PHY 712 Electrodynamics
10-10:50 AM MWF Online

Discussion for Lecture 15:

Start reading Chapter 6

1. **Maxwell's full equations; effects of time varying fields and sources**
2. **Gauge choices and transformations**
3. **Green's function for vector and scalar potentials**

Course schedule for Spring 2021

(Preliminary schedule -- subject to frequent adjustment.)

	Lecture date	JDJ Reading	Topic	HW	Due date
1	Wed: 01/27/2021	Chap. 1 & Appen.	Introduction, units and Poisson equation	#1	01/29/2021
2	Fri: 01/29/2021	Chap. 1	Electrostatic energy calculations	#2	02/01/2021
3	Mon: 02/01/2021	Chap. 1 & 2	Electrostatic potentials and fields	#3	02/03/2021
4	Wed: 02/03/2021	Chap. 1 - 3	Poisson's equation in 2 and 3 dimensions	#4	02/05/2021
5	Fri: 02/05/2021	Chap. 1 - 3	Brief introduction to numerical methods	#5	02/08/2021
6	Mon: 02/08/2021	Chap. 2 & 3	Image charge constructions	#6	02/10/2021
7	Wed: 02/10/2021	Chap. 2 & 3	Cylindrical and spherical geometries		
8	Fri: 02/12/2021	Chap. 3 & 4	Spherical geometry and multipole moments	#7	02/15/2021
9	Mon: 02/15/2021	Chap. 4	Dipoles and Dielectrics	#8	02/19/2021
10	Wed: 02/17/2021	Chap. 4	Dipoles and Dielectrics		
11	Fri: 02/19/2021	Chap. 4	Polarization and Dielectrics	#9	02/24/2021
12	Mon: 02/22/2021	Chap. 5	Magnetostatics	#10	02/26/2021
13	Wed: 02/24/2021	Chap. 5	Magnetic dipoles and hyperfine interaction	#11	03/01/2021
14	Fri: 02/26/2021	Chap. 5	Magnetic dipoles and dipolar fields		
15	Mon: 03/01/2021	Chap. 6	Maxwell's Equations	#12	03/08/2021
16	Wed: 03/03/2021	Chap. 6	Electromagnetic energy and forces		
17	Fri: 03/05/2021	Chap. 7	Electromagnetic plane waves		
18	Mon: 03/08/2021	Chap. 7	Electromagnetic plane waves		
19	Wed: 03/10/2021	Chap. 7	Optical effects of refractive indices		
20	Fri: 03/12/2021	Chap. 1-7	Review		
	Mon: 03/15/2021	No class	*APS March Meeting*	Take Home Exam	
	Wed: 03/17/2021	No class	*APS March Meeting*	Take Home Exam	

PHY 712 – Problem Set #12

Start reading Chaper 6 in **Jackson**

1. In deriving the Liénard Wiechert potentials, the retarded time

$$t_r = t - \frac{|\mathbf{r} - \mathbf{R}_q(t_r)|}{c}$$

was introduced. Here $\mathbf{r}, t$ represent the position and time at which the field is measured, and $\mathbf{R}_q(t')$ represents the trajectory of the charged particle source. The velocty of the particle is given by

$$\mathbf{v} \equiv \mathbf{v}(t') \equiv \frac{d\mathbf{R}_q(t')}{dt'}.$$

Demonstrate the following identities:

(a)

$$\frac{\partial t_r}{\partial t} = \frac{1}{1 - \frac{\mathbf{v}(t_r)\cdot(\mathbf{r}-\mathbf{R}_q(t_r))}{c|\mathbf{r}-\mathbf{R}_q(t_r)|}}$$

(b)

$$-c\nabla t_r = \frac{\frac{\mathbf{r}-\mathbf{R}_q(t_r)}{|\mathbf{r}-\mathbf{R}_q(t_r)|}}{1 - \frac{\mathbf{v}(t_r)\cdot(\mathbf{r}-\mathbf{R}_q(t_r))}{c|\mathbf{r}-\mathbf{R}_q(t_r)|}}$$

Your questions –

From Nick – Concerning HW 11 -- For example, I can only reduce the following and I don't understand exactly what the Div(omega). Moreover, I don't see how you did the first part of Eq 8.

