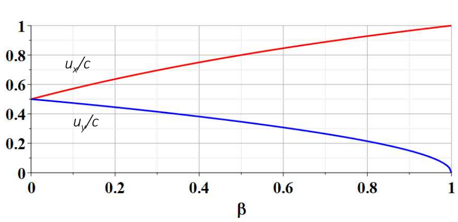
Example of velocity variation with β:
($u'_x/c = u'_y/c = 0.5$)



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Velocity transformations continued:

Consider: $u_x = \frac{u'_x + v}{1 + vu'_x/c^2} \quad u_y = \frac{u'_y}{\gamma_v\left(1 + vu'_x/c^2\right)} \quad u_z = \frac{u'_z}{\gamma_v\left(1 + vu'_x/c^2\right)}.$

Note that $\gamma_u \equiv \frac{1}{\sqrt{1-(u/c)^2}} = \frac{1 + vu'_x/c^2}{\sqrt{1-(u'/c)^2}\sqrt{1-(v/c)^2}} = \gamma_v\gamma_{u'}\left(1 + vu'_x/c^2\right)$

$\Rightarrow \gamma_u c = \gamma_v\left(\gamma_{u'}c + \beta_v\gamma_{u'}u'_x\right)$

$\Rightarrow \gamma_u u_x = \gamma_v\left(\gamma_{u'}u'_x + \gamma_{u'}v\right) = \gamma_v\left(\gamma_{u'}u'_x + \beta_v\gamma_{u'}c\right)$

$\Rightarrow \gamma_u u_y = \gamma_{u'}u'_y \qquad \gamma_u u_z = \gamma_{u'}u'_z$

Velocity 4-vector: $\begin{pmatrix}\gamma_u c\\ \gamma_u u_x\\ \gamma_u u_y\\ \gamma_u u_z\end{pmatrix} = \mathcal{L}_u \begin{pmatrix}\gamma_{u'} c\\ \gamma_{u'} u'_x\\ \gamma_{u'} u'_y\\ \gamma_{u'} u'_z\end{pmatrix}$

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Some details:

$\gamma_u = \gamma_v\gamma_{u'}\left(1 + vu'_x/c^2\right) \quad \Rightarrow \left(1-\frac{v^2}{c^2}\right)\left(1-\frac{u'^2}{c^2}\right) = \left(1-\frac{u^2}{c^2}\right)\left(1+\frac{u_x v}{c^2}\right)^2$

where $u_x = \frac{u'_x + v}{1 + vu'_x/c^2} \quad u_y = \frac{u'_y}{\gamma_v\left(1 + vu'_x/c^2\right)} \quad u_z = \frac{u'_z}{\gamma_v\left(1 + vu'_x/c^2\right)}.$

$\left(\frac{u_x^2}{c^2}+\frac{u_y^2}{c^2}+\frac{u_z^2}{c^2}\right)\left(1+\frac{u_x v}{c^2}\right)^2 = \left(\frac{u'_x}{c}+\frac{v}{c}\right)^2 + \left(\frac{u'^2_y}{c^2}+\frac{u'^2_z}{c^2}\right)\left(1-\frac{v^2}{c^2}\right)$

$\frac{u^2}{c^2}\left(1+\frac{u_x v}{c^2}\right)^2 = \frac{u'^2}{c^2}\left(1-\frac{v^2}{c^2}\right) + \left(1+\frac{u_x v}{c^2}\right)^2 - \left(1-\frac{v^2}{c^2}\right)$

$\Rightarrow \left(1-\frac{u^2}{c^2}\right)\left(1+\frac{u_x v}{c^2}\right)^2 = \left(1-\frac{u'^2}{c^2}\right)\left(1-\frac{v^2}{c^2}\right)$

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Significance of 4-velocity vector: $\begin{pmatrix}\gamma_u c\\ \gamma_u u_x\\ \gamma_u u_y\\ \gamma_u u_z\end{pmatrix}$

Introduce the "rest" mass *m* of particle characterized by velocity **u**:

$mc\begin{pmatrix}\gamma_u c\\ \gamma_u u_x\\ \gamma_u u_y\\ \gamma_u u_z\end{pmatrix} = \begin{pmatrix}\gamma_u mc^2\\ \gamma_u mu_x c\\ \gamma_u mu_y c\\ \gamma_u mu_z c\end{pmatrix} = \begin{pmatrix}E\\ p_x c\\ p_y c\\ p_z c\end{pmatrix}$

