

PHY 712 Electrodynamics

10-10:50 AM Online

Plan for Lecture 18:

Continue reading Chapter 7

- 1. Real and imaginary contributions to electromagnetic response**
- 2. Frequency dependence of dielectric materials; Drude model**
- 3. Kramers-Kronig relationships**

Course schedule for Spring 2021

(Preliminary schedule -- subject to frequent adjustment.)

	Lecture date	JDJ Reading	Topic	HW	Due date
1	Wed: 01/27/2021	Chap. 1 & Appen.	Introduction, units and Poisson equation	#1	01/29/2021
2	Fri: 01/29/2021	Chap. 1	Electrostatic energy calculations	#2	02/01/2021
3	Mon: 02/01/2021	Chap. 1 & 2	Electrostatic potentials and fields	#3	02/03/2021
4	Wed: 02/03/2021	Chap. 1 - 3	Poisson's equation in 2 and 3 dimensions	#4	02/05/2021
5	Fri: 02/05/2021	Chap. 1 - 3	Brief introduction to numerical methods	#5	02/08/2021
6	Mon: 02/08/2021	Chap. 2 & 3	Image charge constructions	#6	02/10/2021
7	Wed: 02/10/2021	Chap. 2 & 3	Cylindrical and spherical geometries		
8	Fri: 02/12/2021	Chap. 3 & 4	Spherical geometry and multipole moments	#7	02/15/2021
9	Mon: 02/15/2021	Chap. 4	Dipoles and Dielectrics	#8	02/19/2021
10	Wed: 02/17/2021	Chap. 4	Dipoles and Dielectrics		
11	Fri: 02/19/2021	Chap. 4	Polarization and Dielectrics	#9	02/24/2021
12	Mon: 02/22/2021	Chap. 5	Magnetostatics	#10	02/26/2021
13	Wed: 02/24/2021	Chap. 5	Magnetic dipoles and hyperfine interaction	#11	03/01/2021
14	Fri: 02/26/2021	Chap. 5	Magnetic dipoles and dipolar fields		
15	Mon: 03/01/2021	Chap. 6	Maxwell's Equations	#12	03/08/2021
16	Wed: 03/03/2021	Chap. 6	Electromagnetic energy and forces		
17	Fri: 03/05/2021	Chap. 7	Electromagnetic plane waves		
18	Mon: 03/08/2021	Chap. 7	Electromagnetic plane waves	#13	03/10/2021
19	Wed: 03/10/2021	Chap. 7	Optical effects of refractive indices	#14	03/12/2021
20	Fri: 03/12/2021	Chap. 1-7	Review		
	Mon: 03/15/2021 03/08/2021	No class	APS March Meeting PHY 712 Spring 2021 -- Lecture 18	Take Home Exam	2

Maxwell's equations

For linear isotropic media and no sources: $\mathbf{D} = \epsilon \mathbf{E}$; $\mathbf{B} = \mu \mathbf{H}$

Coulomb's law: $\nabla \cdot \mathbf{E} = 0$

Ampere-Maxwell's law: $\nabla \times \mathbf{B} - \mu\epsilon \frac{\partial \mathbf{E}}{\partial t} = 0$

Faraday's law: $\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$

No magnetic monopoles: $\nabla \cdot \mathbf{B} = 0$

Plane wave solutions to sourceless Maxwell's equations; extension of analysis to complex dielectric functions

For simplicity assume that $\mu = \mu_0$

Suppose the dielectric function is complex:

$$\varepsilon = \varepsilon_R + i\varepsilon_I \qquad \frac{\varepsilon}{\varepsilon_0} = (n_R + in_I)^2 \equiv \alpha + i\beta$$

$$n_R^2 - n_I^2 = \alpha \qquad 2n_R n_I = \beta$$

$$n_I = \sqrt{n_R^2 - \alpha} \qquad 4n_R^4 - 4n_R^2 \alpha - \beta^2 = 0$$

$$n_R = \left(\frac{\sqrt{\alpha^2 + \beta^2} + \alpha}{2} \right)^{1/2} \qquad n_I = \left(\frac{\sqrt{\alpha^2 + \beta^2} - \alpha}{2} \right)^{1/2}$$

$$\mathbf{E}(\mathbf{r}, t) = \Re \left(\mathbf{E}_0 e^{i \frac{\omega}{c} (n \hat{\mathbf{k}} \cdot \mathbf{r} - ct)} \right) = \Re \left(\mathbf{E}_0 e^{i \frac{\omega}{c} (n_R \hat{\mathbf{k}} \cdot \mathbf{r} - ct)} \right) e^{-\frac{\omega}{c} n_I \hat{\mathbf{k}} \cdot \mathbf{r}}$$

