

PHY 712 Electrodynamics
10-10:50 AM MWF Online
Plan for Lecture 12:

Start reading Chapter 5

A. Magnetostatics

B. Vector potential

C. Example: current loop

Course schedule for Spring 2021

(Preliminary schedule -- subject to frequent adjustment.)

	Lecture date	JDJ Reading	Topic	HW	Due date
1	Wed: 01/27/2021	Chap. 1 & Appen.	Introduction, units and Poisson equation	#1	01/29/2021
2	Fri: 01/29/2021	Chap. 1	Electrostatic energy calculations	#2	02/01/2021
3	Mon: 02/01/2021	Chap. 1 & 2	Electrostatic potentials and fields	#3	02/03/2021
4	Wed: 02/03/2021	Chap. 1 - 3	Poisson's equation in 2 and 3 dimensions	#4	02/05/2021
5	Fri: 02/05/2021	Chap. 1 - 3	Brief introduction to numerical methods	#5	02/08/2021
6	Mon: 02/08/2021	Chap. 2 & 3	Image charge constructions	#6	02/10/2021
7	Wed: 02/10/2021	Chap. 2 & 3	Cylindrical and spherical geometries		
8	Fri: 02/12/2021	Chap. 3 & 4	Spherical geometry and multipole moments	#7	02/15/2021
9	Mon: 02/15/2021	Chap. 4	Dipoles and Dielectrics	#8	02/19/2021
10	Wed: 02/17/2021	Chap. 4	Dipoles and Dielectrics		
11	Fri: 02/19/2021	Chap. 4	Polarization and Dielectrics	#9	02/24/2021
12	Mon: 02/22/2021	Chap. 5	Magnetostatics	#10	02/26/2021
13	Wed: 02/24/2021	Chap. 5	Magnetic dipoles and hyperfine interaction		

PHY 712 -- Assignment #10

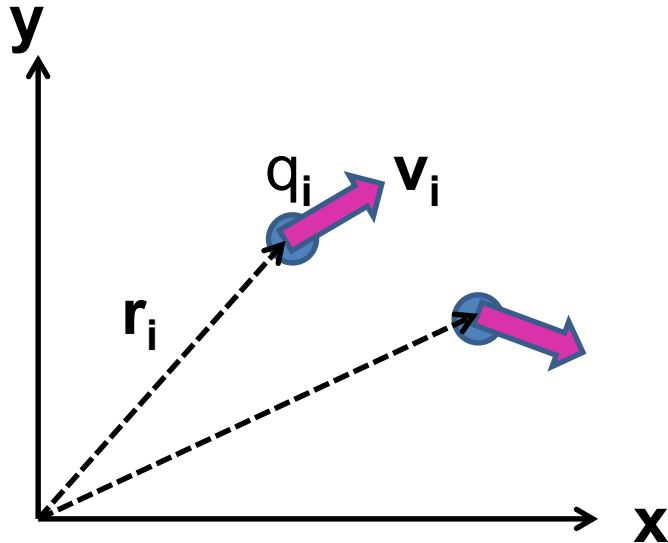
February 22, 2021

Start reading Chapter 5 in **Jackson** .

1. Consider an infinitely long cylindrical wire with radius a , oriented along the z axis. There is a steady uniform current inside the wire. Specifically, in terms of r the radial parameter of the cylindrical coordinates of the system the current density is $\mathbf{J}(r)=\mathbf{J}_0$, where $\mathbf{J}_0$ is a constant vector along the z -axis, for $r \leq a$ and zero otherwise.
 - a. Find the vector potential ($\mathbf{A}$) for all r .
 - b. Find the magnetic flux field ($\mathbf{B}$) for all r .

Magnetostatics

Magnetic flux density or magnetic induction field **B**
Steady state (constant in time) current density **J**



$$\mathbf{J}(\mathbf{r}) = \sum_i q_i \mathbf{v}_i \delta^3(\mathbf{r} - \mathbf{r}_i)$$

Note that "statics" implies that $\nabla \cdot \mathbf{J} = 0$.

