

PHY 712 Electrodynamics

10-10:50 AM MWF Online

Plan for Lecture 16:

Finish reading Chapter 6

- 1. Some details of Liénard-Wiechert results**
- 2. Energy density and flux associated with electromagnetic fields**
- 3. Time harmonic fields**

Online Colloquium: “Artificial Intelligence Captures Language of Life Written in Proteins” — March 4, 2021 at 4 PM

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Thursday, March 4, 2021, 4 PM EST

Via Video Conference (contact wfuphys@wfu.edu for link information)

All interested persons are cordially invited to join the Zoom call.

Course schedule for Spring 2021

(Preliminary schedule -- subject to frequent adjustment.)

	Lecture date	JDJ Reading	Topic	HW	Due date
1	Wed: 01/27/2021	Chap. 1 & Appen.	Introduction, units and Poisson equation	#1	01/29/2021
2	Fri: 01/29/2021	Chap. 1	Electrostatic energy calculations	#2	02/01/2021
3	Mon: 02/01/2021	Chap. 1 & 2	Electrostatic potentials and fields	#3	02/03/2021
4	Wed: 02/03/2021	Chap. 1 - 3	Poisson's equation in 2 and 3 dimensions	#4	02/05/2021
5	Fri: 02/05/2021	Chap. 1 - 3	Brief introduction to numerical methods	#5	02/08/2021
6	Mon: 02/08/2021	Chap. 2 & 3	Image charge constructions	#6	02/10/2021
7	Wed: 02/10/2021	Chap. 2 & 3	Cylindrical and spherical geometries		
8	Fri: 02/12/2021	Chap. 3 & 4	Spherical geometry and multipole moments	#7	02/15/2021
9	Mon: 02/15/2021	Chap. 4	Dipoles and Dielectrics	#8	02/19/2021
10	Wed: 02/17/2021	Chap. 4	Dipoles and Dielectrics		
11	Fri: 02/19/2021	Chap. 4	Polarization and Dielectrics	#9	02/24/2021
12	Mon: 02/22/2021	Chap. 5	Magnetostatics	#10	02/26/2021
13	Wed: 02/24/2021	Chap. 5	Magnetic dipoles and hyperfine interaction	#11	03/01/2021
14	Fri: 02/26/2021	Chap. 5	Magnetic dipoles and dipolar fields		
15	Mon: 03/01/2021	Chap. 6	Maxwell's Equations	#12	03/08/2021
16	Wed: 03/03/2021	Chap. 6	Electromagnetic energy and forces		
17	Fri: 03/05/2021	Chap. 7	Electromagnetic plane waves		
18	Mon: 03/08/2021	Chap. 7	Electromagnetic plane waves		
19	Wed: 03/10/2021	Chap. 7	Optical effects of refractive indices		
20	Fri: 03/12/2021	Chap. 1-7	Review		
	Mon: 03/15/2021 03/03/2021	No class	<i>APS March Meeting</i> PHY 712 Spring 2021 -- Lecture 16	Take Home Exam	3
				Take	

Solution of Maxwell's equations in the Lorentz gauge -- continued

Liénard-Wiechert potentials and fields --

Determination of the scalar and vector potentials for a moving point particle (also see Landau and Lifshitz ***The Classical Theory of Fields***, Chapter 8.)

Consider the fields produced by the following source: a point charge q moving on a trajectory $\mathbf{R}_q(t)$.

Charge density: $\rho(\mathbf{r}, t) = q\delta^3(\mathbf{r} - \mathbf{R}_q(t))$

Current density: $\mathbf{J}(\mathbf{r}, t) = q\dot{\mathbf{R}}_q(t)\delta^3(\mathbf{r} - \mathbf{R}_q(t))$, where $\dot{\mathbf{R}}_q(t) \equiv \frac{d\mathbf{R}_q(t)}{dt}$.



