

PHY 712 Electrodynamics

10-10:50 AM MWF Online

Plan for Lecture 14:

Finish reading Chapter 5

1. Recap of hyperfine interaction
2. Macroscopic magnetization density M
3. H field and its relation to B
4. Magnetic boundary values

Course schedule for Spring 2021

(Preliminary schedule -- subject to frequent adjustment.)

	Lecture date	JDJ Reading	Topic	HW	Due date
1	Wed: 01/27/2021	Chap. 1 & Appen.	Introduction, units and Poisson equation	#1	01/29/2021
2	Fri: 01/29/2021	Chap. 1	Electrostatic energy calculations	#2	02/01/2021
3	Mon: 02/01/2021	Chap. 1 & 2	Electrostatic potentials and fields	#3	02/03/2021
4	Wed: 02/03/2021	Chap. 1 - 3	Poisson's equation in 2 and 3 dimensions	#4	02/05/2021
5	Fri: 02/05/2021	Chap. 1 - 3	Brief introduction to numerical methods	#5	02/08/2021
6	Mon: 02/08/2021	Chap. 2 & 3	Image charge constructions	#6	02/10/2021
7	Wed: 02/10/2021	Chap. 2 & 3	Cylindrical and spherical geometries		
8	Fri: 02/12/2021	Chap. 3 & 4	Spherical geometry and multipole moments	#7	02/15/2021
9	Mon: 02/15/2021	Chap. 4	Dipoles and Dielectrics	#8	02/19/2021
10	Wed: 02/17/2021	Chap. 4	Dipoles and Dielectrics		
11	Fri: 02/19/2021	Chap. 4	Polarization and Dielectrics	#9	02/24/2021
12	Mon: 02/22/2021	Chap. 5	Magnetostatics	#10	02/26/2021
13	Wed: 02/24/2021	Chap. 5	Magnetic dipoles and hyperfine interaction	#11	03/01/2021
14	Fri: 02/26/2021	Chap. 5	Magnetic dipoles and dipolar fields		
15	Mon: 03/01/2021	Chap. 6	Maxwell's Equations		

Summary of hyperfine interaction form:

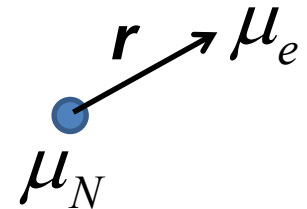
Interactions between magnetic dipoles

Sources of magnetic dipoles and other sources of magnetism in an atom:

- Intrinsic magnetic moment of a nucleus μ_N
- Intrinsic magnetic moment of an electron μ_e
- Magnetic field due to electron orbital current $\mathbf{J}_e(\mathbf{r})$

Interaction energy between a magnetic dipole $\mathbf{m}$ and a magnetic field $\mathbf{B}$:

$$E_{int} = -\mathbf{m} \cdot \mathbf{B}$$



In this case: $E_{int} = -\mu_N \cdot \mathbf{B}_{\mu_e} - \mu_N \cdot \mathbf{B}_{J_e}(0)$

$$\mathbf{B}_{\mu_e}(\mathbf{r}) = \frac{\mu_0}{4\pi} \left\{ \frac{3\hat{\mathbf{r}}(\boldsymbol{\mu}_e \cdot \hat{\mathbf{r}}) - \boldsymbol{\mu}_e}{r^3} + \frac{8\pi}{3} \boldsymbol{\mu}_e \delta^3(\mathbf{r}) \right\}$$

Hyperfine interaction energy: -- continued

$$E_{int} = -\mu_N \cdot \mathbf{B}_{\mu_e} - \mu_N \cdot \mathbf{B}_{J_e}(0) \quad \text{Here we assume that nuclear position is } \mathbf{r}=0.$$

Evaluation of the magnetic field at the nucleus due to the electron current density:

The vector potential associated with an electron in a bound state of an atom as described by a quantum mechanical wavefunction $\psi_{nlm_l}(\mathbf{r})$ can be written:

$$\mathbf{A}_{J_e}(\mathbf{r}) = -\frac{\mu_0}{4\pi} \frac{e\hbar}{m_e} m_l \int d^3r' \frac{\hat{\mathbf{z}} \times \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|} \frac{|\psi_{nlm_l}(\mathbf{r}')|^2}{r'^2 \sin^2 \theta'}$$

We want to evaluate the magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$ in the vicinity of the nucleus ($\mathbf{r} \rightarrow 0$).