$$\begin{aligned}\nabla \times (\boldsymbol{\omega} \times \mathbf{r}) &= \boldsymbol{\omega}(\nabla \cdot \mathbf{r}) - \mathbf{r}(\nabla \cdot \boldsymbol{\omega}) + (\mathbf{r} \cdot \nabla)\boldsymbol{\omega} - (\boldsymbol{\omega} \cdot \nabla)\mathbf{r} \\ &= 3\boldsymbol{\omega} - \mathbf{r}(\nabla \cdot \boldsymbol{\omega}) + (\mathbf{r} \cdot \nabla)\boldsymbol{\omega} - \boldsymbol{\omega} \\ &= 2\boldsymbol{\omega} - \mathbf{r}(\nabla \cdot \boldsymbol{\omega}) + (\mathbf{r} \cdot \nabla)\boldsymbol{\omega}\end{aligned}$$

From example problem -- (uniformly charged sphere)

Note that $\boldsymbol{\omega}$ is a constant vector representing angular velocity

Therefore the vector potential for this system is:

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 \rho_0 \boldsymbol{\omega} \times \mathbf{r}}{3r} \int_0^a dr' \, r'^3 \frac{r_<}{r_>^2}, \tag{6}$$

which can be evaluated as:

$$\mathbf{A}(\mathbf{r}) = \begin{cases} \frac{\mu_0 \rho_0}{3} \boldsymbol{\omega} \times \mathbf{r} \left(\frac{a^2}{2} - \frac{3r^2}{10} \right) & \text{for } r \leq a \\ \\ \frac{\mu_0 \rho_0}{3} \boldsymbol{\omega} \times \mathbf{r} \frac{a^5}{5r^3} & \text{for } r \geq a \end{cases} . \tag{7}$$

In order to evaluate $\mathbf{B} = \nabla \times \mathbf{A}$

$$\nabla \times \left(\boldsymbol{\omega} \times \mathbf{r} \, f(r) \right) = \boldsymbol{\omega} \left(\nabla \cdot \left(\mathbf{r} \, f(r) \right) \right) - \left(\boldsymbol{\omega} \cdot \nabla \right) \left(\mathbf{r} \, f(r) \right)$$

When the dust clears --

$$\mathbf{B}(\mathbf{r}) = \nabla \times \mathbf{A}(\mathbf{r}) = \begin{cases} \frac{\mu_0\rho_0}{3}\left[\boldsymbol{\omega}\left(a^2 - \frac{6}{5}r^2\right) + \frac{3}{5}\mathbf{r}(\boldsymbol{\omega}\cdot\mathbf{r})\right] & \text{for } r \leq a \\ \frac{\mu_0\rho_0}{3}\left[-\boldsymbol{\omega}\frac{a^5}{5r^3} + \frac{3a^5}{5r^5}\mathbf{r}(\boldsymbol{\omega}\cdot\mathbf{r})\right] & \text{for } r \geq a \end{cases} . \qquad (8)$$

Full electrodynamics with time varying fields and sources

Maxwell's equations

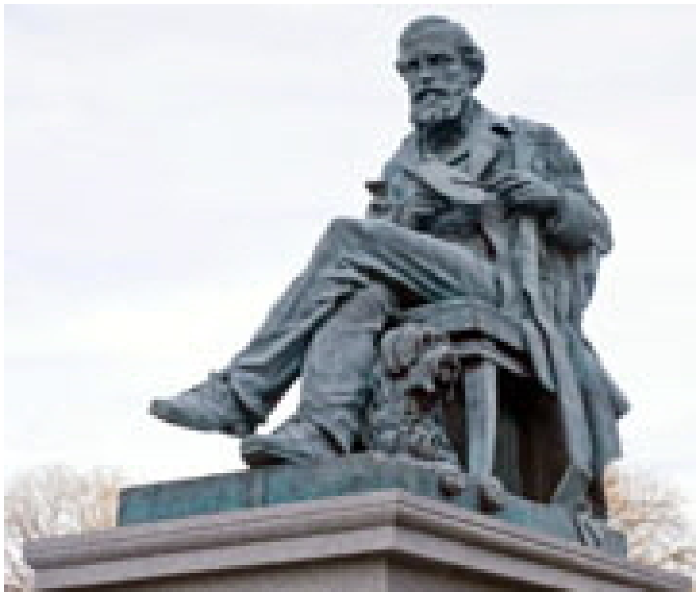

Image of statue of James Clerk-Maxwell (1831-1879) in Edinburgh

"From a long view of the history of mankind - seen from, say, ten thousand years from now - there can be little doubt that the most significant event of the 19th century will be judged as Maxwell's discovery of the laws of electrodynamics"