Solution of Maxwell's equations in the Lorentz gauge -- continued

$$\Phi(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \iint d^3r' dt' \frac{\rho(\mathbf{r}', t')}{|\mathbf{r} - \mathbf{r}'|} \delta(t' - (t - |\mathbf{r} - \mathbf{r}'|/c))$$

$$\mathbf{A}(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0 c^2} \iint d^3r' dt' \frac{\mathbf{J}(\mathbf{r}', t')}{|\mathbf{r} - \mathbf{r}'|} \delta(t' - (t - |\mathbf{r} - \mathbf{r}'|/c)).$$

We performing the integrations over first d^3r' and then dt' making use of the fact that for any function of t' ,

$$\int_{-\infty}^{\infty} dt' f(t') \delta(t' - (t - |\mathbf{r} - \mathbf{R}_q(t')|/c)) = \frac{f(t_r)}{1 - \frac{\dot{\mathbf{R}}_q(t_r) \cdot (\mathbf{r} - \mathbf{R}_q(t_r))}{c |\mathbf{r} - \mathbf{R}_q(t_r)|}},$$

where the "retarded time" is defined to be

$$t_r \equiv t - \frac{|\mathbf{r} - \mathbf{R}_q(t_r)|}{c}.$$

Solution of Maxwell's equations in the Lorentz gauge -- continued

Resulting scalar and vector potentials:

$$\Phi(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0} \frac{1}{R - \frac{\mathbf{v} \cdot \mathbf{R}}{c}},$$

$$\mathbf{A}(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0 c^2} \frac{\mathbf{v}}{R - \frac{\mathbf{v} \cdot \mathbf{R}}{c}},$$

Notation: $\mathbf{R} \equiv \mathbf{r} - \mathbf{R}_q(t_r)$ $t_r \equiv t - \frac{|\mathbf{r} - \mathbf{R}_q(t_r)|}{c}.$

$$\mathbf{v} \equiv \dot{\mathbf{R}}_q(t_r),$$

Solution of Maxwell's equations in the Lorentz gauge -- continued

In order to find the electric and magnetic fields, we need to evaluate

$$\mathbf{E}(\mathbf{r}, t) = -\nabla\Phi(\mathbf{r}, t) - \frac{\partial\mathbf{A}(\mathbf{r}, t)}{\partial t}$$

$$\mathbf{B}(\mathbf{r}, t) = \nabla \times \mathbf{A}(\mathbf{r}, t)$$

The trick of evaluating these derivatives is that the retarded time t_r depends on position $\mathbf{r}$ and on itself. We can show the following results using the shorthand notation:

$$\nabla t_r = -\frac{\mathbf{R}}{c\left(R - \frac{\mathbf{v} \cdot \mathbf{R}}{c}\right)} \quad \text{and} \quad \frac{\partial t_r}{\partial t} = \frac{R}{\left(R - \frac{\mathbf{v} \cdot \mathbf{R}}{c}\right)}.$$

Solution of Maxwell's equations in the Lorentz gauge -- continued

$$-\nabla\Phi(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0} \frac{1}{\left(R - \frac{\mathbf{v} \cdot \mathbf{R}}{c}\right)^3} \left[\mathbf{R} \left(1 - \frac{v^2}{c^2}\right) - \frac{v}{c} \left(R - \frac{\mathbf{v} \cdot \mathbf{R}}{c}\right) + \mathbf{R} \frac{\dot{\mathbf{v}} \cdot \mathbf{R}}{c^2} \right],$$

$$-\frac{\partial \mathbf{A}(\mathbf{r},t)}{\partial t} = \frac{q}{4\pi\epsilon_0} \frac{1}{\left(R - \frac{\mathbf{v} \cdot \mathbf{R}}{c}\right)^3} \left[\frac{\mathbf{v}R}{c} \left(\frac{v^2}{c^2} - \frac{\mathbf{v} \cdot \mathbf{R}}{Rc} - \frac{\dot{\mathbf{v}} \cdot \mathbf{R}}{c^2}\right) - \frac{\dot{\mathbf{v}}R}{c^2} \left(R - \frac{\mathbf{v} \cdot \mathbf{R}}{c}\right) \right].$$

$$\mathbf{E}(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0} \frac{1}{\left(R - \frac{\mathbf{v} \cdot \mathbf{R}}{c}\right)^3} \left[\left(\mathbf{R} - \frac{\mathbf{v}R}{c} \right) \left(1 - \frac{v^2}{c^2}\right) + \left(\mathbf{R} \times \left\{ \left(\mathbf{R} - \frac{\mathbf{v}R}{c} \right) \times \frac{\dot{\mathbf{v}}}{c^2} \right\} \right) \right].$$

$$\mathbf{B}(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0 c^2} \left[\frac{-\mathbf{R} \times \mathbf{v}}{\left(R - \frac{\mathbf{v} \cdot \mathbf{R}}{c}\right)^3} \left(1 - \frac{v^2}{c^2} + \frac{\dot{\mathbf{v}} \cdot \mathbf{R}}{c^2}\right) - \frac{\mathbf{R} \times \dot{\mathbf{v}} / c}{\left(R - \frac{\mathbf{v} \cdot \mathbf{R}}{c}\right)^2} \right] = \frac{\mathbf{R} \times \mathbf{E}(\mathbf{r},t)}{cR}$$