Paul Karl Ludwig Drude 1863-1906



http://photos.aip.org/history/Thumbnails/drude_paul_a1.jpg

Drude model:

Vibrations of charged particles near equilibrium:

$$m\delta \ddot{\mathbf{r}} = q\mathbf{E}_0 e^{-i\omega t} - m\omega_0^2 \delta \mathbf{r} - m\gamma \delta \dot{\mathbf{r}}$$

For $\delta \mathbf{r} \equiv \delta \mathbf{r}_0 e^{-i\omega t}$,
$$\delta \mathbf{r}_0 = \frac{q\mathbf{E}_0}{m} \frac{1}{\omega_0^2 - \omega^2 - i\omega\gamma}$$

Induced dipole:

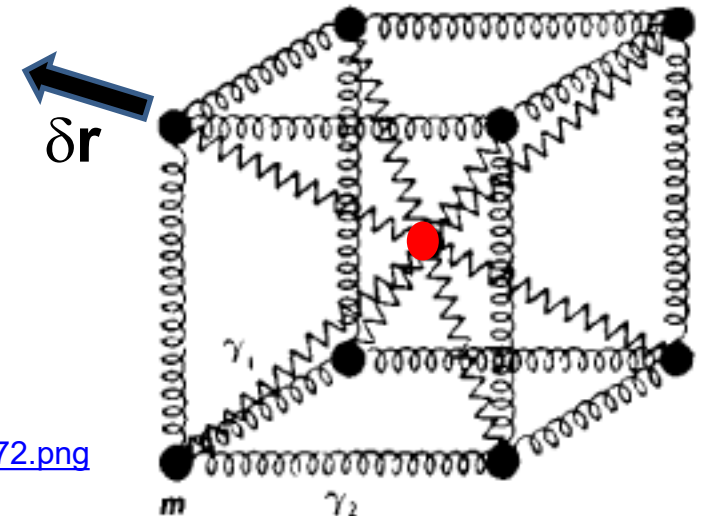
$$\mathbf{p} = q \delta \mathbf{r} = \frac{q^2 \mathbf{E}_0}{m} \frac{1}{\omega_0^2 - \omega^2 - i\omega\gamma} e^{-i\omega t}$$

Displacement field:

$$\mathbf{D} = \epsilon \mathbf{E} = \epsilon_0 \mathbf{E} + \mathbf{P}$$

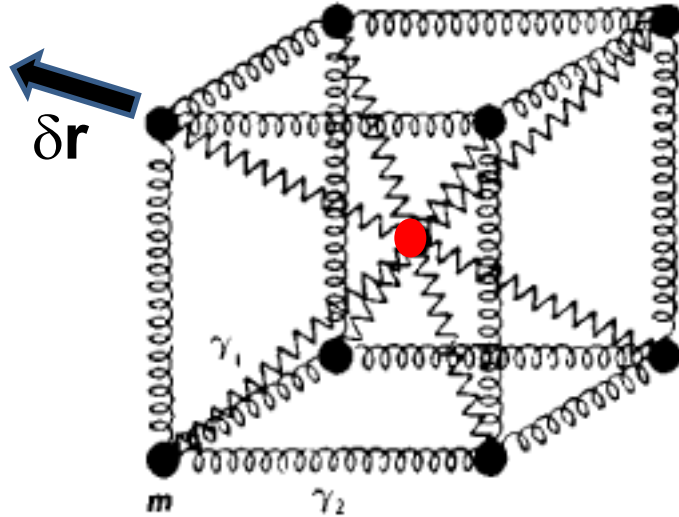
$$\mathbf{P} = \sum_i \mathbf{p}_i \delta^3(\mathbf{r} - \mathbf{r}_i)$$

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Drude model:

Vibration of particle of charge q and mass m near equilibrium:



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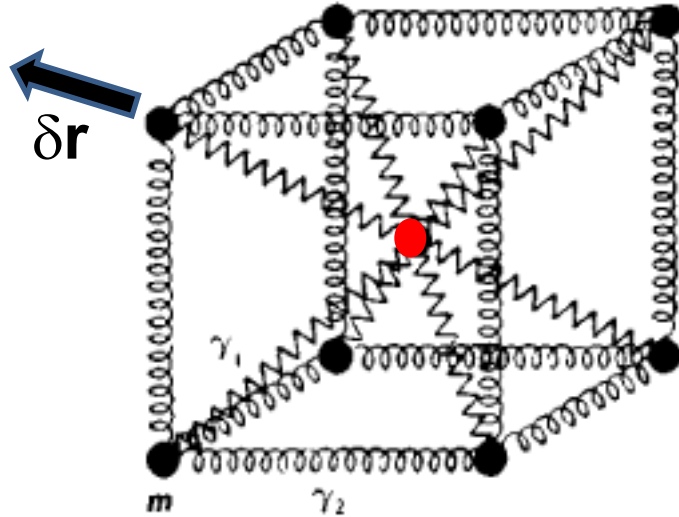
$$m \delta \ddot{\mathbf{r}} = q \mathbf{E}_0 e^{-i\omega t} - m \omega_0^2 \delta \mathbf{r} - m \gamma \delta \dot{\mathbf{r}}$$

Note that:

- ☐ $\gamma > 0$ represents dissipation of energy.
- ☐ ω_0 represents the natural frequency of the vibration; $\omega_0=0$ would represent a free (unbound) particle

Drude model:

Vibration of particle of charge q and mass m near equilibrium:



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$$m \delta \ddot{\mathbf{r}} = q \mathbf{E}_0 e^{-i\omega t} - m \omega_0^2 \delta \mathbf{r} - m \gamma \delta \dot{\mathbf{r}}$$

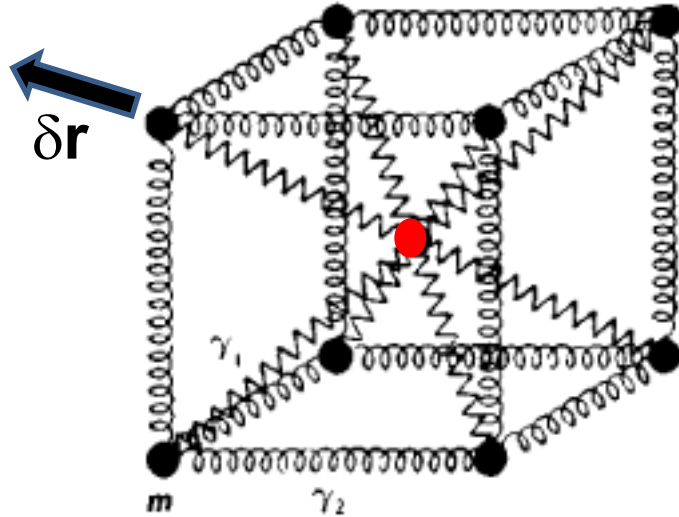
For $\delta \mathbf{r} \equiv \delta \mathbf{r}_0 e^{-i\omega t}$,
$$\delta \mathbf{r}_0 = \frac{q \mathbf{E}_0}{m} \frac{1}{\omega_0^2 - \omega^2 - i\omega\gamma}$$

Induced dipole :

$$\mathbf{p} = q \delta \mathbf{r} = \frac{q^2 \mathbf{E}_0}{m} \frac{1}{\omega_0^2 - \omega^2 - i\omega\gamma} e^{-i\omega t}$$

Drude model:

Vibration of particle of charge q and mass m near equilibrium:



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$$m \delta \ddot{\mathbf{r}} = q \mathbf{E}_0 e^{-i\omega t} - m \omega_0^2 \delta \mathbf{r} - m \gamma \delta \dot{\mathbf{r}}$$