Properties of energy-moment 4-vector:

$\begin{pmatrix}E\\ p_x c\\ p_y c\\ p_z c\end{pmatrix} = \mathcal{L}\begin{pmatrix}E'\\ p'_x c\\ p'_y c\\ p'_z c\end{pmatrix} \qquad \begin{pmatrix}E'\\ p'_x c\\ p'_y c\\ p'_z c\end{pmatrix} = \mathcal{L}^{-1}\begin{pmatrix}E\\ p_x c\\ p_y c\\ p_z c\end{pmatrix} \qquad \text{Note: } E^2 - p^2c^2 = E'^2 - p'^2c^2$

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Properties of Energy-momentum 4-vector -- continued

$$\begin{pmatrix} E \\ p_x c \\ p_y c \\ p_z c \end{pmatrix} = \begin{pmatrix} \gamma_u mc^2 \\ \gamma_u m u_x c \\ \gamma_u m u_y c \\ \gamma_u m u_z c \end{pmatrix}$$

Note: $E^2 - p^2c^2 = \frac{(mc^2)^2}{1-\beta_u^{\,2}}\left(1 - \left(\frac{u_x}{c}\right)^2 - \left(\frac{u_y}{c}\right)^2 - \left(\frac{u_z}{c}\right)^2\right) = (mc^2)^2 = E'^2 - p'^2c^2$

Notion of "rest energy": For $\mathbf{p} \equiv 0$, $E = mc^2$

Define kinetic energy: $E_K \equiv E - mc^2 = \sqrt{p^2c^2 + m^2c^4} - mc^2$

Non-relativistic limit: If $\frac{p}{mc} << 1$, $E_K = mc^2\left(\sqrt{1 + \left(\frac{p}{mc}\right)^2} - 1\right)$

$$\approx \frac{p^2}{2m}$$

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Summary of relativistic energy relationships

$$\begin{pmatrix} E \\ p_x c \\ p_y c \\ p_z c \end{pmatrix} = \begin{pmatrix} \gamma_u mc^2 \\ \gamma_u m u_x c \\ \gamma_u m u_y c \\ \gamma_u m u_z c \end{pmatrix}$$

$E = \sqrt{p^2c^2 + m^2c^4} = \gamma_u mc^2$

Check: $\sqrt{p^2c^2 + m^2c^4} = mc^2\sqrt{\gamma_u^{\,2}\beta_u^{\,2} + 1} = \gamma_u mc^2$

Example: for an electron $mc^2 = 0.5$ MeV

for $E = 200$ GeV

$$\gamma_u = \frac{E}{mc^2} = 4 \times 10^5$$

$$\beta_u = \sqrt{1 - \frac{1}{\gamma_u^{\,2}}} \approx 1 - \frac{1}{2\gamma_u^{\,2}} \approx 1 - 3 \times 10^{-12}$$

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How do these relationships effect quantum mechanics?

Focusing on treatment of Fermi particles

Non-relativistic mechanics	Relativistic mechanics
$E = \frac{\mathbf{p}^2}{2m}$	$E^2 - \mathbf{p}^2c^2 = (mc^2)^2$
$\Downarrow$	$\Downarrow$ (with some license)
$i\hbar\frac{\partial}{\partial t}\Psi = H\Psi$	$(E - \mathbf{p}\cdot\boldsymbol{\sigma}c)(E + \mathbf{p}\cdot\boldsymbol{\sigma}c) = (mc^2)^2$
	$\Downarrow$
	$\left(i\hbar\frac{\partial}{\partial t} - \mathbf{p}\cdot\boldsymbol{\sigma}c\right)\left(i\hbar\frac{\partial}{\partial t} + \mathbf{p}\cdot\boldsymbol{\sigma}c\right)\Psi = (mc^2)^2\Psi$

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Digression on Pauli matrices

$$\boldsymbol{\sigma} = \sigma_x \hat{\mathbf{x}} + \sigma_y \hat{\mathbf{y}} + \sigma_z \hat{\mathbf{z}}$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Note that $\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$(\boldsymbol{\sigma} \cdot \mathbf{p})^2 = p^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

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Relativistic relationships – continued
Ref: J. J. Sakurai, Advanced Quantum Mechanics

$$\left(i\hbar\frac{\partial}{\partial t} - \mathbf{p}\cdot\boldsymbol{\sigma}c \right)\left(i\hbar\frac{\partial}{\partial t} + \mathbf{p}\cdot\boldsymbol{\sigma}c \right)\Psi = \left(mc^2\right)^2 \Psi$$

