

PHY 712 Electrodynamics

10-10:50 AM MWF Online

Class discussion for Lecture 10:

Continue reading Chapter 4

Dipolar fields and dielectrics

A. Electric field due to a dipole

B. Electric polarization $\mathbf{P}$

C. Electric displacement $\mathbf{D}$ and dielectric functions

Course schedule for Spring 2021

(Preliminary schedule -- subject to frequent adjustment.)

	Lecture date	JDJ Reading	Topic	HW	Due date
1	Wed: 01/27/2021	Chap. 1 & Appen.	Introduction, units and Poisson equation	#1	01/29/2021
2	Fri: 01/29/2021	Chap. 1	Electrostatic energy calculations	#2	02/01/2021
3	Mon: 02/01/2021	Chap. 1 & 2	Electrostatic potentials and fields	#3	02/03/2021
4	Wed: 02/03/2021	Chap. 1 - 3	Poisson's equation in 2 and 3 dimensions	#4	02/05/2021
5	Fri: 02/05/2021	Chap. 1 - 3	Brief introduction to numerical methods	#5	02/08/2021
6	Mon: 02/08/2021	Chap. 2 & 3	Image charge constructions	#6	02/10/2021
7	Wed: 02/10/2021	Chap. 2 & 3	Cylindrical and spherical geometries		
8	Fri: 02/12/2021	Chap. 3 & 4	Spherical geometry and multipole moments	#7	02/15/2021
9	Mon: 02/15/2021	Chap. 4	Dipoles and Dielectrics	#8	02/19/2021
10	Wed: 02/17/2021	Chap. 4	Dipoles and Dielectrics		
11	Fri: 02/19/2021	Chap. 5	Magnetostatics		
12	Mon: 02/22/2021	Chap. 5	Magnetic dipoles and hyperfine interaction		
13	Wed: 02/24/2021	Chap. 5	Magnetic dipoles and dipolar fields		
14	Fri: 02/26/2021	Chap. 6	Maxwell's Equations		

PHY 712 -- Assignment #8

February 15, 2021

Continue reading Chapter 4 in **Jackson** .

1. Find the monopole, dipole, and quadrupole moments of the charge distribution shown in the figure in problem 4.1(b) of **Jackson**. You can use either the Cartesian or spherical polar forms for the moments as you prefer.

Colloquium tomorrow --

Online Colloquium: “A Tale of Seven Scientists and a New Philosophy of Science” — February 18, 2021 at 4 PM

Dr. Eric Scerri

Faculty and Lecturer

Department of Chemistry and Biochemistry

University of California, Los Angeles

Thursday, February 18, 2021, 4 PM EST

Via Video Conference (contact wfuphys@wfu.edu for link information)

Your questions –

From Tim -- On slide 5 you call the multipole expansion the "Cartesian expansion". I see that it is in terms of r which is usually used in spherical coordinates. Is it normally used just in terms of r ? Is it possible to get a divergence of the displacement field $D = 0$ even when you have polarizations, but the polarization is only due to dipoles?

From Gao -- Could you go over how these solutions come?

$$\Phi_{<}(\mathbf{r}) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos \theta)$$

$$\Phi_{>}(\mathbf{r}) = \sum_{l=0}^{\infty} \left(B_l r^l + \frac{C_l}{r^{l+1}} \right) P_l(\cos \theta)$$

Notion of multipole moment:

In the spherical harmonic representation --

define the moment q_{lm} of the (confined) charge distribution $\rho(\mathbf{r})$:

$$q_{lm} \equiv \int d^3r' \, r'^l Y_{lm}^*(\theta', \varphi') \rho(\mathbf{r}')$$

In the Cartesian representation --

define the monopole moment q :

$$q \equiv \int d^3r' \, \rho(\mathbf{r}')$$

define the dipole moment $\mathbf{p}$:

$$\mathbf{p} \equiv \int d^3r' \, \mathbf{r}' \rho(\mathbf{r}')$$

define the quadrupole moment components Q_{ij} ($i, j \rightarrow x, y, z$):

$$Q_{ij} \equiv \int d^3r' \, (3r'_i r'_j - r'^2 \delta_{ij}) \rho(\mathbf{r}')$$