Richard P Feynman

http://www.clerkmaxwellfoundation.org/

Maxwell's equations

Coulomb's law: $\nabla \cdot \mathbf{D} = \rho_{free}$

Ampere-Maxwell's law: $\nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{J}_{free}$

Faraday's law: $\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$

No magnetic monopoles: $\nabla \cdot \mathbf{B} = 0$

Maxwell's equations

Microscopic or vacuum form $(\mathbf{P}=0;\ \mathbf{M}=0)$:

Coulomb's law: $\nabla\cdot\mathbf{E}=\rho/\varepsilon_0$

Ampere-Maxwell's law: $\nabla\times\mathbf{B}-\frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t}=\mu_0\mathbf{J}$

Faraday's law: $\nabla\times\mathbf{E}+\frac{\partial\mathbf{B}}{\partial t}=0$

No magnetic monopoles: $\nabla\cdot\mathbf{B}=0$

$$\Rightarrow c^2=\frac{1}{\varepsilon_0\mu_0}$$

Formulation of Maxwell's equations in terms of vector and scalar potentials

$$\nabla \cdot \mathbf{B} = 0 \qquad \Rightarrow \mathbf{B} = \nabla \times \mathbf{A}$$

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \quad \Rightarrow \nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} \right) = 0$$

$$\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\nabla \Phi$$

$$\text{or} \quad \mathbf{E} = -\nabla \Phi - \frac{\partial \mathbf{A}}{\partial t}$$

Formulation of Maxwell's equations in terms of vector and scalar potentials -- continued

$$\nabla \cdot \mathbf{E} = \rho / \varepsilon_0 :$$

$$-\nabla^2 \Phi - \frac{\partial (\nabla \cdot \mathbf{A})}{\partial t} = \rho / \varepsilon_0$$

$$\nabla \times \mathbf{B} - \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{J}$$

$$\nabla \times (\nabla \times \mathbf{A}) + \frac{1}{c^2}\left(\frac{\partial (\nabla \Phi)}{\partial t} + \frac{\partial^2 \mathbf{A}}{\partial t^2} \right) = \mu_0 \mathbf{J}$$

Formulation of Maxwell's equations in terms of vector and scalar potentials -- continued

General form for the scalar and vector potential equations:

$$-\nabla^2\Phi - \frac{\partial(\nabla\cdot\mathbf{A})}{\partial t} = \rho/\varepsilon_0$$

$$\nabla\times(\nabla\times\mathbf{A}) + \frac{1}{c^2}\left(\frac{\partial(\nabla\Phi)}{\partial t} + \frac{\partial^2\mathbf{A}}{\partial t^2}\right) = \mu_0\mathbf{J}$$

Coulomb gauge form -- require $\nabla\cdot\mathbf{A}_C = 0$

$$-\nabla^2\Phi_C = \rho/\varepsilon_0$$

$$-\nabla^2\mathbf{A}_C + \frac{1}{c^2}\frac{\partial^2\mathbf{A}_C}{\partial t^2} + \frac{1}{c^2}\frac{\partial(\nabla\Phi_C)}{\partial t} = \mu_0\mathbf{J}$$

Note that $\mathbf{J} = \mathbf{J}_l + \mathbf{J}_t$ with $\nabla\times\mathbf{J}_l = 0$ and $\nabla\cdot\mathbf{J}_t = 0$

Formulation of Maxwell's equations in terms of vector and scalar potentials -- continued