This follows from the continuity equation :

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0$$

Comparison of electrostatics and magnetostatics

Electrostatic field due to charge density $\rho(\mathbf{r})$:

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int d^3r' \rho(\mathbf{r}') \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3}$$

Magnetostatic field due to current density $\mathbf{J}(\mathbf{r})$:

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int d^3r' \mathbf{J}(\mathbf{r}') \times \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3}$$

Alternative forms magnetostatic equations

Magnetostatic field due to current density $\mathbf{J}(\mathbf{r})$:

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int d^3r' \mathbf{J}(\mathbf{r}') \times \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} = \frac{\mu_0}{4\pi} \int d^3r' \left(\nabla \frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) \times \mathbf{J}(\mathbf{r}')$$

Note that: $\nabla \frac{1}{|\mathbf{r} - \mathbf{r}'|} = -\frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3}$

Also note that: $\nabla \times (s(\mathbf{r}) \mathbf{V}(\mathbf{r})) = \nabla s(\mathbf{r}) \times \mathbf{V}(\mathbf{r}) + s(\mathbf{r}) \nabla \times \mathbf{V}(\mathbf{r})$

let $s(\mathbf{r}) = \frac{1}{|\mathbf{r} - \mathbf{r}'|}$ and $\mathbf{V}(\mathbf{r}) = \mathbf{J}(\mathbf{r}')$, where $\mathbf{r}'$ is fixed

$$\left(\nabla \frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) \times \mathbf{J}(\mathbf{r}') = \nabla s(\mathbf{r}) \times \mathbf{V}(\mathbf{r}) = \nabla \times (s(\mathbf{r}) \mathbf{V}(\mathbf{r})) - s(\mathbf{r}) \nabla \times \mathbf{V}(\mathbf{r})$$

Magnetostatic field due to current density $\mathbf{J}(\mathbf{r})$:

$$\begin{aligned}\mathbf{B}(\mathbf{r}) &= \frac{\mu_0}{4\pi} \int d^3r' \left(\nabla \frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) \times \mathbf{J}(\mathbf{r}') \\ &= \frac{\mu_0}{4\pi} \int d^3r' \left(\nabla \times \left(\frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \right) - \frac{1}{|\mathbf{r} - \mathbf{r}'|} \nabla \times \mathbf{J}(\mathbf{r}') \right) \\ \Rightarrow \mathbf{B}(\mathbf{r}) &= \frac{\mu_0}{4\pi} \nabla \times \int d^3r' \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}\end{aligned}$$

Alternative forms magnetostatic equations -- continued

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \nabla \times \int d^3 r' \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}$$

$$\Rightarrow \nabla \cdot \mathbf{B}(\mathbf{r}) = 0 \quad \text{No magnetic monopoles}$$

$$\Rightarrow \nabla \times \mathbf{B}(\mathbf{r}) = \mu_0 \mathbf{J}(\mathbf{r}) \quad \text{Ampere's law}$$

"Proof" of Ampere's law for magnetostatic system :

$$\nabla \times \mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \nabla \times \nabla \times \int d^3 r' \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}$$

$$\text{Note that : } \nabla \times \nabla \times \mathbf{V} = \nabla(\nabla \cdot \mathbf{V}) - \nabla^2 \mathbf{V}$$

$$\text{Recall that : } \nabla^2 \frac{1}{|\mathbf{r} - \mathbf{r}'|} = -4\pi \delta^3(\mathbf{r} - \mathbf{r}') \quad \text{and} \quad \nabla \cdot \mathbf{J}(\mathbf{r}) = 0$$

Differential forms of magnetostatic equations:

$$\Rightarrow \nabla \cdot \mathbf{B}(\mathbf{r}) = 0 \quad \text{No magnetic monopoles}$$

$$\Rightarrow \nabla \times \mathbf{B}(\mathbf{r}) = \mu_0 \mathbf{J}(\mathbf{r}) \quad \text{Ampere's law}$$

Magnetostatic vector potential

$$\mathbf{B}(\mathbf{r}) = \nabla \times \mathbf{A}(\mathbf{r})$$

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \nabla \times \int d^3 r' \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}$$

$$\Rightarrow \mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int d^3 r' \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} + \nabla s(\mathbf{r})$$

Non uniqueness of the magnetostatic vector potential

Note that : $\mathbf{B}(\mathbf{r}) = \nabla \times \mathbf{A}(\mathbf{r}) = \nabla \times \mathbf{A}'(\mathbf{r})$
if $\mathbf{A}'(\mathbf{r}) = \mathbf{A}(\mathbf{r}) + \nabla s(\mathbf{r})$