Hyperfine interaction energy: -- continued

$$\mathbf{B}_{\mathbf{J}_e}(0) = \nabla \times \mathbf{A}_{\mathbf{J}_e} \Big|_{\mathbf{r} \rightarrow 0} = -\frac{\mu_0}{4\pi} \frac{e\hbar}{m_e} m_l \int d^3 r' \nabla \times \frac{\hat{\mathbf{z}} \times \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|} \frac{|\psi_{nlm_l}(\mathbf{r}')|^2}{r'^2 \sin^2 \theta'} \Big|_{\mathbf{r} \rightarrow 0}$$

$$\mathbf{B}_o(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{e\hbar}{m_e} m_l \int d^3 r' \frac{(\mathbf{r} - \mathbf{r}') \times (\hat{\mathbf{z}} \times \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \frac{|\psi_{nlm_l}(\mathbf{r}')|^2}{r'^2 \sin^2 \theta'} \Big|_{\mathbf{r} \rightarrow 0}$$

$$\mathbf{B}_o(\mathbf{0}) = -\frac{\mu_0}{4\pi} \frac{e\hbar}{m_e} m_l \int d^3 r' \frac{\mathbf{r}' \times (\hat{\mathbf{z}} \times \mathbf{r}')}{r'^3} \frac{|\psi_{nlm_l}(\mathbf{r}')|^2}{r'^2 \sin^2 \theta'}$$

$$\hat{\mathbf{r}}' \times (\hat{\mathbf{z}} \times \hat{\mathbf{r}}') = \hat{\mathbf{z}}(1 - \cos^2 \theta') - \hat{\mathbf{x}} \cos \theta' \sin \theta' \cos \phi' - \hat{\mathbf{y}} \cos \theta' \sin \theta' \sin \phi'.$$

$$\begin{aligned} \mathbf{B}_o(\mathbf{0}) &= -\frac{\mu_0}{4\pi} \frac{e\hbar}{m_e} m_l \int d^3 r' \frac{\hat{\mathbf{z}} r'^2 \sin^2 \theta'}{r'^3} \frac{|\psi_{nlm_l}(\mathbf{r}')|^2}{r'^2 \sin^2 \theta'} = -\frac{\mu_0}{4\pi} \frac{e\hbar}{m_e} m_l \hat{\mathbf{z}} \int d^3 r' \frac{|\psi_{nlm_l}(\mathbf{r}')|^2}{r'^3} \\ &= -\frac{\mu_0}{4\pi} \frac{e\hbar}{m_e} m_l \hat{\mathbf{z}} \left\langle \frac{1}{r'^3} \right\rangle \end{aligned}$$

Hyperfine interaction energy: -- continued

$$E_{int} \equiv H_{HF} = -\mu_N \cdot \mathbf{B}_{\mu_e} - \mu_N \cdot \mathbf{B}_{J_e}(0)$$

Putting all of the terms together:

$$H_{HF} = -\frac{\mu_0}{4\pi} \left(\left\langle \frac{3(\boldsymbol{\mu}_N \cdot \hat{\mathbf{r}})(\boldsymbol{\mu}_e \cdot \hat{\mathbf{r}}) - \boldsymbol{\mu}_N \cdot \boldsymbol{\mu}_e}{r^3} + \frac{8\pi}{3} \boldsymbol{\mu}_N \cdot \boldsymbol{\mu}_e \delta^3(\mathbf{r}) \right\rangle + \frac{e}{m_e} \left\langle \frac{\mathbf{L} \cdot \boldsymbol{\mu}_N}{r^3} \right\rangle \right).$$

In this expression the brackets $\langle \rangle$ indicate evaluating the expectation value relative to the electronic state.

Macroscopic dipolar effects --
Magnetic dipole moment

$$\mathbf{m} = \frac{1}{2} \int d^3r \, \mathbf{r} \times \mathbf{J}(\mathbf{r})$$

Note that the intrinsic spin of elementary particles is associated with a magnetic dipole moment, but we often do not have a detailed knowledge of its $\mathbf{J}(\mathbf{r})$.

Vector potential for magnetic dipole moment

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{\mathbf{m} \times \mathbf{r}}{|\mathbf{r}|^3}$$

Valid outside the
extent of $\mathbf{J}(\mathbf{r})$

Macroscopic magnetization

$$\mathbf{M}(\mathbf{r}) = \sum_i \mathbf{m}_i \delta^3(\mathbf{r} - \mathbf{r}_i)$$

Vector potential due to “free” current $\mathbf{J}_{\text{free}}(\mathbf{r})$ and macroscopic magnetization $\mathbf{M}(\mathbf{r})$. Note: the designation $\mathbf{J}_{\text{free}}(\mathbf{r})$ implies that this current does not also contribute to the magnetization density.

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int d^3 r' \left(\frac{\mathbf{J}_{\text{free}}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} + \frac{\mathbf{M}(\mathbf{r}') \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \right)$$

Vector potential contributions from macroscopic magnetization -- continued

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int d^3r' \left(\frac{\mathbf{J}_{free}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} + \frac{\mathbf{M}(\mathbf{r}') \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \right)$$

Note that :

$$\begin{aligned} \frac{\mathbf{M}(\mathbf{r}') \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} &= \mathbf{M}(\mathbf{r}') \times \nabla' \frac{1}{|\mathbf{r} - \mathbf{r}'|} \\ &= -\nabla' \times \left(\frac{\mathbf{M}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \right) + \frac{\nabla' \times \mathbf{M}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \\ \Rightarrow \mathbf{A}(\mathbf{r}) &= \frac{\mu_0}{4\pi} \int d^3r' \frac{\mathbf{J}_{free}(\mathbf{r}') + \nabla' \times \mathbf{M}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \end{aligned}$$