Coulomb gauge form -- require $\nabla \cdot \mathbf{A}_C = 0$

$$-\nabla^2 \Phi_C = \rho / \varepsilon_0$$

$$-\nabla^2 \mathbf{A}_C + \frac{1}{c^2}\frac{\partial^2 \mathbf{A}_C}{\partial t^2} + \frac{1}{c^2}\frac{\partial(\nabla\Phi_C)}{\partial t} = \mu_0 \mathbf{J}$$

Note that $\mathbf{J} = \mathbf{J}_l + \mathbf{J}_t$ with $\nabla \times \mathbf{J}_l = 0$ and $\nabla \cdot \mathbf{J}_t = 0$

Continuity equation for charge and current density:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J}_l = 0 \qquad \Rightarrow \frac{\partial \rho}{\partial t} = -\nabla \cdot \mathbf{J}_l = -\varepsilon_0 \nabla \cdot \frac{\partial(\nabla\Phi_C)}{\partial t}$$

$$\Rightarrow \frac{1}{c^2}\frac{\partial(\nabla\Phi_C)}{\partial t} = \varepsilon_0\mu_0 \frac{\partial(\nabla\Phi_C)}{\partial t} = \mu_0 \mathbf{J}_l$$

$$-\nabla^2 \mathbf{A}_C + \frac{1}{c^2}\frac{\partial^2 \mathbf{A}_C}{\partial t^2} = \mu_0 \mathbf{J}_t$$

Formulation of Maxwell's equations in terms of vector and scalar potentials -- continued

Review of the general equations:

$$-\nabla^2\Phi - \frac{\partial\left(\nabla\cdot\mathbf{A}\right)}{\partial t} = \rho/\varepsilon_0$$

$$\nabla\times\left(\nabla\times\mathbf{A}\right) + \frac{1}{c^2}\left(\frac{\partial\left(\nabla\Phi\right)}{\partial t} + \frac{\partial^2\mathbf{A}}{\partial t^2}\right) = \mu_0\mathbf{J}$$

Lorentz gauge form -- require $\nabla\cdot\mathbf{A}_L + \frac{1}{c^2}\frac{\partial\Phi_L}{\partial t} = 0$

$$-\nabla^2\Phi_L + \frac{1}{c^2}\frac{\partial^2\Phi_L}{\partial t^2} = \rho/\varepsilon_0$$

$$-\nabla^2\mathbf{A}_L + \frac{1}{c^2}\frac{\partial^2\mathbf{A}_L}{\partial t^2} = \mu_0\mathbf{J}$$

Formulation of Maxwell's equations in terms of vector and scalar potentials -- continued

Lorentz gauge form -- require $\nabla \cdot \mathbf{A}_L + \frac{1}{c^2}\frac{\partial \Phi_L}{\partial t} = 0$

$$-\nabla^2 \Phi_L + \frac{1}{c^2}\frac{\partial^2 \Phi_L}{\partial t^2} = \rho / \varepsilon_0$$

$$-\nabla^2 \mathbf{A}_L + \frac{1}{c^2}\frac{\partial^2 \mathbf{A}_L}{\partial t^2} = \mu_0 \mathbf{J}$$

Alternate potentials: $\mathbf{A}'_L = \mathbf{A}_L + \nabla\Lambda$ and $\Phi'_L = \Phi_L - \frac{\partial \Lambda}{\partial t}$

Yields same physics provided that: $\nabla^2 \Lambda - \frac{1}{c^2}\frac{\partial^2 \Lambda}{\partial t^2} = 0$

Solution of Maxwell's equations in the Lorentz gauge

$$\nabla^2 \Phi_L - \frac{1}{c^2}\frac{\partial^2 \Phi_L}{\partial t^2} = -\rho / \varepsilon_0$$

$$\nabla^2 \mathbf{A}_L - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}_L}{\partial t^2} = -\mu_0 \mathbf{J}$$

Consider the general form of the 3 - dimensional wave equation :

$$\nabla^2 \Psi - \frac{1}{c^2}\frac{\partial^2 \Psi}{\partial t^2} = -4\pi f$$

$\Psi(\mathbf{r}, t) \Rightarrow$ wave field $\quad f(\mathbf{r}, t) \Rightarrow$ source