Displacement field:

$$\mathbf{D} = \epsilon \mathbf{E} = \epsilon_0 \mathbf{E} + \mathbf{P}$$

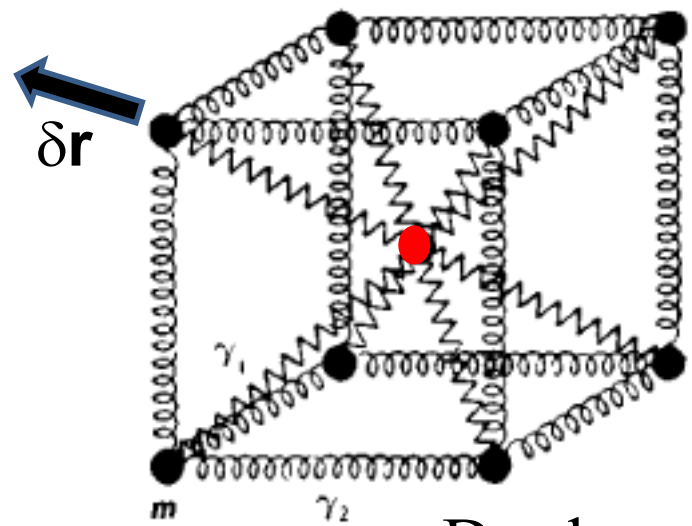
$$\mathbf{P} = \sum_i \mathbf{p}_i \delta^3(\mathbf{r} - \mathbf{r}_i) \approx N \sum_i f_i \mathbf{p}_i$$

$N \equiv$ number of dipoles/volume

$f_i \equiv$ fraction of type i dipoles

Drude model:

Vibration of particle of charge q and mass m near equilibrium:



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Drude model expression for permittivity:

$$\mathbf{D} = \epsilon \mathbf{E} = \epsilon_0 \mathbf{E} + \mathbf{P} = \epsilon_0 \mathbf{E} + N \sum_i f_i \mathbf{p}_i$$

$$\mathbf{p}_i = q_i \delta \mathbf{r} = \frac{q_i^2 \mathbf{E}_0}{m_i} \frac{1}{\omega_i^2 - \omega^2 - i \omega \gamma_i} e^{-i \omega t}$$

$$\epsilon \mathbf{E} = \epsilon_0 \mathbf{E}_0 e^{-i \omega t} \left(1 + N \sum_i f_i \frac{q_i^2}{\epsilon_0 m_i} \frac{1}{\omega_i^2 - \omega^2 - i \omega \gamma_i} \right)$$