Vector potential contributions from macroscopic magnetization -- continued

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int d^3 r' \frac{\mathbf{J}_{free}(\mathbf{r}') + \nabla' \times \mathbf{M}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}$$

Note that for the case that $\nabla \cdot \mathbf{A} = 0$:

$$\begin{aligned} \nabla \times \mathbf{B}(\mathbf{r}) &= \nabla \times (\nabla \times \mathbf{A}(\mathbf{r})) = -\nabla^2 \mathbf{A}(\mathbf{r}) \\ &= \frac{\mu_0}{4\pi} \int d^3 r' (4\pi \delta^3(\mathbf{r} - \mathbf{r}')) (\mathbf{J}_{free}(\mathbf{r}') + \nabla' \times \mathbf{M}(\mathbf{r}')) \\ &= \mu_0 (\mathbf{J}_{free}(\mathbf{r}) + \nabla \times \mathbf{M}(\mathbf{r})) \\ \Rightarrow \nabla \times (\mathbf{B}(\mathbf{r}) - \mu_0 \mathbf{M}(\mathbf{r})) &= \mu_0 \mathbf{J}_{free}(\mathbf{r}) \end{aligned}$$

Magnetic field contributions

$$\nabla \times (\mathbf{B}(\mathbf{r}) - \mu_0 \mathbf{M}(\mathbf{r})) = \mu_0 \mathbf{J}_{free}(\mathbf{r})$$

Define "new" magnetic field vector:

$$\mu_0 \mathbf{H}(\mathbf{r}) \equiv \mathbf{B}(\mathbf{r}) - \mu_0 \mathbf{M}(\mathbf{r})$$

$$\Rightarrow \nabla \times \mathbf{H}(\mathbf{r}) = \mathbf{J}_{free}(\mathbf{r})$$

$$\nabla \times (\mathbf{B}(\mathbf{r}) - \mu_0 \mathbf{M}(\mathbf{r})) = \mu_0 \mathbf{J}_{free}(\mathbf{r})$$

Note that $\mathbf{B}(\mathbf{r}) \equiv$ the magnetic flux density

Define $\mathbf{H}(\mathbf{r}) \equiv$ the magnetic field

$$\mu_0 \mathbf{H}(\mathbf{r}) \equiv \mathbf{B}(\mathbf{r}) - \mu_0 \mathbf{M}(\mathbf{r})$$

$$\Rightarrow \nabla \times \mathbf{H}(\mathbf{r}) = \mathbf{J}_{free}(\mathbf{r})$$

Energy associated with magnetic fields

Note: We previously used without proof --

the force on a magnetic dipole $\mathbf{m}$ in an external $\mathbf{B}$ field is:

$$\mathbf{F} = \nabla (\mathbf{m} \cdot \mathbf{B})$$

This implies that energy associated with aligning a magnetic dipole $\mathbf{m}$ in an external $\mathbf{B}$ field is given by:

$$U = -\mathbf{m} \cdot \mathbf{B}$$

Macroscopic energies --

It can be shown that:
$$W_B = \frac{1}{2} \int d^3r \mathbf{B}(\mathbf{r}) \cdot \mathbf{H}(\mathbf{r})$$

In analogy to:
$$W_E = \frac{1}{2} \int d^3r \mathbf{E}(\mathbf{r}) \cdot \mathbf{D}(\mathbf{r})$$

Summary of equations of magnetostatics :

$$\nabla \times \mathbf{B}(\mathbf{r}) = \mu_0 \mathbf{J}_{total}(\mathbf{r})$$

$$\nabla \times \mathbf{H}(\mathbf{r}) = \mathbf{J}_{free}(\mathbf{r})$$

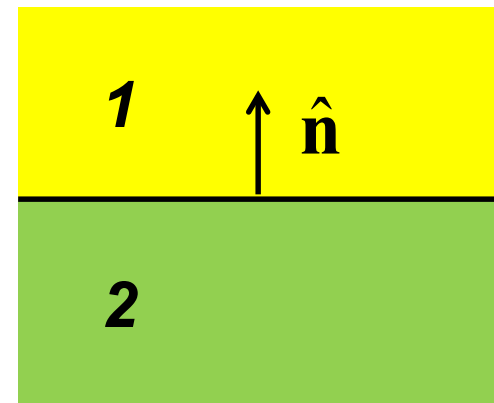
$$\mathbf{B}(\mathbf{r}) = \mu_0 (\mathbf{H}(\mathbf{r}) + \mathbf{M}(\mathbf{r}))$$

$$\nabla \cdot \mathbf{B}(\mathbf{r}) = 0$$

For the case that $\mathbf{J}_{free}(\mathbf{r}) = 0$:

$$\nabla \times \mathbf{H}(\mathbf{r}) = 0$$

$$\nabla \cdot \mathbf{B}(\mathbf{r}) = 0$$



For the case that $\mathbf{J}_{free}(\mathbf{r}) = 0$:

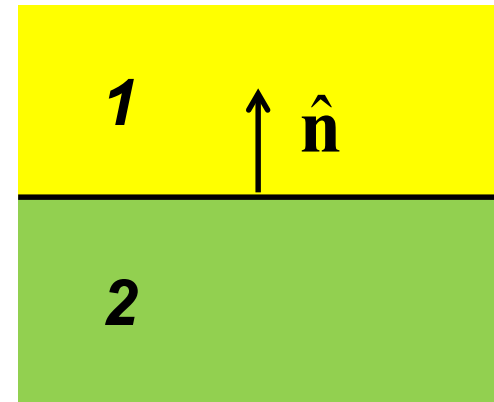
$$\nabla \times \mathbf{H}(\mathbf{r}) = 0$$

$$\nabla \cdot \mathbf{B}(\mathbf{r}) = 0$$

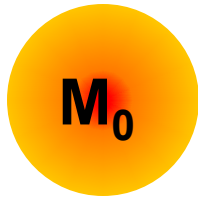
At boundary:

$$\mathbf{H}_1 \times \hat{\mathbf{n}} = \mathbf{H}_2 \times \hat{\mathbf{n}}$$

$$\mathbf{B}_1 \cdot \hat{\mathbf{n}} = \mathbf{B}_2 \cdot \hat{\mathbf{n}}$$



Example magnetostatic boundary value problem



$$\mathbf{M}(\mathbf{r}) = \begin{cases} M_0 \hat{\mathbf{z}} & r \leq a \\ 0 & r > a \end{cases}$$

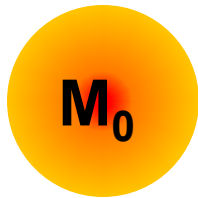
$$\nabla \times \mathbf{H}(\mathbf{r}) = 0 \quad \Rightarrow \quad \mathbf{H}(\mathbf{r}) = -\nabla \Phi_H(\mathbf{r})$$

$$\mathbf{B}(\mathbf{r}) = \mu_0 (\mathbf{H}(\mathbf{r}) + \mathbf{M}(\mathbf{r}))$$

$$\nabla \cdot \mathbf{B}(\mathbf{r}) = 0 = \mu_0 \nabla \cdot (\mathbf{H}(\mathbf{r}) + \mathbf{M}(\mathbf{r}))$$

$$\Rightarrow \nabla^2 \Phi_H(\mathbf{r}) = \nabla \cdot \mathbf{M}(\mathbf{r})$$

Example magnetostatic boundary value problem -- continued



$$\mathbf{M}(\mathbf{r}) = \begin{cases} M_0 \hat{\mathbf{z}} & r \leq a \\ 0 & r > a \end{cases}$$

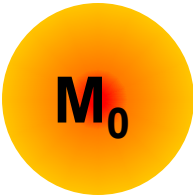
$$\nabla^2 \Phi_H(\mathbf{r}) = \nabla \cdot \mathbf{M}(\mathbf{r})$$

$$\Rightarrow \Phi_H(\mathbf{r}) = -\frac{1}{4\pi} \int d^3 r' \frac{\nabla' \cdot \mathbf{M}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}$$

$$= -\frac{1}{4\pi} \int d^3 r' \left[\nabla' \cdot \left(\frac{\mathbf{M}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \right) - \mathbf{M}(\mathbf{r}') \cdot \nabla' \frac{1}{|\mathbf{r} - \mathbf{r}'|} \right]$$

$$= -\frac{1}{4\pi} \nabla \cdot \int d^3 r' \frac{\mathbf{M}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}$$

Example magnetostatic boundary value problem -- continued

 $\mathbf{M}(\mathbf{r}) = \begin{cases} M_0 \hat{\mathbf{z}} & r \leq a \\ 0 & r > a \end{cases} \quad \Phi_H(\mathbf{r}) = -\frac{1}{4\pi} \nabla \cdot \int d^3 r' \frac{\mathbf{M}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}$

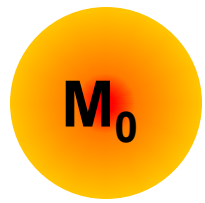
For this example:

$$\Phi_H(\mathbf{r}) = -\frac{M_0}{4\pi} \frac{\partial}{\partial z} \left(4\pi \int_0^a r'^2 dr' \frac{1}{r_{>}} \right)$$

$$\text{For } r \leq a: \quad \Phi_H(\mathbf{r}) = -M_0 \frac{\partial}{\partial z} \left(\frac{a^2}{2} - \frac{r^2}{6} \right) = \frac{M_0 z}{3}$$

$$\text{For } r > a: \quad \Phi_H(\mathbf{r}) = -M_0 \frac{\partial}{\partial z} \left(\frac{a^3}{3r} \right) = \frac{M_0 a^3 z}{3r^3}$$