Example: for $\mathbf{B}(\mathbf{r}) = B_0 \hat{\mathbf{z}}$

$$\mathbf{A}(\mathbf{r}) = \frac{1}{2} B_0 (x \hat{\mathbf{y}} - y \hat{\mathbf{x}})$$

or $\mathbf{A}(\mathbf{r}) = B_0 x \hat{\mathbf{y}}$

or $\mathbf{A}(\mathbf{r}) = -B_0 y \hat{\mathbf{x}}$

Differential form of Ampere's law in terms of vector potential:

$$\nabla \times \mathbf{B}(\mathbf{r}) = \nabla \times \nabla \times \mathbf{A}(\mathbf{r}) = \mu_0 \mathbf{J}(\mathbf{r})$$

$$\Rightarrow \nabla(\nabla \cdot \mathbf{A}(\mathbf{r})) - \nabla^2 \mathbf{A}(\mathbf{r}) = \mu_0 \mathbf{J}(\mathbf{r})$$

$$\text{If } \nabla \cdot \mathbf{A}(\mathbf{r}) = 0 \text{ (Coulomb gauge)} \Rightarrow \nabla^2 \mathbf{A}(\mathbf{r}) = -\mu_0 \mathbf{J}(\mathbf{r})$$

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int d^3 r' \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}$$

Comment on determining $\mathbf{A}(\mathbf{r})$ in the Coulomb gauge:

Differential form: $\nabla^2 \mathbf{A}(\mathbf{r}) = -\mu_0 \mathbf{J}(\mathbf{r})$

Integral form: $\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int d^3 r' \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}$

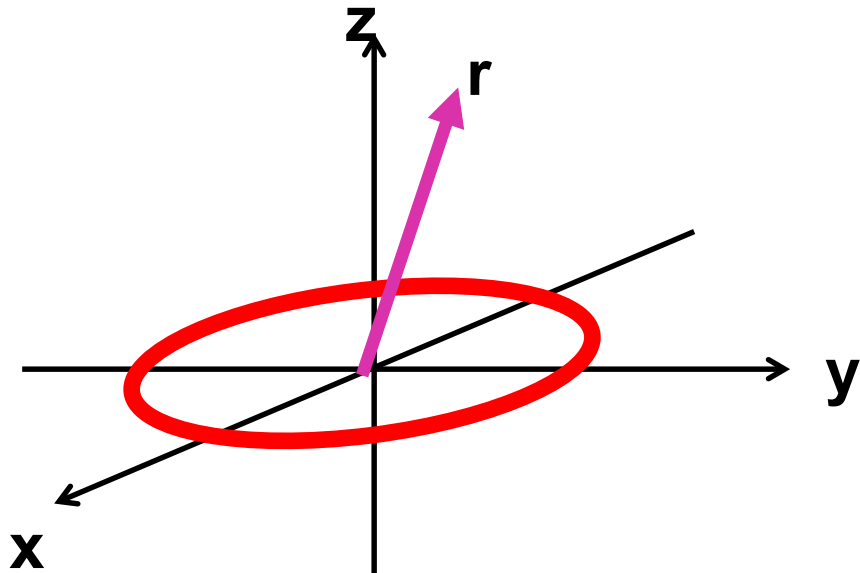
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Magnetostatics example: current loop



$$\mathbf{J}(\mathbf{r}') = \frac{I}{a} \sin \theta' \delta(\cos \theta') \delta(r' - a) (-\sin \phi' \hat{\mathbf{x}} + \cos \phi' \hat{\mathbf{y}})$$

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int d^3 r' \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}$$

Magnetostatics example: current loop -- continued

$$\mathbf{J}(\mathbf{r}') = \frac{I}{a} \sin \theta' \delta(\cos \theta') \delta(r' - a) (-\sin \varphi' \hat{\mathbf{x}} + \cos \varphi' \hat{\mathbf{y}})$$

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int d^3 r' \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}$$

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I}{4\pi a} \int r'^2 dr' d\cos \theta' d\varphi' \frac{\sin \theta' \delta(\cos \theta') \delta(r' - a) (-\sin \varphi' \hat{\mathbf{x}} + \cos \varphi' \hat{\mathbf{y}})}{(r^2 + r'^2 - 2rr'(\cos \theta \cos \theta' + \sin \theta \sin \theta' \cos(\varphi - \varphi')))^{1/2}}$$