Note that here we mean $r'_i = \{x', y', z'\}$ for $i = 1, 2, 3$

$$Q_{xx} \equiv \int d^3r' \, (3x'^2 - (x'^2 + y'^2 + z'^2)) \rho(\mathbf{r}') \quad Q_{xy} \equiv \int d^3r' \, (3x' y') \rho(\mathbf{r}')$$

General form of electrostatic potential in terms of multipole moments:

For r outside the extent of $\rho(\mathbf{r})$:

$$\begin{aligned}\Phi(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0} \sum_{lm} \frac{4\pi}{2l+1} \frac{Y_{lm}(\theta, \varphi)}{r^{l+1}} \left(\int d^3r' r'^l Y_{lm}^*(\theta', \varphi') \rho(\mathbf{r}') \right) \\ &= \frac{1}{4\pi\epsilon_0} \sum_{lm} \frac{4\pi q_{lm}}{2l+1} \frac{Y_{lm}(\theta, \varphi)}{r^{l+1}}\end{aligned}$$

In terms of Cartesian expansion :

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \left(\frac{q}{r} + \frac{\mathbf{p} \cdot \mathbf{r}}{r^3} + \frac{1}{2} \sum_{i,j} Q_{ij} \frac{r_i r_j}{r^5} \dots \right)$$

Focus on dipolar contributions:

For r outside the extent of $\rho(\mathbf{r})$:

Electrostatic potential:

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \left(\frac{\mathbf{p} \cdot \mathbf{r}}{r^3} \right)$$

Electrostatic field:

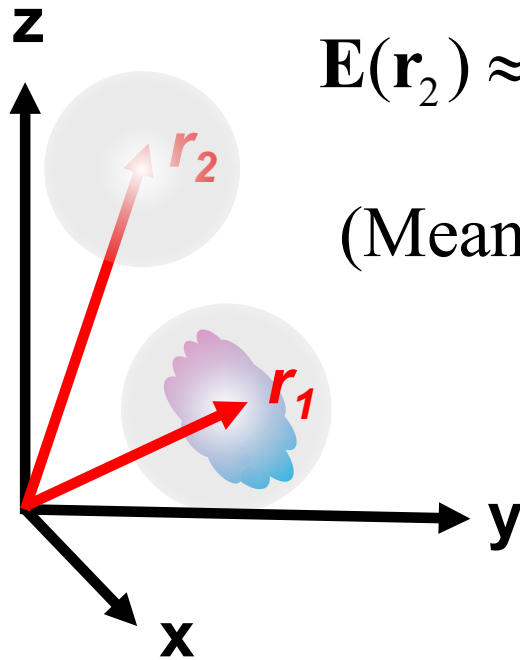
$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \left(\frac{3\mathbf{r} (\mathbf{p} \cdot \mathbf{r}) - r^2 \mathbf{p}}{r^5} - \frac{4\pi}{3} \mathbf{p} \delta^3(\mathbf{r}) \right)$$


Poorly defined for $r \rightarrow 0$

Correct value for $r \rightarrow 0$

This result was “derived” in Lecture 9.

Summary of key argument:



$$\mathbf{E}(\mathbf{r}_2) \approx \frac{3}{4\pi R^3} \int_{r \leq R} d^3 r \mathbf{E}(\mathbf{r}_2 + \mathbf{r}) = \mathbf{E}(\mathbf{r}_2)$$

(Mean value theorem for Laplace equation)

$$\begin{aligned} \mathbf{E}(\mathbf{r}_1) &\approx \frac{3}{4\pi R^3} \int_{r \leq R} d^3 r \mathbf{E}(\mathbf{r}_1 + \mathbf{r}) \\ &\approx \frac{3}{4\pi R^3} \left(-\frac{\mathbf{p}}{3\epsilon_0} \right) \Rightarrow -\frac{\mathbf{p}}{3\epsilon_0} \delta^3(\mathbf{r} - \mathbf{r}_1) \end{aligned}$$

Summary:

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \left(\frac{3\mathbf{r} (\mathbf{p} \cdot \mathbf{r}) - r^2 \mathbf{p}}{r^5} - \frac{4\pi}{3} \mathbf{p} \delta^3(\mathbf{r}) \right)$$

Now consider a complicated distribution of charges --

Coarse grain representation of macroscopic distribution of dipoles:

Electric polarization $\mathbf{P}(\mathbf{r})$ due to collection of dipoles :

$$\mathbf{P}(\mathbf{r}) \equiv \sum_i \mathbf{p}_i \delta^3(\mathbf{r} - \mathbf{r}_i)$$

Monopole electric charge density $\rho_{\text{mono}}(\mathbf{r})$:

$$\rho_{\text{mono}}(\mathbf{r}) \equiv \sum_i q_i \delta^3(\mathbf{r} - \mathbf{r}_i)$$

Electrostatic potential for a single monopole charge q
and a single dipole $\mathbf{p}$:

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \left(\frac{q}{r} + \frac{\mathbf{p} \cdot \mathbf{r}}{r^3} \right)$$

Coarse grain representation of macroscopic distribution of dipoles -- continued:

Electrostatic potential for a single monopole charge q
and a single dipole $\mathbf{p}$:

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \left(\frac{q}{r} + \frac{\mathbf{p} \cdot \mathbf{r}}{r^3} \right)$$

Electrostatic potential for collections of monopole charges q_i
and dipoles $\mathbf{p}_i$:

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \left(\int d^3r' \frac{\rho_{mono}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} + \int d^3r' \frac{\mathbf{P}(\mathbf{r}') \cdot (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \right)$$

$$\text{Note: } \int d^3r' \frac{\mathbf{P}(\mathbf{r}') \cdot (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} = \int d^3r' \mathbf{P}(\mathbf{r}') \cdot \nabla' \frac{1}{|\mathbf{r} - \mathbf{r}'|} = - \int d^3r' \frac{\nabla' \cdot \mathbf{P}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}$$

Coarse grain representation of macroscopic distribution of dipoles -- continued:

Electrostatic potential for collections of monopole charges q_i and dipoles $\mathbf{p}_i$:

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \left(\int d^3r' \frac{\rho_{mono}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} - \int d^3r' \frac{\nabla' \cdot \mathbf{P}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \right)$$

$$-\nabla^2 \Phi(\mathbf{r}) = \nabla \cdot \mathbf{E}(\mathbf{r}) = \frac{1}{\epsilon_0} (\rho_{mono}(\mathbf{r}) - \nabla \cdot \mathbf{P}(\mathbf{r}))$$

$$\Rightarrow \nabla \cdot (\epsilon_0 \mathbf{E}(\mathbf{r}) + \mathbf{P}(\mathbf{r})) = \rho_{mono}(\mathbf{r})$$

Define Displacement field: $\mathbf{D}(\mathbf{r}) \equiv \epsilon_0 \mathbf{E}(\mathbf{r}) + \mathbf{P}(\mathbf{r})$

Macroscopic form of Gauss's law: $\nabla \cdot \mathbf{D}(\mathbf{r}) = \rho_{mono}(\mathbf{r})$

Coarse grain representation of macroscopic distribution of dipoles -- continued:

Many materials are polarizable and produce a polarization field in the presence of an electric field with a proportionality constant χ_e :

$$\mathbf{P}(\mathbf{r}) = \varepsilon_0 \chi_e \mathbf{E}(\mathbf{r})$$

$$\mathbf{D}(\mathbf{r}) \equiv \varepsilon_0 \mathbf{E}(\mathbf{r}) + \mathbf{P}(\mathbf{r}) = \varepsilon_0 (1 + \chi_e) \mathbf{E}(\mathbf{r}) \equiv \varepsilon \mathbf{E}(\mathbf{r})$$