Example magnetostatic boundary value problem -- continued



$$\mathbf{M}(\mathbf{r}) = \begin{cases} M_0 \hat{\mathbf{z}} & r \leq a \\ 0 & r > a \end{cases}$$

$$\text{For } r \leq a: \quad \Phi_H(\mathbf{r}) = \frac{M_0 z}{3} \quad \mathbf{H}(\mathbf{r}) = -\nabla \Phi_H(\mathbf{r}) = -\frac{M_0}{3} \hat{\mathbf{z}}$$

$$\text{For } r > a: \quad \Phi_H(\mathbf{r}) = \frac{M_0 a^3 z}{3r^3} \quad \mathbf{H}(\mathbf{r}) = -\nabla \Phi_H(\mathbf{r}) = -\frac{M_0 a^3}{3} \left(\frac{\hat{\mathbf{z}}}{r^3} - \frac{3z\mathbf{r}}{r^5} \right)$$

$$\mathbf{B}(\mathbf{r}) = \mu_0 (\mathbf{H}(\mathbf{r}) + \mathbf{M}(\mathbf{r}))$$

$$\text{For } r \leq a: \quad \mathbf{H}(\mathbf{r}) = -\frac{M_0 \hat{\mathbf{z}}}{3} \quad \mathbf{B}(\mathbf{r}) = \mu_0 \frac{2M_0 \hat{\mathbf{z}}}{3}$$

$$\text{For } r > a: \quad \mathbf{H}(\mathbf{r}) = -\frac{M_0 a^3}{3} \left(\frac{\hat{\mathbf{z}}}{r^3} - \frac{3z\mathbf{r}}{r^5} \right)$$

$$\mathbf{B}(\mathbf{r}) = -\mu_0 \frac{M_0 a^3}{3} \left(\frac{\hat{\mathbf{z}}}{r^3} - \frac{3z\mathbf{r}}{r^5} \right)$$

Check boundary values:

$$\text{For } r \leq a: \quad \mathbf{H}(\mathbf{r}) = -\frac{M_0 \hat{\mathbf{z}}}{3} \quad \mathbf{H}(a\hat{\mathbf{r}}) \times \hat{\mathbf{r}} = -\frac{M_0}{3} \hat{\mathbf{z}} \times \hat{\mathbf{r}}$$

$$\text{For } r > a: \quad \mathbf{H}(\mathbf{r}) = -\frac{M_0 a^3}{3} \left(\frac{\hat{\mathbf{z}}}{r^3} - \frac{3z\mathbf{r}}{r^5} \right)$$

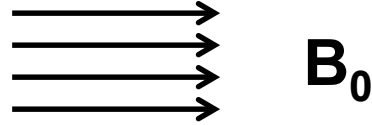
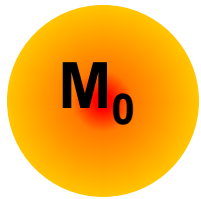
$$\mathbf{H}(a\hat{\mathbf{r}}) \times \hat{\mathbf{r}} = -\frac{M_0 a^3}{3} \frac{\hat{\mathbf{z}} \times \hat{\mathbf{r}}}{a^3}$$

$$\text{For } r \leq a: \quad \mathbf{B}(\mathbf{r}) = \mu_0 \frac{2M_0 \hat{\mathbf{z}}}{3} \quad \mathbf{B}(a\hat{\mathbf{r}}) \cdot \hat{\mathbf{r}} = \mu_0 \frac{2M_0}{3} \hat{\mathbf{z}} \cdot \hat{\mathbf{r}}$$

$$\text{For } r > a: \quad \mathbf{B}(\mathbf{r}) = -\mu_0 \frac{M_0 a^3}{3} \left(\frac{\hat{\mathbf{z}}}{r^3} - \frac{3z\mathbf{r}}{r^5} \right)$$

$$\mathbf{B}(a\hat{\mathbf{r}}) \cdot \hat{\mathbf{r}} = -\mu_0 \frac{M_0 a^3}{3} \hat{\mathbf{z}} \cdot \hat{\mathbf{r}} \left(\frac{1}{a^3} - \frac{3a^2}{a^5} \right)$$

Variation; magnetic sphere plus external field $\mathbf{B}_0$



$$\mathbf{M}(\mathbf{r}) = \begin{cases} \mathbf{M}_0 & r \leq a \\ 0 & r > a \end{cases}$$

By superposition :

For $r \leq a$:

$$\mathbf{B}(\mathbf{r}) = \mathbf{B}_0 + \mu_0 \frac{2}{3} \mathbf{M}_0$$

$$\mathbf{H}(\mathbf{r}) = \frac{1}{\mu_0} \mathbf{B}_0 - \frac{1}{3} \mathbf{M}_0$$

$$\mathbf{B}(\mathbf{r}) + 2\mu_0 \mathbf{H}(\mathbf{r}) = 3\mathbf{B}_0$$

For an isotropic "paramagnetic" material, $\mathbf{B}(\mathbf{r}) = \mu \mathbf{H}(\mathbf{r})$

$$\mathbf{M}_0 = \frac{3}{\mu_0} \left(\frac{\mu - \mu_0}{\mu + 2\mu_0} \right) \mathbf{B}_0$$