Maxwell's equations

Coulomb's law :

$$\nabla \cdot \mathbf{D} = \rho_{free}$$

Ampere - Maxwell's law :

$$\nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{J}_{free}$$

Faraday's law :

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$$

No magnetic monopoles : $\nabla \cdot \mathbf{B} = 0$

Energy analysis of electromagnetic fields and sources
Rate of work done on source $\mathbf{J}(\mathbf{r}, t)$ by electromagnetic field:

$$\frac{dW_{mech}}{dt} \equiv \frac{dE_{mech}}{dt} = \int d^3r \quad \mathbf{E} \cdot \mathbf{J}_{free}$$

Expressing source current in terms of fields it produces:

$$\frac{dW_{mech}}{dt} = \int d^3r \quad \mathbf{E} \cdot \left(\nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} \right)$$

Energy analysis of electromagnetic fields and sources - continued

$$\begin{aligned}\frac{dW_{mech}}{dt} &= \int d^3r \quad \mathbf{E} \cdot \mathbf{J}_{free} = \int d^3r \quad \mathbf{E} \cdot \left(\nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} \right) \\ &= - \int d^3r \quad \left(\nabla \cdot (\mathbf{E} \times \mathbf{H}) + \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} + \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} \right)\end{aligned}$$

Let $\mathbf{S} \equiv \mathbf{E} \times \mathbf{H}$ "Poynting vector"

$$u \equiv \frac{1}{2} (\mathbf{E} \cdot \mathbf{D} + \mathbf{H} \cdot \mathbf{B}) \quad \text{energy density}$$

$$\Rightarrow \frac{\partial u}{\partial t} + \nabla \cdot \mathbf{S} = -\mathbf{E} \cdot \mathbf{J}_{free}$$

Assuming that $\mathbf{D} = \epsilon \mathbf{E}$
and that $\mathbf{B} = \mu \mathbf{H}$

Energy analysis of electromagnetic fields and sources - continued

$$\frac{dE_{mech}}{dt} \equiv \int d^3r \quad \mathbf{E} \cdot \mathbf{J}_{free}$$

Electromagnetic energy density: $u \equiv \frac{1}{2}(\mathbf{E} \cdot \mathbf{D} + \mathbf{H} \cdot \mathbf{B})$

$$E_{field} \equiv \int d^3r \quad u(\mathbf{r}, t)$$

Poynting vector: $\mathbf{S} \equiv \mathbf{E} \times \mathbf{H}$

From the previous energy analysis: $\frac{\partial u}{\partial t} + \nabla \cdot \mathbf{S} = -\mathbf{E} \cdot \mathbf{J}_{free}$

$$\Rightarrow \frac{dE_{mech}}{dt} + \frac{dE_{field}}{dt} = -\int d^3r \quad \nabla \cdot \mathbf{S}(\mathbf{r}, t) = -\oint d^2r \quad \hat{\mathbf{r}} \cdot \mathbf{S}(\mathbf{r}, t)$$

Momentum analysis of electromagnetic fields and sources

$$\frac{d\mathbf{P}_{mech}}{dt} \equiv \int d^3r \left(\rho \mathbf{E} + \mathbf{J} \times \mathbf{B} \right)$$

Follows by analogy with Lorentz force:

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

$$\mathbf{P}_{field} = \varepsilon_0 \int d^3r \left(\mathbf{E} \times \mathbf{B} \right)$$

Expression for vacuum fields:

$$\left(\frac{d\mathbf{P}_{mech}}{dt} + \frac{d\mathbf{P}_{field}}{dt} \right)_i = \sum_j \int d^3r \frac{\partial T_{ij}}{\partial r_j}$$

Maxwell stress tensor:

$$T_{ij} \equiv \varepsilon_0 \left(E_i E_j + c^2 B_i B_j - \delta_{ij} \frac{1}{2} (\mathbf{E} \cdot \mathbf{E} + c^2 \mathbf{B} \cdot \mathbf{B}) \right)$$