ε represents the dielectric function of the material

Boundary value problems in dielectric materials

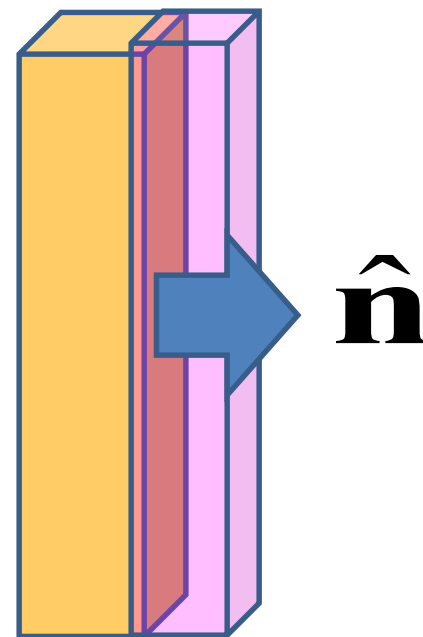
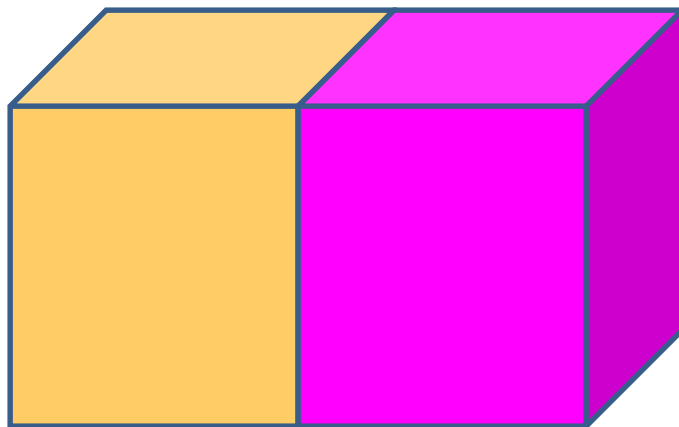
$$\text{For } \rho_{\text{mono}}(\mathbf{r}) = 0$$

$$\nabla \cdot \mathbf{D}(\mathbf{r}) = 0 \quad \text{and} \quad \nabla \times \mathbf{E}(\mathbf{r}) = 0$$

$\Rightarrow$ At a surface between two dielectrics, in terms of surface normal $\hat{\mathbf{r}}$:

$$\hat{\mathbf{r}} \cdot \mathbf{D}(\mathbf{r}) = \text{continuous} = \hat{\mathbf{r}} \times \mathbf{E}(\mathbf{r})$$

Some details --



Electrostatic equations:

$$\nabla \cdot \mathbf{D}(\mathbf{r}) = \rho_{\text{mono}}(\mathbf{r}) \quad \text{and} \quad \nabla \times \mathbf{E}(\mathbf{r}) = 0$$

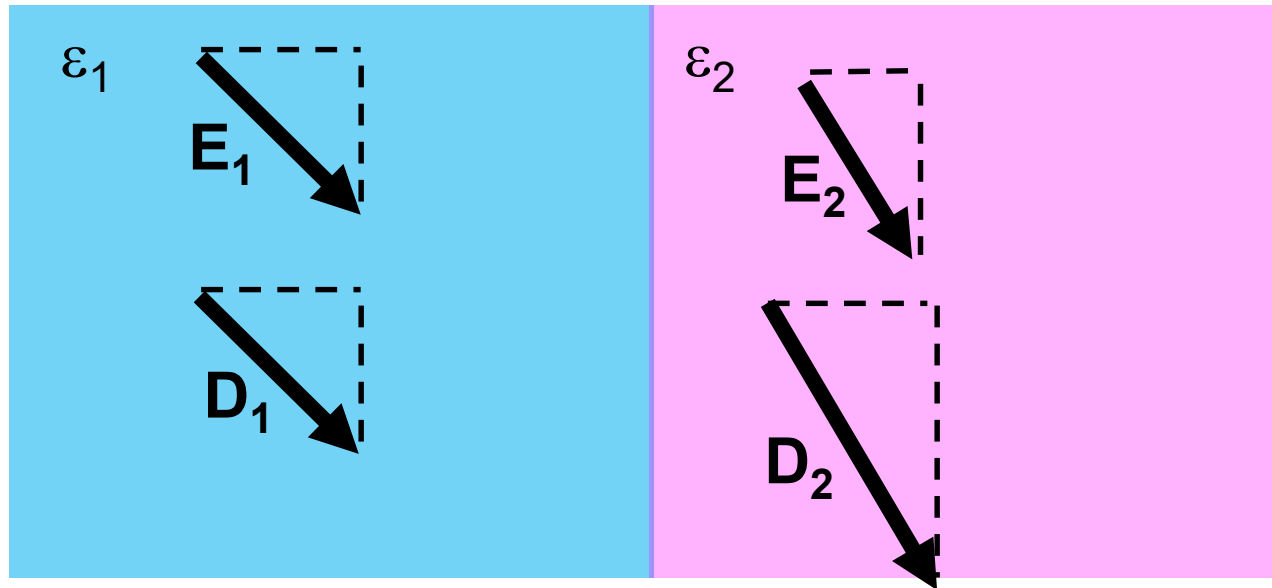
Under electrostatic conditions, these equations are true everywhere, including at the boundary between two materials.