Completing integration over r' and θ' :

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I a^2}{4\pi a} \int_0^{2\pi} d\varphi' \frac{(-\sin \varphi' \hat{\mathbf{x}} + \cos \varphi' \hat{\mathbf{y}})}{(r^2 + a^2 - 2ra(\sin \theta \cos(\varphi - \varphi')))^{1/2}}$$

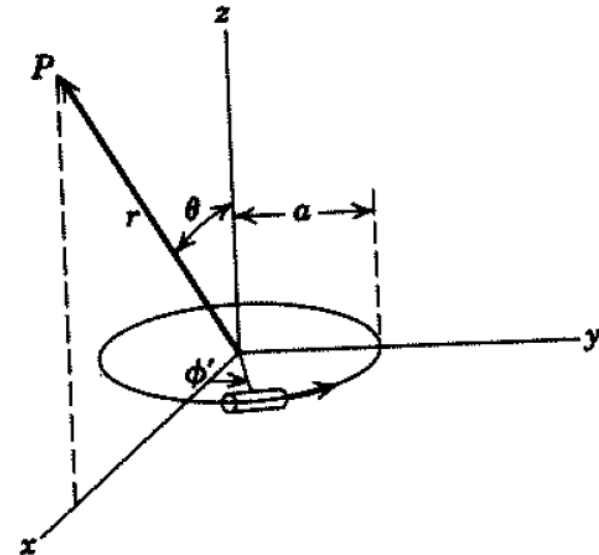
Let $\varphi - \varphi' \equiv \phi$

$$\sin \varphi' = \sin(\varphi - \phi) = \sin \varphi \cos \phi - \cos \varphi \sin \phi$$

$$\cos \varphi' = \cos(\varphi - \phi) = \cos \varphi \cos \phi + \sin \varphi \sin \phi$$

Remaining non-trivial terms

$$\mathbf{A}(\mathbf{r}) = -\frac{\mu_0 I a}{4\pi} (\sin \varphi \hat{\mathbf{x}} - \cos \varphi \hat{\mathbf{y}}) \int_0^{2\pi} d\phi \frac{\cos \phi}{(r^2 + a^2 - 2ra(\sin \theta \cos \phi))^{1/2}}$$



Magnetostatics example: current loop -- continued

$$\mathbf{A}(\mathbf{r}) = -\frac{\mu_0 I a}{4\pi} (\sin \varphi \hat{\mathbf{x}} - \cos \varphi \hat{\mathbf{y}}) \int_0^{2\pi} d\phi \frac{\cos \phi}{\left(r^2 + a^2 - 2ra(\sin \theta \cos \phi)\right)^{1/2}}$$

Elliptic integrals:

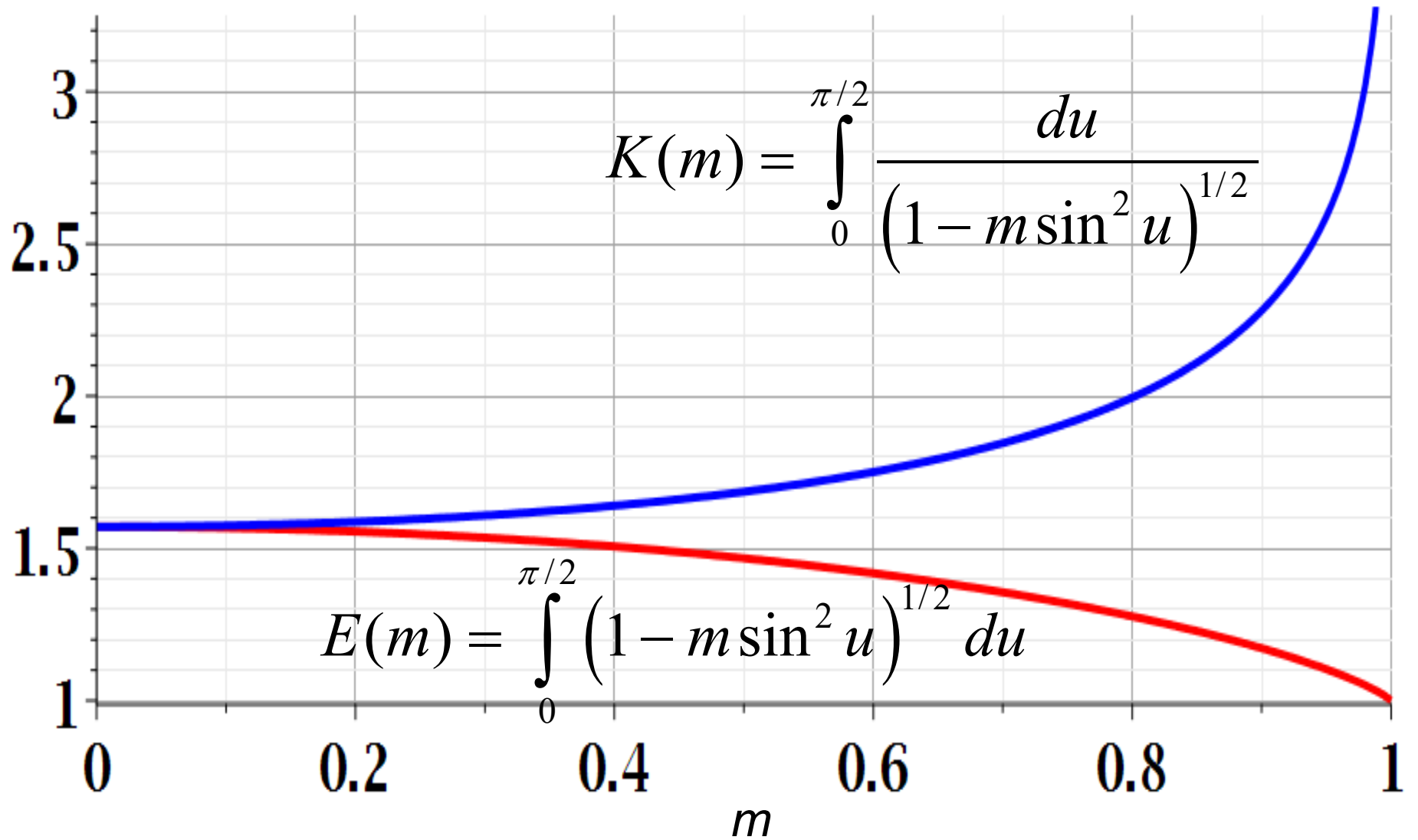
$$K(m) = \int_0^{\pi/2} \frac{du}{\left(1 - m \sin^2 u\right)^{1/2}}$$

$$E(m) = \int_0^{\pi/2} \left(1 - m \sin^2 u\right)^{1/2} du$$

$$\mathbf{A}(\mathbf{r}) = -\frac{\mu_0}{4\pi} 4Ia \frac{(\sin \varphi \hat{\mathbf{x}} - \cos \varphi \hat{\mathbf{y}})}{\left(r^2 + a^2 + 2ra \sin \theta\right)^{1/2}} \left[\frac{(2 - k^2)K(k) - 2E(k)}{k^2} \right]$$

where: $k^2 \equiv \frac{4ar \sin \theta}{r^2 + a^2 + 2ra \sin \theta}$

$$\mathbf{B}(\mathbf{r}) = \nabla \times \mathbf{A}(\mathbf{r})$$



Magnetostatics example: current loop -- continued

$$\mathbf{A}(\mathbf{r}) = -\frac{\mu_0}{4\pi} 4Ia \frac{(\sin \varphi \hat{\mathbf{x}} - \cos \varphi \hat{\mathbf{y}})}{(r^2 + a^2 + 2ra \sin \theta)^{1/2}} \left[\frac{(2 - k^2)K(k) - 2E(k)}{k^2} \right]$$

where: $k^2 \equiv \frac{4ar \sin \theta}{r^2 + a^2 + 2ra \sin \theta} = \frac{4ax}{x^2 + z^2 + a^2 + 2ax}$