Comment on treatment of time-harmonic fields
Fourier transformation in time domain :

$$\mathbf{E}(\mathbf{r}, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \tilde{\mathbf{E}}(\mathbf{r}, \omega) e^{-i\omega t}$$

$$\tilde{\mathbf{E}}(\mathbf{r}, \omega) = \int_{-\infty}^{\infty} dt \mathbf{E}(\mathbf{r}, t) e^{i\omega t}$$

Note that $\mathbf{E}(\mathbf{r}, t)$ is real $\Rightarrow \tilde{\mathbf{E}}(\mathbf{r}, \omega) = \tilde{\mathbf{E}}^*(\mathbf{r}, -\omega)$

These relations and the notion of the superposition principle,
lead to the common treatment:

$$\mathbf{E}(\mathbf{r}, t) = \Re \left(\tilde{\mathbf{E}}(\mathbf{r}, \omega) e^{-i\omega t} \right) \equiv \frac{1}{2} \left(\tilde{\mathbf{E}}(\mathbf{r}, \omega) e^{-i\omega t} + \tilde{\mathbf{E}}^*(\mathbf{r}, \omega) e^{i\omega t} \right)$$

Comment on treatment of time-harmonic fields -- continued

Equations for time harmonic fields :

$$\mathbf{E}(\mathbf{r}, t) = \Re \left(\tilde{\mathbf{E}}(\mathbf{r}, \omega) e^{-i\omega t} \right) \equiv \frac{1}{2} \left(\tilde{\mathbf{E}}(\mathbf{r}, \omega) e^{-i\omega t} + \tilde{\mathbf{E}}^*(\mathbf{r}, \omega) e^{i\omega t} \right)$$

Equations	in time domain	in frequency domain
Coulomb's law :	$\nabla \cdot \mathbf{D} = \rho_{free}$	$\nabla \cdot \tilde{\mathbf{D}} = \tilde{\rho}_{free}$
Ampere - Maxwell's law :	$\nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{J}_{free}$	$\nabla \times \tilde{\mathbf{H}} + i\omega \tilde{\mathbf{D}} = \tilde{\mathbf{J}}_{free}$
Faraday's law :	$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$	$\nabla \times \tilde{\mathbf{E}} - i\omega \tilde{\mathbf{B}} = 0$
No magnetic monopoles :	$\nabla \cdot \mathbf{B} = 0$	$\nabla \cdot \tilde{\mathbf{B}} = 0$

Note -- in all of these, the real part is taken at the end of the calculation.

Comment on treatment of time-harmonic fields -- continued

Equations for time harmonic fields :

$$\mathbf{E}(\mathbf{r}, t) = \Re \left(\tilde{\mathbf{E}}(\mathbf{r}, \omega) e^{-i\omega t} \right) \equiv \frac{1}{2} \left(\tilde{\mathbf{E}}(\mathbf{r}, \omega) e^{-i\omega t} + \tilde{\mathbf{E}}^*(\mathbf{r}, \omega) e^{i\omega t} \right)$$

Poynting vector: $\mathbf{S}(\mathbf{r}, t) = \mathbf{E}(\mathbf{r}, t) \times \mathbf{H}(\mathbf{r}, t)$

$$\begin{aligned} \mathbf{S}(\mathbf{r}, t) &= \frac{1}{4} \left(\tilde{\mathbf{E}}(\mathbf{r}, \omega) e^{-i\omega t} + \tilde{\mathbf{E}}^*(\mathbf{r}, \omega) e^{i\omega t} \right) \times \left(\tilde{\mathbf{H}}(\mathbf{r}, \omega) e^{-i\omega t} + \tilde{\mathbf{H}}^*(\mathbf{r}, \omega) e^{i\omega t} \right) \\ &= \frac{1}{4} \left(\tilde{\mathbf{E}}(\mathbf{r}, \omega) \times \tilde{\mathbf{H}}^*(\mathbf{r}, \omega) + \tilde{\mathbf{E}}^*(\mathbf{r}, \omega) \times \tilde{\mathbf{H}}(\mathbf{r}, \omega) \right) \\ &\quad + \frac{1}{4} \left(\tilde{\mathbf{E}}(\mathbf{r}, \omega) \times \tilde{\mathbf{H}}(\mathbf{r}, \omega) e^{-2i\omega t} + \tilde{\mathbf{E}}^*(\mathbf{r}, \omega) \times \tilde{\mathbf{H}}^*(\mathbf{r}, \omega) e^{2i\omega t} \right) \end{aligned}$$

$$\langle \mathbf{S}(\mathbf{r}, t) \rangle_{t \text{ avg}} = \Re \left(\frac{1}{2} \left(\tilde{\mathbf{E}}(\mathbf{r}, \omega) \times \tilde{\mathbf{H}}^*(\mathbf{r}, \omega) \right) \right)$$