$$\text{For } \rho_{\text{mono}}(r) = 0: \quad \nabla \cdot \mathbf{D} = 0 \quad \text{and} \quad \nabla \times \mathbf{E} = 0$$

$$\hat{\mathbf{n}} \cdot \mathbf{D} = \text{continuous} = \hat{\mathbf{n}} \times \mathbf{E}$$

Boundary value problems in the presence of dielectrics

– example:



$$\text{For } \frac{\epsilon_2}{\epsilon_1} = 2$$

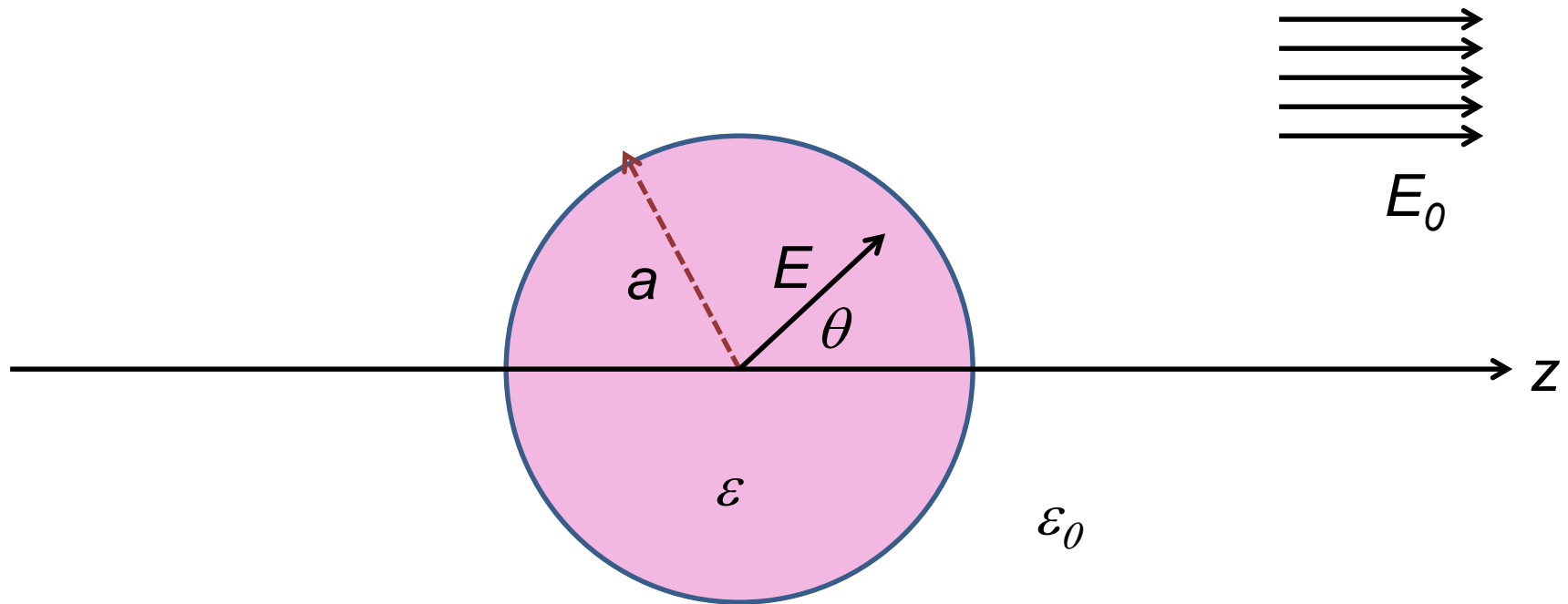
For isotropic dielectrics:

$$D_{1n} = D_{2n} \quad \epsilon_1 E_{1n} = \epsilon_2 E_{2n}$$

$$E_{1t} = E_{2t} \quad \frac{D_{1t}}{\epsilon_1} = \frac{D_{2t}}{\epsilon_2}$$

Boundary value problems in the presence of dielectrics

– example:



$$\nabla \cdot \mathbf{D}(\mathbf{r}) = 0 \quad \text{and} \quad \nabla \times \mathbf{E}(\mathbf{r}) = 0 \quad \text{At } r = a: \quad \epsilon \frac{\partial \Phi_{<}(\mathbf{r})}{\partial r} = \epsilon_0 \frac{\partial \Phi_{>}(\mathbf{r})}{\partial r}$$

$$\text{For } r \leq a \quad \mathbf{D}(\mathbf{r}) = -\epsilon \nabla \Phi(\mathbf{r})$$

$$\text{For } r > a \quad \mathbf{D}(\mathbf{r}) = -\epsilon_0 \nabla \Phi(\mathbf{r})$$

$$\frac{\partial \Phi_{<}(\mathbf{r})}{\partial \theta} = \frac{\partial \Phi_{>}(\mathbf{r})}{\partial \theta}$$

Note that we are introducing the scalar potential as before --

$\mathbf{E}(\mathbf{r}) = -\nabla\Phi(\mathbf{r})$ which automatically satisfies $\nabla \times \mathbf{E}(\mathbf{r}) = 0$

$\mathbf{D}(\mathbf{r}) = -\epsilon\nabla\Phi(\mathbf{r})$ or $\mathbf{D}(\mathbf{r}) = -\epsilon_0\nabla\Phi(\mathbf{r})$

Note that $\Phi(\mathbf{r})$ must satisfy $\nabla^2\Phi(\mathbf{r}) = 0$

Boundary value problems in the presence of dielectrics

– example -- continued:

$$\Phi_{<}(\mathbf{r}) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos \theta)$$

$$\Phi_{>}(\mathbf{r}) = \sum_{l=0}^{\infty} \left(B_l r^l + \frac{C_l}{r^{l+1}} \right) P_l(\cos \theta)$$

$$\text{At } r = a : \quad \varepsilon \frac{\partial \Phi_{<}(\mathbf{r})}{\partial r} = \varepsilon_0 \frac{\partial \Phi_{>}(\mathbf{r})}{\partial r}$$

$$\frac{\partial \Phi_{<}(\mathbf{r})}{\partial \theta} = \frac{\partial \Phi_{>}(\mathbf{r})}{\partial \theta}$$

$$\text{For } r \rightarrow \infty \quad \Phi_{>}(\mathbf{r}) = -E_0 r \cos \theta$$

Solution -- only $l = 1$ contributes

$$B_1 = -E_0$$

$$A_1 = -\left(\frac{3}{2 + \varepsilon / \varepsilon_0} \right) E_0$$

$$C_1 = \left(\frac{\varepsilon / \varepsilon_0 - 1}{2 + \varepsilon / \varepsilon_0} \right) a^3 E_0$$

Boundary value problems in the presence of dielectrics

– example -- continued:

$$\Phi_{<}(\mathbf{r}) = -\left(\frac{3}{2 + \epsilon / \epsilon_0}\right) E_0 r \cos \theta$$

$$\Phi_{>}(\mathbf{r}) = -\left(r - \left(\frac{\epsilon / \epsilon_0 - 1}{2 + \epsilon / \epsilon_0}\right) \frac{a^3}{r^2}\right) E_0 \cos \theta$$