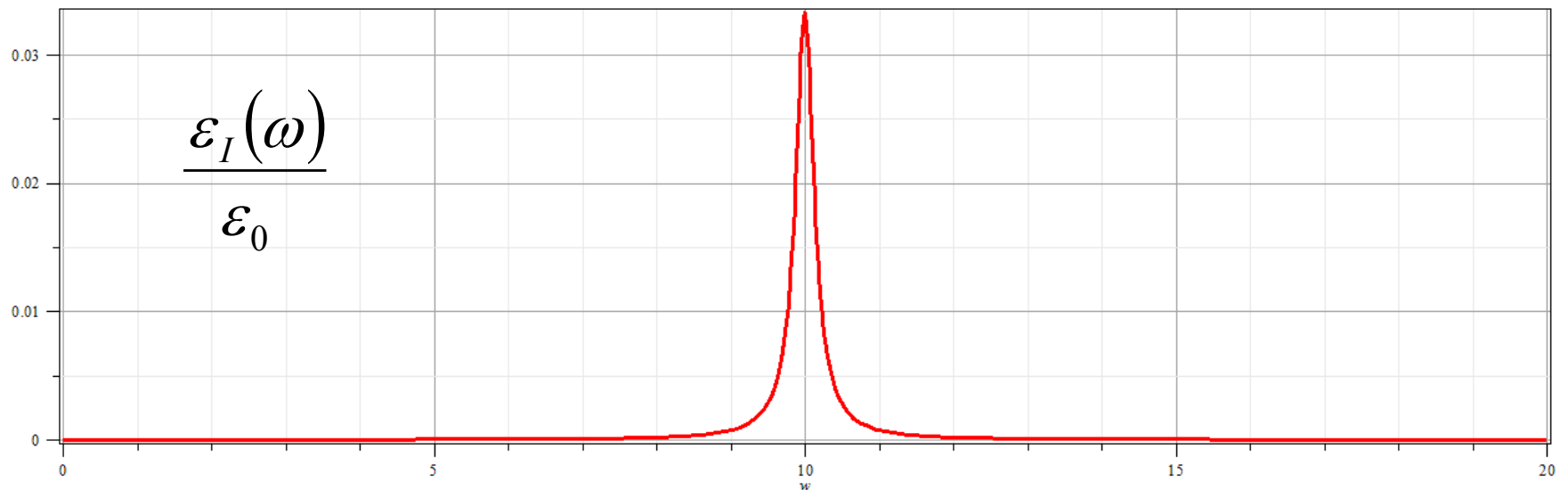
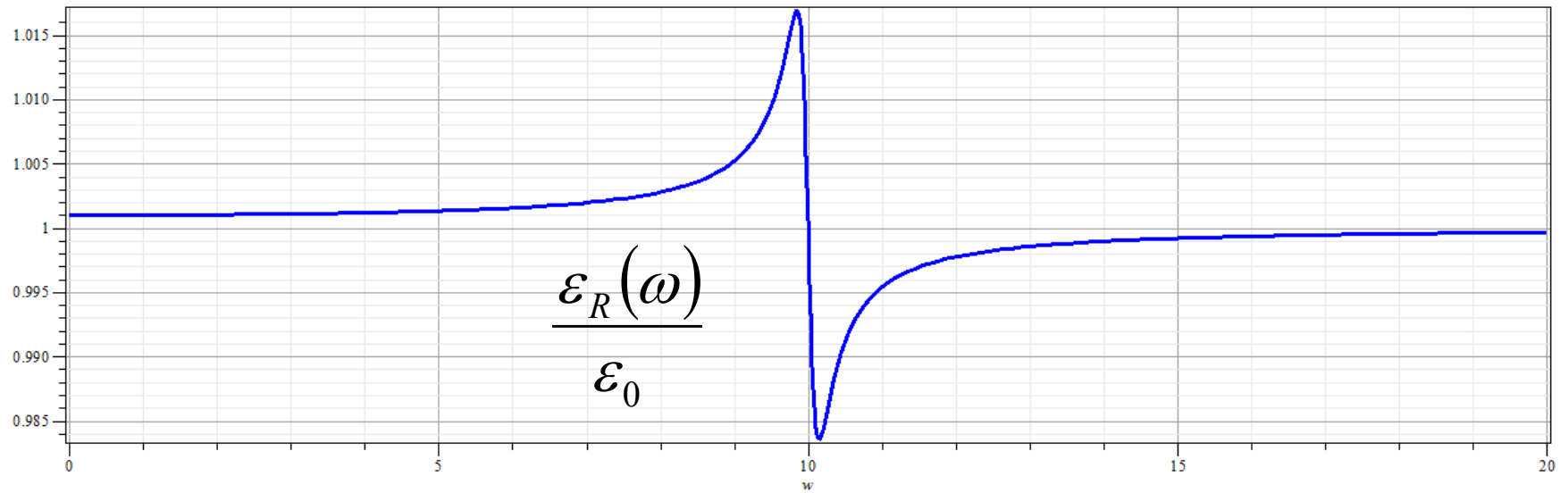
Drude model dielectric function:

$$\begin{aligned}\frac{\varepsilon(\omega)}{\varepsilon_0} &= 1 + N \sum_i f_i \frac{q_i^2}{\varepsilon_0 m_i} \frac{1}{\omega_i^2 - \omega^2 - i\omega\gamma_i} \\ &= \frac{\varepsilon_R(\omega)}{\varepsilon_0} + i \frac{\varepsilon_I(\omega)}{\varepsilon_0}\end{aligned}$$

$$\frac{\varepsilon_R(\omega)}{\varepsilon_0} = 1 + N \sum_i f_i \frac{q_i^2}{\varepsilon_0 m_i} \frac{\omega_i^2 - \omega^2}{(\omega_i^2 - \omega^2)^2 + \omega^2 \gamma_i^2}$$

$$\frac{\varepsilon_I(\omega)}{\varepsilon_0} = N \sum_i f_i \frac{q_i^2}{\varepsilon_0 m_i} \frac{\omega \gamma_i}{(\omega_i^2 - \omega^2)^2 + \omega^2 \gamma_i^2}$$

Drude model dielectric function:



Drude model dielectric function – some analytic properties:

$$\frac{\varepsilon(\omega)}{\varepsilon_0} = 1 + N \sum_i f_i \frac{q_i^2}{\varepsilon_0 m_i} \frac{1}{\omega_i^2 - \omega^2 - i\omega\gamma_i}$$