Let $\Psi \equiv \Psi^L \qquad \left(i\hbar\frac{\partial}{\partial t} + \mathbf{p}\cdot\boldsymbol{\sigma}c \right)\Psi^L \equiv mc^2 \Psi^R$

Factored equations:

$$\left(i\hbar\frac{\partial}{\partial t} + \mathbf{p}\cdot\boldsymbol{\sigma}c \right)\Psi^L = mc^2 \Psi^R$$

$$\left(i\hbar\frac{\partial}{\partial t} - \mathbf{p}\cdot\boldsymbol{\sigma}c \right)\Psi^R = mc^2 \Psi^L$$

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Relativistic relationships – continued
Ref: J. J. Sakurai, Advanced Quantum Mechanics

Factored equations:

$$\left(i\hbar\frac{\partial}{\partial t} + \mathbf{p}\cdot\boldsymbol{\sigma}c \right)\Psi^L = mc^2 \Psi^R$$

$$\left(i\hbar\frac{\partial}{\partial t} - \mathbf{p}\cdot\boldsymbol{\sigma}c \right)\Psi^R = mc^2 \Psi^L$$

Dirac's rearrangement: $\varphi^U = \Psi^R + \Psi^L$

$\varphi^L = \Psi^R - \Psi^L$

$$\begin{pmatrix} i\hbar\frac{\partial}{\partial t} & -\mathbf{p}\cdot\boldsymbol{\sigma}c \\ \mathbf{p}\cdot\boldsymbol{\sigma}c & -i\hbar\frac{\partial}{\partial t} \end{pmatrix} \begin{pmatrix} \varphi^U \\ \varphi^L \end{pmatrix} = mc^2 \begin{pmatrix} \varphi^U \\ \varphi^L \end{pmatrix}$$

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Relativistic relationships – continued

$$\begin{pmatrix} i\hbar\frac{\partial}{\partial t} & -\mathbf{p}\cdot\boldsymbol{\sigma}c \\ \mathbf{p}\cdot\boldsymbol{\sigma}c & -i\hbar\frac{\partial}{\partial t} \end{pmatrix}\begin{pmatrix} \varphi^U \\ \varphi^L \end{pmatrix} = mc^2 \begin{pmatrix} \varphi^U \\ \varphi^L \end{pmatrix}$$

Further rearrangements:

$$i\hbar\frac{\partial}{\partial t}\begin{pmatrix} \varphi^U \\ \varphi^L \end{pmatrix} = \begin{pmatrix} mc^2 & \mathbf{p}\cdot\boldsymbol{\sigma}c \\ \mathbf{p}\cdot\boldsymbol{\sigma}c & -mc^2 \end{pmatrix}\begin{pmatrix} \varphi^U \\ \varphi^L \end{pmatrix}$$

$$i\hbar\frac{\partial}{\partial t}\Psi = H\Psi$$

$$\Psi = \begin{pmatrix} \varphi^U \\ \varphi^L \end{pmatrix}$$

$$H = \begin{pmatrix} mc^2 & \mathbf{p}\cdot\boldsymbol{\sigma}c \\ \mathbf{p}\cdot\boldsymbol{\sigma}c & -mc^2 \end{pmatrix}$$

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Four component wavefunction of free Fermi particle

$$i\hbar\frac{\partial}{\partial t}\begin{pmatrix} \varphi^U \\ \varphi^L \end{pmatrix} = \begin{pmatrix} mc^2 & \mathbf{p}\cdot\boldsymbol{\sigma}c \\ \mathbf{p}\cdot\boldsymbol{\sigma}c & -mc^2 \end{pmatrix}\begin{pmatrix} \varphi^U \\ \varphi^L \end{pmatrix}$$

Assume $\begin{pmatrix} \varphi^U \\ \varphi^L \end{pmatrix} = \begin{pmatrix} \chi^U(\mathbf{k}) \\ \chi^L(\mathbf{k}) \end{pmatrix} e^{i\mathbf{k}\cdot\mathbf{r}-iEt/\hbar}$

$$\Rightarrow \chi^U(\mathbf{k}) = \frac{\hbar\mathbf{k}\cdot\boldsymbol{\sigma}c}{E-mc^2}\chi^L(\mathbf{k})$$

$$\chi^L(\mathbf{k}) = \frac{\hbar\mathbf{k}\cdot\boldsymbol{\sigma}c}{E+mc^2}\chi^U(\mathbf{k})$$