Summary of equations of magnetostatics :

$$\nabla \times \mathbf{B}(\mathbf{r}) = \mu_0 \mathbf{J}_{total}(\mathbf{r})$$

$$\nabla \times \mathbf{H}(\mathbf{r}) = \mathbf{J}_{free}(\mathbf{r})$$

$$\mathbf{B}(\mathbf{r}) = \mu_0 (\mathbf{H}(\mathbf{r}) + \mathbf{M}(\mathbf{r}))$$

$$\nabla \cdot \mathbf{B}(\mathbf{r}) = 0$$

For the case that $\mathbf{J}_{free}(\mathbf{r})$:

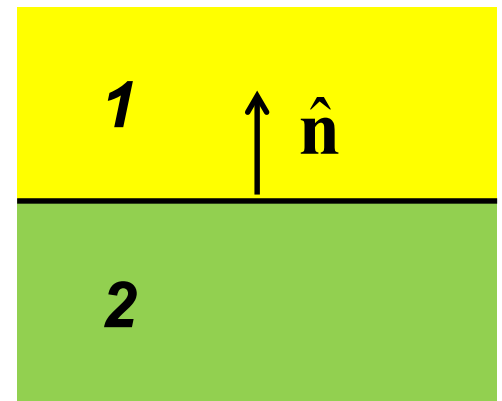
$$\nabla \times \mathbf{H}(\mathbf{r}) = 0$$

$$\nabla \cdot \mathbf{B}(\mathbf{r}) = 0$$

At boundary :

$$\mathbf{H}_1 \times \hat{\mathbf{n}} = \mathbf{H}_2 \times \hat{\mathbf{n}}$$

$$\mathbf{B}_1 \cdot \hat{\mathbf{n}} = \mathbf{B}_2 \cdot \hat{\mathbf{n}}$$



Magnetism in materials

$$\mathbf{B}(\mathbf{r}) = \mu_0 (\mathbf{H}(\mathbf{r}) + \mathbf{M}(\mathbf{r}))$$

For materials with linear magnetism :

$$\mathbf{B} = \mu \mathbf{H}$$

$\mu > \mu_0 \Rightarrow$ paramagnetic material

$\mu < \mu_0 \Rightarrow$ diamagnetic material

For ferromagnetic, antiferromagnetic materials

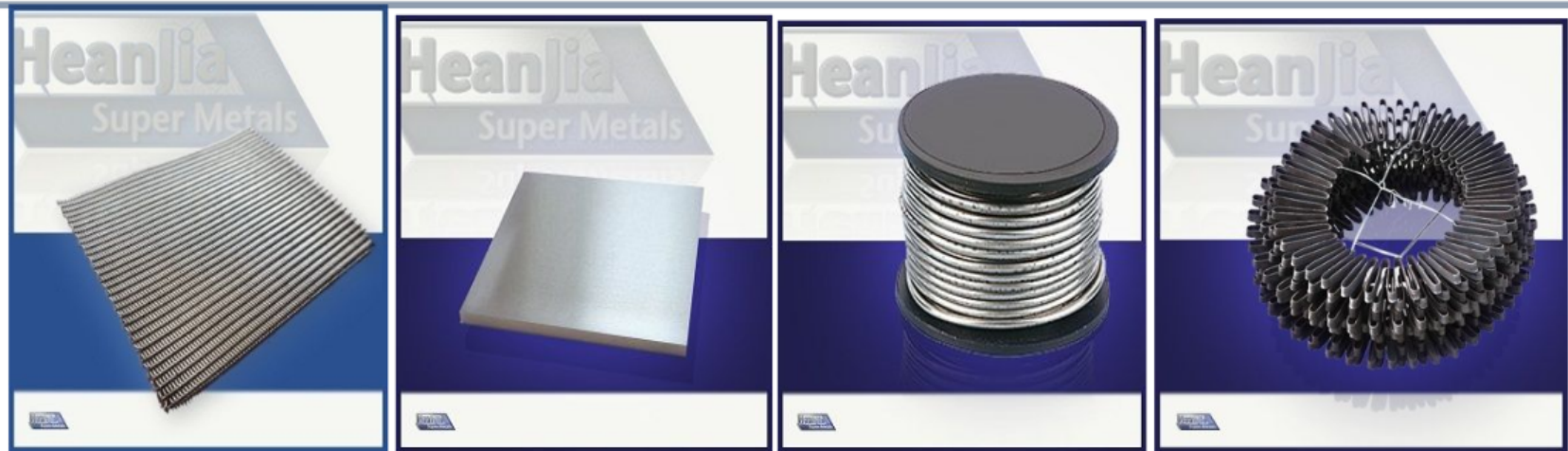
$$\mathbf{B} = f(\mathbf{H}) \quad (\text{with hysteresis})$$

[https://en.wikipedia.org/wiki/Permeability_\(electromagnetism\)](https://en.wikipedia.org/wiki/Permeability_(electromagnetism))