Solution of Maxwell's equations in the Lorentz gauge -- continued

Let Ψ represent Φ, A_x, A_y, A_z Let f represent ρ, J_x, J_y, J_z

$$\nabla^2\Psi(\mathbf{r},t) - \frac{1}{c^2}\frac{\partial^2\Psi(\mathbf{r},t)}{\partial t^2} = -4\pi f(\mathbf{r},t)$$

Green's function :

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)G(\mathbf{r},t;\mathbf{r}',t') = -4\pi\delta^3(\mathbf{r}-\mathbf{r}')\delta(t-t')$$

Formal solution for field $\Psi(\mathbf{r},t)$:

$$\Psi(\mathbf{r},t) = \Psi_{f=0}(\mathbf{r},t) + \int d^3r' \int dt' G(\mathbf{r},t;\mathbf{r}',t') f(\mathbf{r}',t')$$

Solution of Maxwell's equations in the Lorentz gauge -- continued

Determination of the form for the Green's function:

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)G(\mathbf{r},t;\mathbf{r}',t') = -4\pi\delta^3(\mathbf{r}-\mathbf{r}')\delta(t-t')$$

For the case of isotropic boundary values at infinity:

$$G(\mathbf{r},t;\mathbf{r}',t') = \frac{1}{|\mathbf{r}-\mathbf{r}'|}\delta\left(t' - \left(t - \frac{1}{c}|\mathbf{r}-\mathbf{r}'|\right)\right)$$

Formal solution for field $\Psi(\mathbf{r},t)$:

$$\Psi(\mathbf{r},t) = \Psi_{f=0}(\mathbf{r},t) +$$

$$\int d^3r' \int dt' \frac{1}{|\mathbf{r}-\mathbf{r}'|}\delta\left(t' - \left(t - \frac{1}{c}|\mathbf{r}-\mathbf{r}'|\right)\right) f(\mathbf{r}',t')$$

Solution of Maxwell's equations in the Lorentz gauge -- continued

Analysis of the Green's function:

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) G(\mathbf{r},t;\mathbf{r}',t') = -4\pi\delta^3(\mathbf{r}-\mathbf{r}')\delta(t-t')$$

"Proof" -- Fourier analysis in the time domain -- note that

$$\delta(t-t') = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{i\omega(t-t')}$$

Define:

$$G(\mathbf{r},t;\mathbf{r}',t') = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{i\omega(t-t')}\; \tilde{G}(\mathbf{r},\mathbf{r}',\omega)$$

$$\Rightarrow \quad \left(\nabla^2 + \frac{\omega^2}{c^2}\right)\tilde{G}(\mathbf{r},\mathbf{r}',\omega) = -4\pi\delta^3(\mathbf{r}-\mathbf{r}')$$

Solution of Maxwell's equations in the Lorentz gauge -- continued

Analysis of the Green's function (continued):

$$\left(\nabla^2 + \frac{\omega^2}{c^2}\right)\tilde{G}(\mathbf{r},\mathbf{r}',\omega) = -4\pi\delta^3(\mathbf{r}-\mathbf{r}')$$

For the case of isotropic boundary values at infinity:

$$\tilde{G}(\mathbf{r},\mathbf{r}',\omega) = \tilde{G}(\mathbf{r}-\mathbf{r}',\omega)$$

Further assuming that $\tilde{G}(\mathbf{r}-\mathbf{r}',\omega)$ is isotropic in $|\mathbf{r}-\mathbf{r}'| \equiv R$:

$$\left(\frac{1}{R}\frac{d^2}{dR^2}R + \frac{\omega^2}{c^2}\right)\tilde{G}(\mathbf{r},\mathbf{r}',\omega) = -4\pi\delta^3(\mathbf{r}-\mathbf{r}')$$

Solution: $\tilde{G}(\mathbf{r},\mathbf{r}',\omega) = \frac{1}{R}e^{\pm i\omega R/c}$