For $\omega \gg \omega_i$

$$\frac{\varepsilon(\omega)}{\varepsilon_0} \approx 1 - \frac{1}{\omega^2} \left(N \sum_i f_i \frac{q_i^2}{\varepsilon_0 m_i} \right)$$
$$\equiv 1 - \frac{\omega_P^2}{\omega^2}$$

Drude model dielectric function – some analytic properties:

$$\frac{\varepsilon(\omega)}{\varepsilon_0} = 1 + N \sum_i f_i \frac{q_i^2}{\varepsilon_0 m_i} \frac{1}{\omega_i^2 - \omega^2 - i\omega\gamma_i}$$

For $\omega_0 = 0$ (representing a free particle of charge q_0 , mass m_0)

$$\begin{aligned} \frac{\varepsilon(\omega)}{\varepsilon_0} &= 1 + N \sum_{i>0} f_i \frac{q_i^2}{\varepsilon_0 m_i} \frac{1}{\omega_i^2 - \omega^2 - i\omega\gamma_i} + iNf_0 \frac{q_0^2}{\varepsilon_0 m_0} \frac{1}{\omega(\gamma_0 - i\omega)} \\ &\equiv \frac{\varepsilon_b(\omega)}{\varepsilon_0} + i \frac{\sigma(\omega)}{\varepsilon_0 \omega} \end{aligned}$$

Some details:

$$\mathbf{D} = \varepsilon_b \mathbf{E} \quad \mathbf{J} = \sigma \mathbf{E}$$

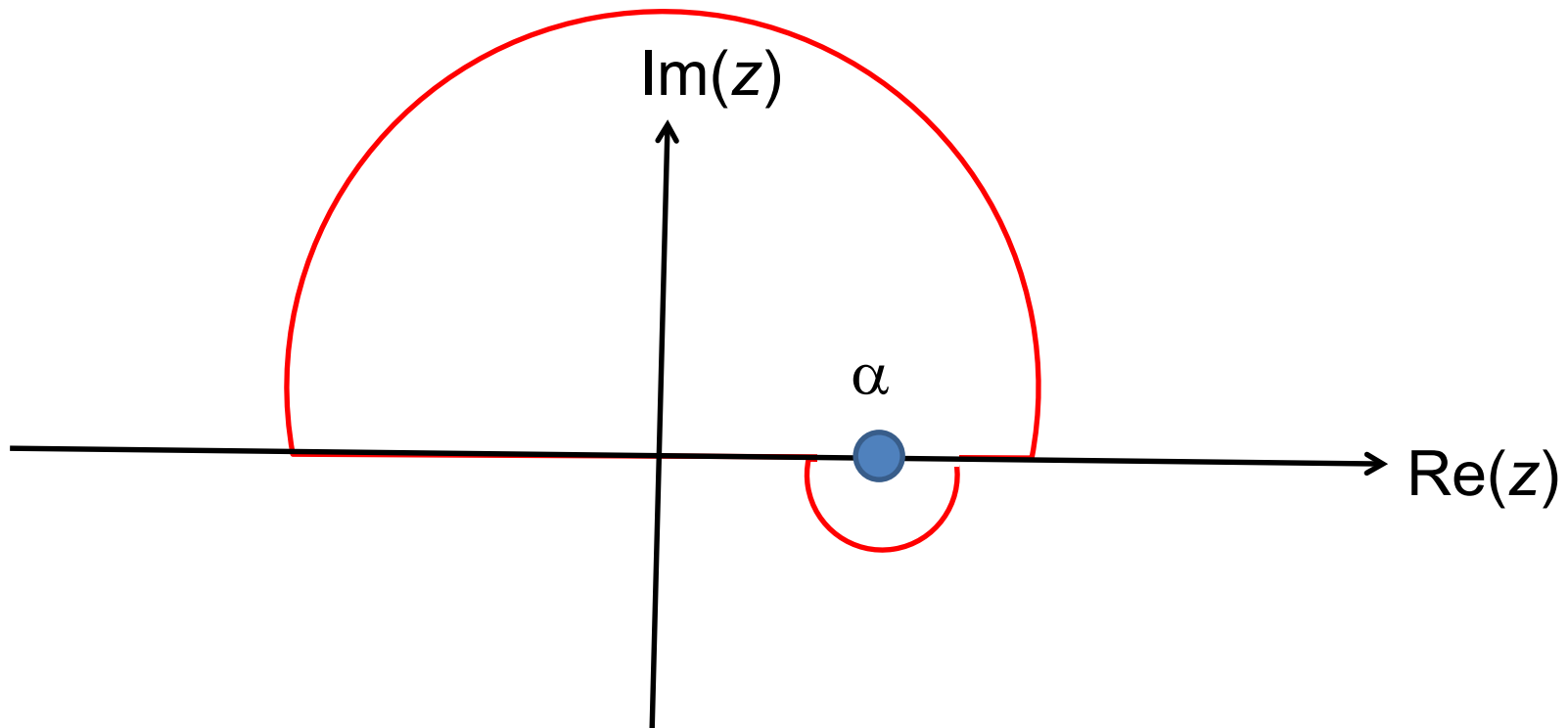
$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} = (\sigma - i\omega\varepsilon_b) \mathbf{E} = \varepsilon \frac{\partial \mathbf{E}}{\partial t} = -i\omega \left(\varepsilon_b + \frac{i\sigma}{\omega} \right) \mathbf{E}$$