Magnetic susceptibility and permeability data for selected materials

Medium	Susceptibility, volumetric, SI, χ_m	Permeability, μ (H/m)	Relative permeability, max., μ/μ_0	Magnetic field
Metglas 2714A (annealed)		1.26×10^0	1 000 000 ^[10]	At 0.5 T
Iron (99.95% pure Fe annealed in H)		2.5×10^{-1}	200 000 ^[11]	
NANOPERM®		1.0×10^{-1}	80 000 ^[12]	At 0.5 T
Mu-metal		2.5×10^{-2}	20 000 ^[13]	At 0.002 T
Mu-metal		6.3×10^{-2}	50 000 ^[14]	
Cobalt-iron (high permeability strip material)		2.3×10^{-2}	18 000 ^[15]	
Permalloy	8000	1.0×10^{-2}	8000 ^[13]	At 0.002 T

Mumetal Magnetic Shielding



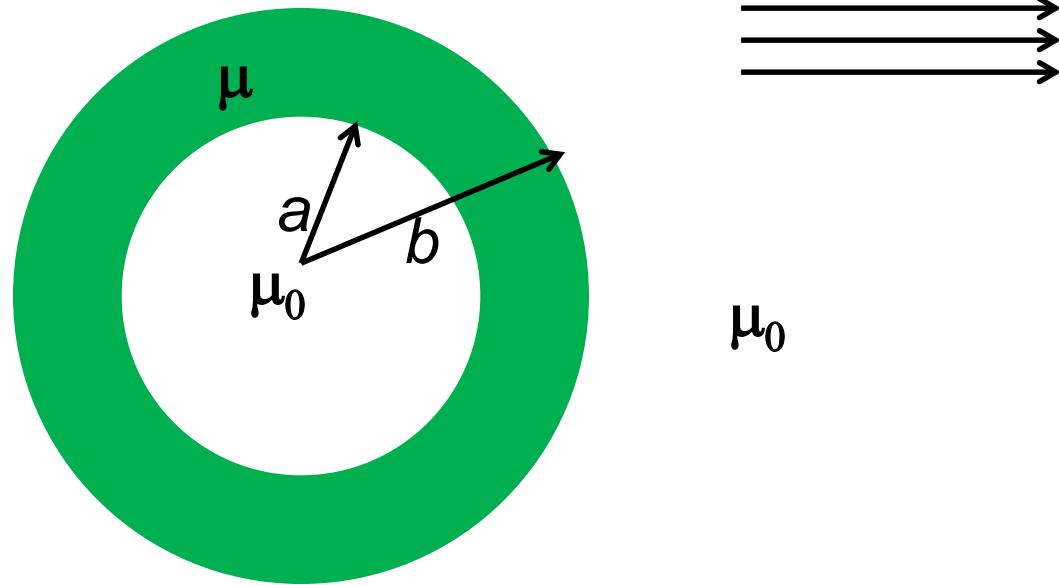
Mu metal is a soft ferromagnetic alloy that has extremely high initial and maximum magnetic permeability. It is used in electric transformer, storage disks, magnetic phonographs, resonance devices and superconducting circuits.

Mumetal alloy generally attributes relative permeability about 80,000 to 100,000 than the normal steel alloy. It is also called as soft magnetic alloy and offers low magnetic anisotropy and magnetostriction providing low core loss to saturate the low magnetic fields. It provides nominal hysteresis losses when the alloy is employed in the AC magnetic circuits.

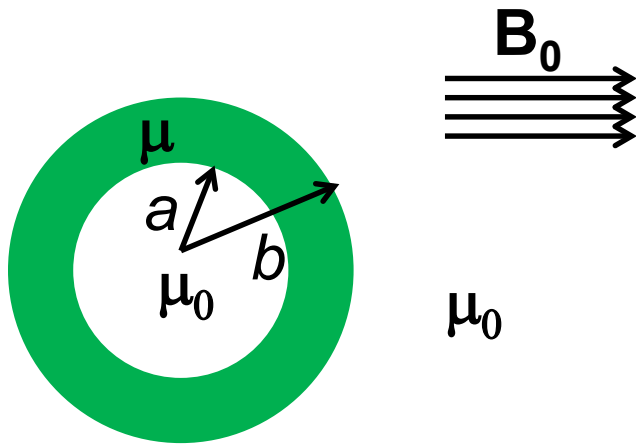
Composed of 80% Ni, 15% Fe, 5% Mo+other materials

Example: permalloy, mumetal $\mu/\mu_0 \sim 10^4$

Spherical shell $a < r < b$:



Example: permalloy, mumetal $\mu/\mu_0 \sim 10^4$ -- continued



For this case :

$$\nabla \times \mathbf{H}(\mathbf{r}) = 0$$

$$\nabla \cdot \mathbf{B}(\mathbf{r}) = 0$$

$$\mathbf{B}(\mathbf{r}) = \mu \mathbf{H}(\mathbf{r})$$

Continuity at boundaries :