Solution of Maxwell's equations in the Lorentz gauge -- continued

Analysis of the Green's function (continued):

$$\tilde{G}(\mathbf{r},\mathbf{r}',\omega)=\frac{1}{|\mathbf{r}-\mathbf{r}'|}e^{\pm i\omega|\mathbf{r}-\mathbf{r}'|/c}$$

$$G(\mathbf{r},t;\mathbf{r}',t')=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\omega\, e^{i\omega(t-t')}\,\tilde{G}(\mathbf{r},\mathbf{r}',\omega)$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\omega\, e^{i\omega(t-t')}\frac{1}{|\mathbf{r}-\mathbf{r}'|}e^{\pm i\omega|\mathbf{r}-\mathbf{r}'|/c}$$

$$=\frac{1}{|\mathbf{r}-\mathbf{r}'|}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}d\omega\, e^{i\omega\left(t-t'\pm|\mathbf{r}-\mathbf{r}'|/c\right)}\right)$$

$$=\frac{1}{|\mathbf{r}-\mathbf{r}'|}\delta\left(t-t'\pm|\mathbf{r}-\mathbf{r}'|/c\right)=\frac{1}{|\mathbf{r}-\mathbf{r}'|}\delta\left(t'-t\mp|\mathbf{r}-\mathbf{r}'|/c\right)$$

Solution of Maxwell's equations in the Lorentz gauge -- continued

$$G(\mathbf{r},t;\mathbf{r}',t') = \frac{1}{|\mathbf{r}-\mathbf{r}'|}\,\delta\left(t' - \left(t \pm |\mathbf{r}-\mathbf{r}'|/c\right)\right)$$

Solution for field $\Psi(\mathbf{r},t)$:

$$\Psi(\mathbf{r},t) = \Psi_{f=0}(\mathbf{r},t) + \int d^3r' \int dt' \frac{1}{|\mathbf{r}-\mathbf{r}'|}\,\delta\left(t' - \left(t - \frac{1}{c}|\mathbf{r}-\mathbf{r}'|\right)\right) f(\mathbf{r}',t')$$

Solution of Maxwell's equations in the Lorentz gauge -- continued

Liènard-Wiechert potentials and fields --
Determination of the scalar and vector potentials for a moving point particle (also see Landau and Lifshitz ***The Classical Theory of Fields***, Chapter 8.)

Consider the fields produced by the following source: a point charge *q* moving on a trajectory $R_q(t)$.

$$\text{Charge density: } \rho(\mathbf{r},t) = q\delta^3(\mathbf{r}-\mathbf{R}_q(t))$$

$$\text{Current density: } \mathbf{J}(\mathbf{r},t) = q\,\dot{\mathbf{R}}_q(t)\delta^3(\mathbf{r}-\mathbf{R}_q(t)), \quad \text{where } \dot{\mathbf{R}}_q(t) \equiv \frac{d\mathbf{R}_q(t)}{dt}.$$

Solution of Maxwell's equations in the Lorentz gauge -- continued

$$\Phi(\mathbf{r},t)=\frac{1}{4\pi\epsilon_0}\int\int d^3r'dt'\frac{\rho(\mathbf{r}',t')}{|\mathbf{r}-\mathbf{r}']|}\delta\left(t'-(t-|\mathbf{r}-\mathbf{r}'|/c)\right)$$

$$\mathbf{A}(\mathbf{r},t)=\frac{1}{4\pi\epsilon_0 c^2}\int\int d^3r'dt'\frac{\mathbf{J}(\mathbf{r}',\mathrm{t}')}{|\mathbf{r}-\mathbf{r}'|}\delta\left(t'-(t-|\mathbf{r}-\mathbf{r}'|/c)\right).$$

We performing the integrations over first d^3r' and then dt' making use of the fact that for any function of t',

$$\int_{-\infty}^{\infty} dt'\, f(t')\delta\left(t'-(t-|\mathbf{r}-\mathbf{R}_q(t')|/c)\right)=\frac{f(t_r)}{1-\frac{\dot{\mathbf{R}}_q(t_r)\cdot(\mathbf{r}-\mathbf{R}_q(t_r))}{c|\mathbf{r}-\mathbf{R}_q(t_r)|}},$$

where the ``retarded time'' is defined to be

$$t_r\equiv t-\frac{|\mathbf{r}-\ \mathbf{R}_q(t_r)|}{c}.$$

Solution of Maxwell's equations in the Lorentz gauge -- continued

Resulting scalar and vector potentials:

$$\Phi(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0} \frac{1}{R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}},$$

$$\mathbf{A}(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0 c^2} \frac{\mathbf{v}}{R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}},$$