Maxwell's equations

Coulomb's law :

$$\nabla \cdot \mathbf{D} = \rho_{free}$$

Ampere - Maxwell's law :

$$\nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{J}_{free}$$

Faraday's law :

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$$

No magnetic monopoles :

$$\nabla \cdot \mathbf{B} = 0$$

Maxwell's equations

For linear isotropic media - - $\mathbf{D} = \epsilon\mathbf{E}$; $\mathbf{B} = \mu\mathbf{H}$
and no sources :

Coulomb's law : $\nabla \cdot \mathbf{E} = 0$

Ampere - Maxwell's law : $\nabla \times \mathbf{B} - \mu\epsilon \frac{\partial \mathbf{E}}{\partial t} = 0$

Faraday's law : $\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$

No magnetic monopoles : $\nabla \cdot \mathbf{B} = 0$

Analysis of Maxwell's equations without sources -- continued:

Coulomb's law : $\nabla \cdot \mathbf{E} = 0$

Ampere - Maxwell's law : $\nabla \times \mathbf{B} - \mu\epsilon \frac{\partial \mathbf{E}}{\partial t} = 0$

Faraday's law : $\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$

No magnetic monopoles : $\nabla \cdot \mathbf{B} = 0$

$$\begin{aligned}\nabla \times \left(\nabla \times \mathbf{B} - \mu\epsilon \frac{\partial \mathbf{E}}{\partial t} \right) &= -\nabla^2 \mathbf{B} - \mu\epsilon \frac{\partial (\nabla \times \mathbf{E})}{\partial t} \\ &= -\nabla^2 \mathbf{B} + \mu\epsilon \frac{\partial^2 \mathbf{B}}{\partial t^2} = 0\end{aligned}$$

$$\begin{aligned}\nabla \times \left(\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} \right) &= -\nabla^2 \mathbf{E} + \frac{\partial (\nabla \times \mathbf{B})}{\partial t} \\ &= -\nabla^2 \mathbf{E} + \mu\epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0\end{aligned}$$

Analysis of Maxwell's equations without sources -- continued:
Both $\mathbf{E}$ and $\mathbf{B}$ fields are solutions to a wave equation:

$$\nabla^2 \mathbf{B} - \frac{1}{v^2} \frac{\partial^2 \mathbf{B}}{\partial t^2} = 0$$

$$\nabla^2 \mathbf{E} - \frac{1}{v^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0$$

$$\text{where } v^2 \equiv c^2 \frac{\mu_0 \epsilon_0}{\mu \epsilon} \equiv \frac{c^2}{n^2}$$

Plane wave solutions to wave equation :

$$\mathbf{B}(\mathbf{r}, t) = \Re(\mathbf{B}_0 e^{i\mathbf{k} \cdot \mathbf{r} - i\omega t}) \quad \mathbf{E}(\mathbf{r}, t) = \Re(\mathbf{E}_0 e^{i\mathbf{k} \cdot \mathbf{r} - i\omega t})$$

Analysis of Maxwell's equations without sources -- continued:
Plane wave solutions to wave equation :

$$\mathbf{B}(\mathbf{r}, t) = \Re(\mathbf{B}_0 e^{i\mathbf{k} \cdot \mathbf{r} - i\omega t}) \quad \mathbf{E}(\mathbf{r}, t) = \Re(\mathbf{E}_0 e^{i\mathbf{k} \cdot \mathbf{r} - i\omega t})$$

$$|\mathbf{k}|^2 = \left(\frac{\omega}{v}\right)^2 = \left(\frac{n\omega}{c}\right)^2 \quad \text{where } n \equiv \sqrt{\frac{\mu\epsilon}{\mu_0\epsilon_0}}$$

Note: ϵ, μ, n, k can all be complex; for the moment we will assume that they are all real (no dissipation).