For $\varphi = 0$: $x = r \sin \theta$, $y = 0$, $z = r \cos \theta$

$$\mathbf{A}(\mathbf{r}) = A_y(x, z) \hat{\mathbf{y}} = \frac{\mu_0}{4\pi} 4Ia \hat{\mathbf{y}} \frac{1}{(x^2 + z^2 + a^2 + 2ax)^{1/2}} \left[\frac{(2 - k^2)K(k) - 2E(k)}{k^2} \right]$$

where: $k^2 \equiv \frac{4ax}{x^2 + z^2 + a^2 + 2ax}$

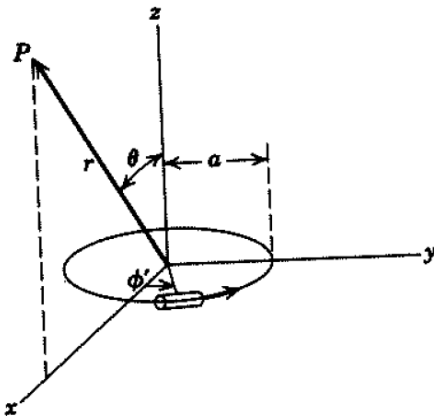
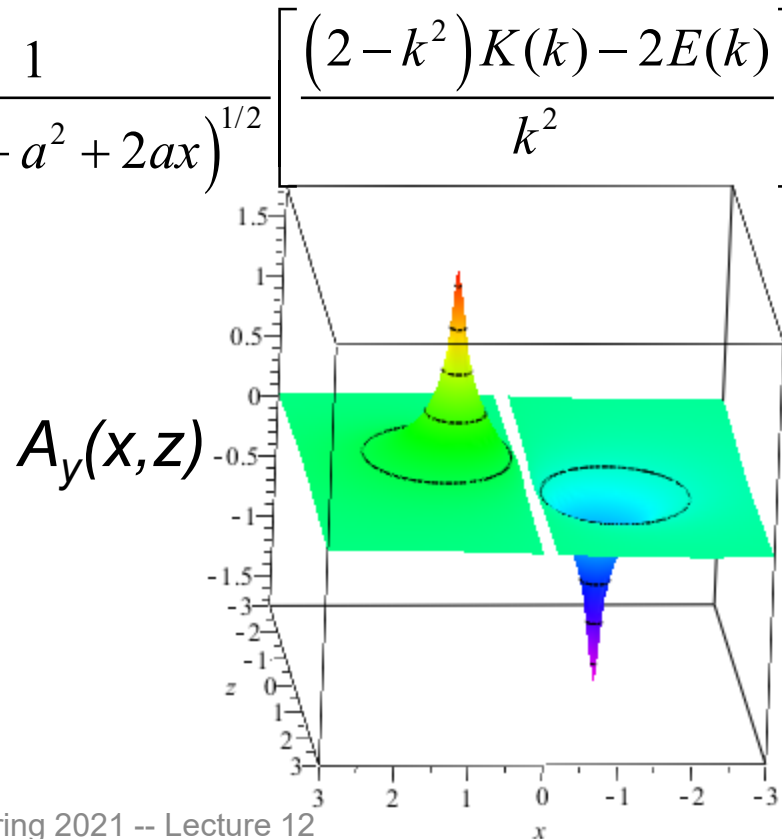


Figure 5.5



Magnetostatics example: current loop -- continued

$$\begin{aligned}\mathbf{A}(\mathbf{r}) &= -\frac{\mu_0}{4\pi} 4Ia \frac{(\sin \varphi \hat{\mathbf{x}} - \cos \varphi \hat{\mathbf{y}})}{(r^2 + a^2 + 2ra \sin \theta)^{1/2}} \left[\frac{(2 - k^2)K(k) - 2E(k)}{k^2} \right] \\ &= \hat{\boldsymbol{\phi}} \frac{\mu_0}{4\pi} \frac{4Ia}{(r^2 + a^2 + 2ra \sin \theta)^{1/2}} \left[\frac{(2 - k^2)K(k) - 2E(k)}{k^2} \right]\end{aligned}$$

where: $k^2 \equiv \frac{4ar \sin \theta}{r^2 + a^2 + 2ra \sin \theta}$

$$\mathbf{B} = \nabla \times \mathbf{A} = \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (r \sin \theta A_\phi(r, \theta)) \hat{\mathbf{r}} - \frac{1}{r} \frac{\partial (r A_\phi(r, \theta))}{\partial r} \hat{\boldsymbol{\theta}}$$