Notation: $\mathbf{R} \equiv \mathbf{r} - \mathbf{R}_q(t_r)$

$$\mathbf{v} \equiv \dot{\mathbf{R}}_q(t_r),$$

$$t_r \equiv t - \frac{|\mathbf{r} - \mathbf{R}_q(t_r)|}{c}.$$

Comment on Lienard-Wiechert potential results

$$\int_{-\infty}^{\infty} dt' \; f(t')\delta\left(t' - (t - |\mathbf{r} - \mathbf{R}_q(t')|/c)\right) = \frac{f(t_r)}{1 - \frac{\dot{\mathbf{R}}_q(t_r)\cdot(\mathbf{r}-\mathbf{R}_q(t_r))}{c|\mathbf{r}-\mathbf{R}_q(t_r)|}},$$

where the "retarded time" is defined to be

$$t_r \equiv t - \frac{|\mathbf{r} - \mathbf{R}_q(t_r)|}{c}.$$

Note that for any function *F(x)*:

$$\int_{-\infty}^{\infty} F(x)\delta(x - x_0)dx = F(x_0)$$

Now consider a function *p(x)*, for which $p(x_i) = 0$ for $i = 1, 2, \cdots$

$$\int_{-\infty}^{\infty} F(x)\delta(p(x))dx = \int_{-\infty}^{\infty} F(x)\left(\sum_i \delta\left((x - x_i)\left.\frac{dp}{dx}\right|_{x_i}\right)\right)dx$$

$$= \sum_i \frac{F(x_i)}{\left|\frac{dp}{dx}\right|_{x_i}}$$

Comment on Lienard-Wiechert potential results -- continued

$$\int_{-\infty}^{\infty} dt' \; f(t')\delta\left(t' - (t - |\mathbf{r} - \mathbf{R}_q(t')|/c)\right) = \frac{f(t_r)}{1 - \frac{\dot{\mathbf{R}}_q(t_r)\cdot(\mathbf{r}-\mathbf{R}_q(t_r))}{c|\mathbf{r}-\mathbf{R}_q(t_r)|}},$$

where the "retarded time" is defined to be

$$t_r \equiv t - \frac{|\mathbf{r} - \mathbf{R}_q(t_r)|}{c}.$$

In this case we have: $\displaystyle\int_{-\infty}^{\infty} f(t')\delta\left(p(t')\right)dt' = \frac{f(t_r)}{\left|1 - \frac{\dot{\mathbf{R}}_q(t_r)\cdot\left(\mathbf{r}-\mathbf{R}_q(t')\right)}{c\left|\mathbf{r}-\mathbf{R}_q(t_r)\right|}\right|}$

where: $\displaystyle p(t') \equiv t' - \left(t - \frac{\left|\mathbf{r}-\mathbf{R}_q(t')\right|}{c}\right)$

$$\frac{dp(t')}{dt'} = 1 - \frac{\frac{d\mathbf{R}_q(t')}{dt'}\cdot\left(\mathbf{r}-\mathbf{R}_q(t')\right)}{c\left|\mathbf{r}-\mathbf{R}_q(t')\right|} \equiv 1 - \frac{\dot{\mathbf{R}}_q(t')\cdot\left(\mathbf{r}-\mathbf{R}_q(t')\right)}{c\left|\mathbf{r}-\mathbf{R}_q(t')\right|}$$

Summary of results for fields due to moving charge – Liénard Wiechert potentials

Resulting scalar and vector potentials:

$$\Phi(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0}\frac{1}{R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}},$$

$$\mathbf{A}(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0 c^2}\frac{\mathbf{v}}{R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}},$$

Notation: $\mathbf{R} \equiv \mathbf{r} - \mathbf{R}_q(t_r)$

$$\mathbf{v} \equiv \dot{\mathbf{R}}_q(t_r),$$

$$t_r \equiv t - \frac{|\mathbf{r} - \mathbf{R}_q(t_r)|}{c}.$$