$$\mathbf{H} \times \hat{\mathbf{n}} = \text{continuous}$$

$$\mathbf{B} \cdot \hat{\mathbf{n}} = \text{continuous}$$

Example: permalloy, mumetal $\mu/\mu_0 \sim 10^4$ -- continued

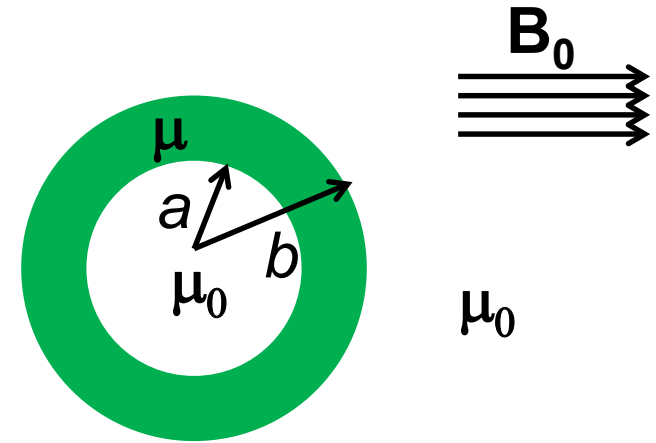
Let: $\mathbf{H}(\mathbf{r}) = -\nabla\Phi_H(\mathbf{r})$

$\nabla \cdot \mathbf{B}(\mathbf{r}) = 0 \quad \Rightarrow \quad \nabla^2\Phi_H(\mathbf{r}) = 0$

For $0 \leq r \leq a$ $\Phi_H(\mathbf{r}) = \sum_l \delta_l r^l P_l(\cos\theta)$

For $a \leq r \leq b$ $\Phi_H(\mathbf{r}) = \sum_l \left(\beta_l r^l + \frac{\gamma_l}{r^{l+1}} \right) P_l(\cos\theta)$

For $r \geq b$ $\Phi_H(\mathbf{r}) = -\frac{B_0}{\mu_0} r \cos\theta + \sum_l \frac{\alpha_l}{r^{l+1}} P_l(\cos\theta)$



Example: permalloy, mumetal $\mu/\mu_0 \sim 10^4$ -- continued

Applying boundary conditions

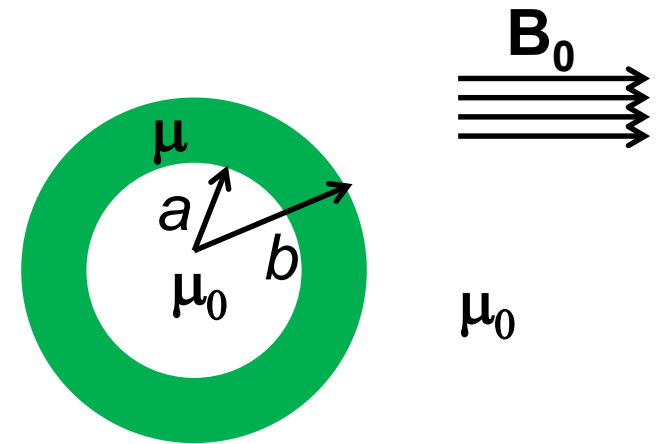
(only $l = 1$ terms contribute):

$$\text{At } r = a \quad \delta_1 = \frac{\mu}{\mu_0} \left(\beta_1 - 2 \frac{\gamma_1}{a^3} \right)$$

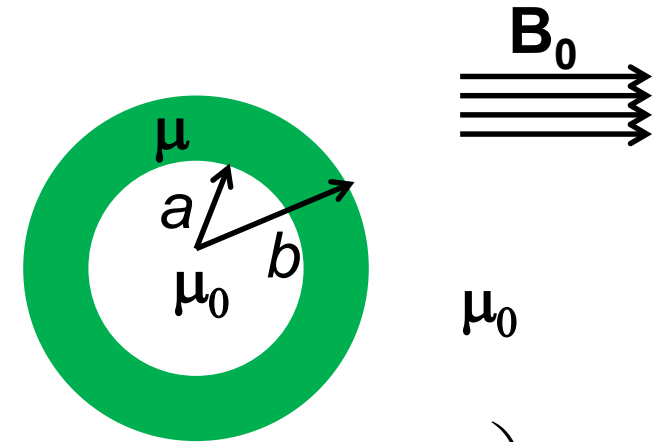
$$a\delta_1 = a\beta_1 + \frac{\gamma_1}{a^2}$$

$$\text{At } r = b \quad \frac{\mu}{\mu_0} \left(\beta_1 - 2 \frac{\gamma_1}{b^3} \right) = -\frac{B_0}{\mu_0} - 2 \frac{\alpha_1}{b^3}$$

$$b\beta_1 + \frac{\gamma_1}{b^2} = -b \frac{B_0}{\mu_0} + \frac{\alpha_1}{b^2}$$



Example: permalloy, mumetal $\mu/\mu_0 \sim 10^4$ -- continued



When the dust clears :

$$\delta_1 = \left(\frac{-9\mu / \mu_0}{(2\mu / \mu_0 + 1)(\mu / \mu_0 + 2) - 2(a/b)^3 (\mu / \mu_0 - 1)^2} \right) \frac{B_0}{\mu_0}$$

$$\approx \frac{1}{\mu / \mu_0} \left(\frac{-9/2}{\left(1 - (a/b)^3\right)} \frac{B_0}{\mu_0} \right)$$