Evaluation for special cases

For $k^2 \rightarrow 0$:

$$\frac{(2 - k^2)K(k) - 2E(k)}{k^2} \approx \frac{\pi}{16} k^2$$

Evaluation for special case $k^2 \rightarrow 0$

$$\mathbf{A}(\mathbf{r}) = \hat{\boldsymbol{\phi}} \frac{\mu_0 I a^2}{4} \frac{r \sin \theta}{(r^2 + a^2 + 2ra \sin \theta)^{3/2}}$$

$$\begin{aligned} \mathbf{B} = \nabla \times \mathbf{A} &= \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (r \sin \theta A_\phi(r, \theta)) \hat{\mathbf{r}} - \frac{1}{r} \frac{\partial (r A_\phi(r, \theta))}{\partial r} \hat{\boldsymbol{\theta}} \\ &= \frac{\mu_0 I a^2}{4} \frac{(\hat{\mathbf{r}} \cos \theta (2a^2 + 2r^2 + ar \sin \theta) - \hat{\boldsymbol{\theta}} \sin \theta (2a^2 - r^2 + ar \sin \theta))}{(r^2 + a^2 + 2ra \sin \theta)^{5/2}} \end{aligned}$$

$$\text{where } k^2 \equiv \frac{4ar \sin \theta}{r^2 + a^2 + 2ra \sin \theta} \rightarrow 0$$

For $\theta = 0$: $r = z$

$$\begin{aligned} \mathbf{B} = \nabla \times \mathbf{A} &= \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (r \sin \theta A_\phi(r, \theta)) \hat{\mathbf{r}} - \frac{1}{r} \frac{\partial (r A_\phi(r, \theta))}{\partial r} \hat{\boldsymbol{\theta}} \\ &= \hat{\mathbf{z}} \frac{\mu_0 I a^2}{2} \frac{1}{(z^2 + a^2)^{3/2}} \end{aligned}$$

Evaluation for special case $k^2 \rightarrow 0$

$$\mathbf{A}(\mathbf{r}) = \hat{\boldsymbol{\phi}} \frac{\mu_0 I a^2}{4} \frac{r \sin \theta}{(r^2 + a^2 + 2ra \sin \theta)^{3/2}}$$

$$\begin{aligned} \mathbf{B} &= \nabla \times \mathbf{A} = \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (r \sin \theta A_\phi(r, \theta)) \hat{\mathbf{r}} - \frac{1}{r} \frac{\partial (r A_\phi(r, \theta))}{\partial r} \hat{\boldsymbol{\theta}} \\ &= \frac{\mu_0 I a^2}{4} \frac{(\hat{\mathbf{r}} \cos \theta (2a^2 + 2r^2 + ar \sin \theta) - \hat{\boldsymbol{\theta}} \sin \theta (2a^2 - r^2 + ar \sin \theta))}{(r^2 + a^2 + 2ra \sin \theta)^{5/2}} \end{aligned}$$

$$\text{where } k^2 \equiv \frac{4ar \sin \theta}{r^2 + a^2 + 2ra \sin \theta} \rightarrow 0$$

For $r \rightarrow \infty$

$$\begin{aligned} \mathbf{B} &= \nabla \times \mathbf{A} = \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (r \sin \theta A_\phi(r, \theta)) \hat{\mathbf{r}} - \frac{1}{r} \frac{\partial (r A_\phi(r, \theta))}{\partial r} \hat{\boldsymbol{\theta}} \\ &= \frac{\mu_0 I a^2}{4r^3} (\hat{\mathbf{r}} (2 \cos \theta) + \hat{\boldsymbol{\theta}} \sin \theta) \end{aligned}$$