Note that $\mathbf{E}_0$ and $\mathbf{B}_0$ are not independent;

$$\text{from Faraday's law : } \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$$

$$\Rightarrow \mathbf{B}_0 = \frac{\mathbf{k} \times \mathbf{E}_0}{\omega} = \frac{n\hat{\mathbf{k}} \times \mathbf{E}_0}{c}$$

$$\text{also note : } \hat{\mathbf{k}} \cdot \mathbf{E}_0 = 0 \quad \text{and} \quad \hat{\mathbf{k}} \cdot \mathbf{B}_0 = 0$$

Analysis of Maxwell's equations without sources -- continued:

Summary of plane electromagnetic waves :

$$\mathbf{B}(\mathbf{r}, t) = \Re \left(\frac{n \hat{\mathbf{k}} \times \mathbf{E}_0}{c} e^{i\mathbf{k} \cdot \mathbf{r} - i\omega t} \right) \quad \mathbf{E}(\mathbf{r}, t) = \Re \left(\mathbf{E}_0 e^{i\mathbf{k} \cdot \mathbf{r} - i\omega t} \right)$$

$$|\mathbf{k}|^2 = \left(\frac{\omega}{v} \right)^2 = \left(\frac{n\omega}{c} \right)^2 \quad \text{where } n \equiv \sqrt{\frac{\mu\epsilon}{\mu_0\epsilon_0}} \quad \text{and } \hat{\mathbf{k}} \cdot \mathbf{E}_0 = 0$$

Poynting vector for plane electromagnetic waves :

$$\langle \mathbf{S} \rangle_{avg} = \frac{1}{2} \Re \left(\mathbf{E}_0 e^{i\mathbf{k} \cdot \mathbf{r} - i\omega t} \times \frac{1}{\mu} \left(\frac{n \hat{\mathbf{k}} \times \mathbf{E}_0}{c} e^{i\mathbf{k} \cdot \mathbf{r} - i\omega t} \right)^* \right)$$

$$= \frac{n |\mathbf{E}_0|^2}{2\mu c} \hat{\mathbf{k}} = \frac{1}{2} \sqrt{\frac{\epsilon}{\mu}} |\mathbf{E}_0|^2 \hat{\mathbf{k}}$$

Note that:

$$\begin{aligned} \mathbf{E}_0 \times (\hat{\mathbf{k}} \times \mathbf{E}_0) &= \hat{\mathbf{k}} (\mathbf{E}_0 \cdot \mathbf{E}_0) - \mathbf{E}_0 (\hat{\mathbf{k}} \cdot \mathbf{E}_0) \\ &= \hat{\mathbf{k}} |\mathbf{E}_0|^2 \end{aligned}$$

Analysis of Maxwell's equations without sources -- continued:

Transverse Electric and Magnetic (TEM) waves

Summary of plane electromagnetic waves :

$$\mathbf{B}(\mathbf{r}, t) = \Re \left(\frac{n \hat{\mathbf{k}} \times \mathbf{E}_0}{c} e^{i\mathbf{k} \cdot \mathbf{r} - i\omega t} \right) \quad \mathbf{E}(\mathbf{r}, t) = \Re \left(\mathbf{E}_0 e^{i\mathbf{k} \cdot \mathbf{r} - i\omega t} \right)$$

$$|\mathbf{k}|^2 = \left(\frac{\omega}{v} \right)^2 = \left(\frac{n\omega}{c} \right)^2 \quad \text{where } n \equiv \sqrt{\frac{\mu\epsilon}{\mu_0\epsilon_0}} \quad \text{and } \hat{\mathbf{k}} \cdot \mathbf{E}_0 = 0$$

Energy density for plane electromagnetic waves :

$$\begin{aligned} \langle u \rangle_{avg} &= \frac{1}{4} \Re \left(\epsilon \mathbf{E}_0 e^{i\mathbf{k} \cdot \mathbf{r} - i\omega t} \cdot \left(\mathbf{E}_0 e^{i\mathbf{k} \cdot \mathbf{r} - i\omega t} \right)^* \right) + \\ &\quad \frac{1}{4} \Re \left(\frac{1}{\mu} \frac{n \hat{\mathbf{k}} \times \mathbf{E}_0}{c} e^{i\mathbf{k} \cdot \mathbf{r} - i\omega t} \cdot \left(\frac{n \hat{\mathbf{k}} \times \mathbf{E}_0}{c} e^{i\mathbf{k} \cdot \mathbf{r} - i\omega t} \right)^* \right) \\ &= \frac{1}{2} \epsilon |\mathbf{E}_0|^2 \end{aligned}$$