$$E^2 = \hbar^2c^2\mathbf{k}^2 + m^2c^4$$

$$E = \pm\sqrt{\hbar^2c^2\mathbf{k}^2 + m^2c^4}$$

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Pauli matrices: $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$$\mathbf{k}\cdot\boldsymbol{\sigma} = \begin{pmatrix} k_z & k_x - ik_y \\ k_x + ik_y & -k_z \end{pmatrix}$$

$$\chi^U(\mathbf{k}) = \frac{\hbar\mathbf{k}\cdot\boldsymbol{\sigma}c}{E-mc^2}\chi^L(\mathbf{k}) \qquad \chi^L(\mathbf{k}) = \frac{\hbar\mathbf{k}\cdot\boldsymbol{\sigma}c}{E+mc^2}\chi^U(\mathbf{k})$$

Positive energy solutions: $E = \sqrt{\hbar^2c^2\mathbf{k}^2 + m^2c^4}$

$$\chi^U_\uparrow(\mathbf{k}) = \mathcal{N}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \chi^L_\uparrow(\mathbf{k}) = \mathcal{N}\begin{pmatrix} \kappa_z \\ \kappa_+ \end{pmatrix} \qquad \kappa_z \equiv \frac{\hbar k_z c}{E+mc^2}$$

$$\chi^U_\downarrow(\mathbf{k}) = \mathcal{N}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \chi^L_\downarrow(\mathbf{k}) = \mathcal{N}\begin{pmatrix} \kappa_- \\ -\kappa_z \end{pmatrix} \qquad \kappa_\pm \equiv \frac{\hbar k_\pm c}{E+mc^2}$$

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Pauli matrices: $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$$\mathbf{k}\cdot\boldsymbol{\sigma} = \begin{pmatrix} k_z & k_x - ik_y \\ k_x + ik_y & -k_z \end{pmatrix}$$

$$\chi^U(\mathbf{k}) = \frac{\hbar\mathbf{k}\cdot\boldsymbol{\sigma}c}{E - mc^2}\chi^L(\mathbf{k}) \qquad \chi^L(\mathbf{k}) = \frac{\hbar\mathbf{k}\cdot\boldsymbol{\sigma}c}{E + mc^2}\chi^U(\mathbf{k})$$

Negative energy solutions: $E = -\sqrt{\hbar^2 c^2 \mathbf{k}^2 + m^2 c^4}$

$$\chi^U_\uparrow(\mathbf{k}) = \mathcal{N}\begin{pmatrix} \kappa_z \\ \kappa_+ \end{pmatrix} \qquad \chi^L_\uparrow(\mathbf{k}) = \mathcal{N}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \kappa_z \equiv \frac{\hbar k_z c}{E - mc^2}$$

$$\chi^U_\downarrow(\mathbf{k}) = \mathcal{N}\begin{pmatrix} \kappa_- \\ -\kappa_z \end{pmatrix} \qquad \chi^L_\downarrow(\mathbf{k}) = \mathcal{N}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \kappa_\pm \equiv \frac{\hbar k_\pm c}{E - mc^2}$$

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What does this all mean?

Positive energy solutions: $E = \sqrt{\hbar^2 c^2 \mathbf{k}^2 + m^2 c^4}$

$$\chi^U_\uparrow(\mathbf{k}) = \mathcal{N}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \chi^L_\uparrow(\mathbf{k}) = \mathcal{N}\begin{pmatrix} \kappa_z \\ \kappa_+ \end{pmatrix} \qquad \kappa_z \equiv \frac{\hbar k_z c}{E + mc^2}$$

$$\chi^U_\downarrow(\mathbf{k}) = \mathcal{N}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \chi^L_\downarrow(\mathbf{k}) = \mathcal{N}\begin{pmatrix} \kappa_- \\ -\kappa_z \end{pmatrix} \qquad \kappa_\pm \equiv \frac{\hbar k_\pm c}{E + mc^2}$$

For $\hbar c|\mathbf{k}| \ll mc^2$ $\quad E \approx mc^2 + \frac{\hbar^2|\mathbf{k}|^2}{2m}$

$$\chi^U_\uparrow(\mathbf{k}) = \mathcal{N}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \chi^L_\uparrow(\mathbf{k}) \approx \mathcal{N}\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\chi^U_\downarrow(\mathbf{k}) = \mathcal{N}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \chi^L_\downarrow(\mathbf{k}) \approx \mathcal{N}\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

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