Magnetostatics example: current loop -- continued

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} 4Ia \frac{(\sin \phi \hat{\mathbf{x}} - \cos \phi \hat{\mathbf{y}})}{(r^2 + a^2 + 2ra \sin \theta)^{1/2}} \left[\frac{(2 - k^2)K(k) - 2E(k)}{k^2} \right]$$

where: $k \equiv \frac{4ar \sin \theta}{r^2 + a^2 + 2ra \sin \theta}$

$$\mathbf{B}(\mathbf{r}) = \nabla \times \mathbf{A}(\mathbf{r})$$

Note that for spherical polar coordinates: $\hat{\phi} = \sin \phi \hat{\mathbf{x}} - \cos \phi \hat{\mathbf{y}}$

$$\mathbf{A}(\mathbf{r}) = A_\phi(\mathbf{r}) \hat{\phi}$$

where $A_\phi(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{4Ia}{(r^2 + a^2 + 2ra \sin \theta)^{1/2}} \left[\frac{(2 - k^2)K(k) - 2E(k)}{k^2} \right]$

$$\mathbf{B}(\mathbf{r}) = \frac{1}{r \sin \theta} \frac{\partial(\sin \theta A_\phi(\mathbf{r}))}{\partial \theta} \hat{\mathbf{r}} - \frac{1}{r} \frac{\partial(r A_\phi(\mathbf{r}))}{\partial r} \hat{\theta}$$

For $r \rightarrow \infty$:

$$\mathbf{B}(\mathbf{r}) \approx \frac{\mu_0}{4\pi} \frac{I\pi a^2}{r^3} (2 \cos \theta \hat{\mathbf{r}} + \sin \theta \hat{\theta})$$

Other examples of current density sources:

Quantum mechanical expression for current density

for a particle of mass M and charge e and of probability amplitude $\Psi(\mathbf{r})$:

$$\mathbf{J}(\mathbf{r}) = -\frac{e\hbar}{2Mi} (\Psi^*(\mathbf{r})\nabla\Psi(\mathbf{r}) - \Psi(\mathbf{r})\nabla\Psi^*(\mathbf{r}))$$

For an electron in a spherical potential (such as in an atom):

$$\Psi(\mathbf{r}) \equiv \Psi_{nlm_l}(\mathbf{r}) = R_{nl}(r)Y_{lm_l}(\hat{\mathbf{r}})$$

$$\begin{aligned}\mathbf{J}(\mathbf{r}) &= \frac{e\hbar}{2Mi} |R_{nl}(r)|^2 \frac{1}{r \sin \theta} \left(Y_{lm_l}^*(\hat{\mathbf{r}}) \frac{\partial Y_{lm_l}(\hat{\mathbf{r}})}{\partial \phi} - Y_{lm_l}(\hat{\mathbf{r}}) \frac{\partial Y_{lm_l}^*(\hat{\mathbf{r}})}{\partial \phi} \right) \hat{\phi} \\ &= \frac{e\hbar}{M} \frac{m_l}{r \sin \theta} |\Psi_{nlm_l}(\mathbf{r})|^2 \hat{\phi}\end{aligned}$$

$$\text{Note that : } \hat{\phi} = -\sin \phi \hat{\mathbf{x}} + \cos \phi \hat{\mathbf{y}} = \frac{\hat{\mathbf{z}} \times \mathbf{r}}{r \sin \theta}$$

$$\mathbf{J}(\mathbf{r}) = \frac{e\hbar}{M} \frac{m_l}{r^2 \sin^2 \theta} |\Psi_{nlm_l}(\mathbf{r})|^2 (\hat{\mathbf{z}} \times \mathbf{r})$$

Magnetic vector potential for this case:

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int d^3 r' \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}$$

$$\mathbf{J}(\mathbf{r}') = \frac{e\hbar}{M} \frac{m_l}{r'^2 \sin^2 \theta'} |\Psi_{nlm_l}(\mathbf{r}')|^2 (\hat{\mathbf{z}} \times \mathbf{r}')$$

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{e\hbar m_l}{M} \int d^3 r' \frac{(\hat{\mathbf{z}} \times \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \frac{|\Psi_{nlm_l}(\mathbf{r}')|^2}{r'^2 \sin^2 \theta'}$$

For example: electron in the $nlm_l = 211$ state of H:

$$|\Psi_{211}(\mathbf{r}')|^2 = \frac{1}{64\pi a^3} \left(\frac{r'}{a}\right)^2 e^{-r'/a} \sin^2 \theta'$$

$$\mathbf{A}(\mathbf{r}) = -\frac{\mu_0}{8\pi} \frac{e\hbar}{M} \frac{(\hat{\mathbf{z}} \times \mathbf{r})}{r^3} \left[1 - e^{-r/a} \left(1 + \frac{r}{a} + \frac{r^2}{2a^2} + \frac{r^3}{8a^3} \right) \right]$$