$$\Rightarrow \sigma(\omega) = Nf_0 \frac{q_0^2}{m_0} \frac{1}{(\gamma_0 - i\omega)}$$

Analytic properties of the dielectric function (in the Drude model or from “first principles” -- Kramers-Kronig transform

Consider Cauchy's integral formula for an analytic function $f(z)$:

$$\oint dz f(z) = 0 \qquad f(\alpha) = \frac{1}{2\pi i} \oint_{\text{includes } \alpha} dz \frac{f(z)}{z - \alpha}$$



Kramers-Kronig transform -- continued

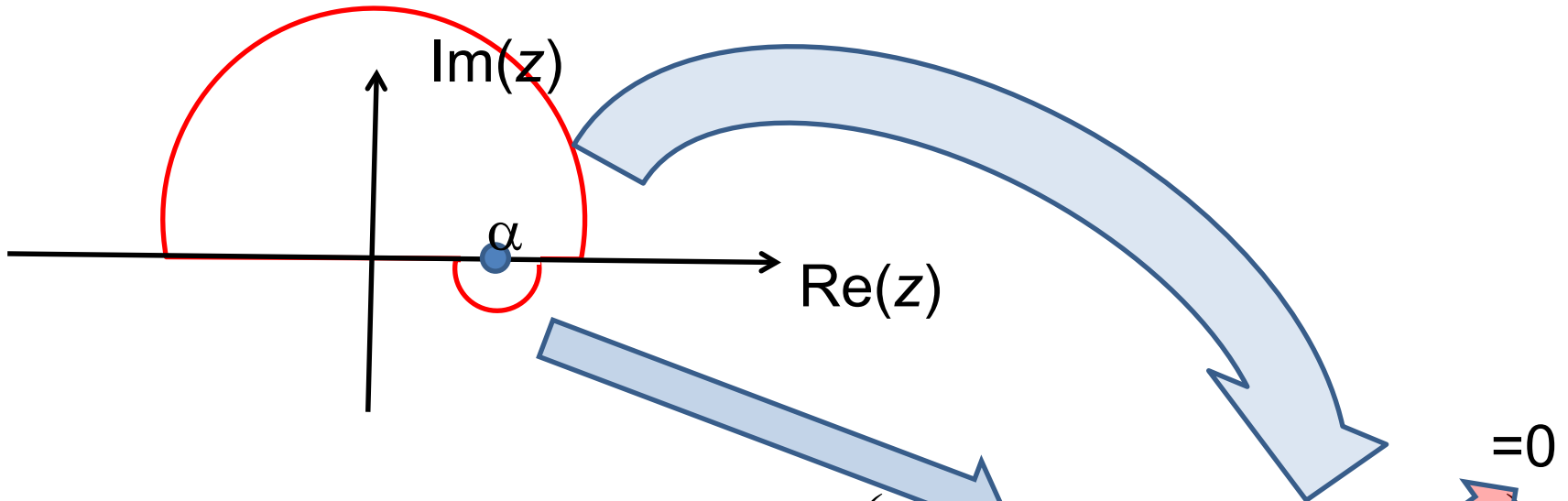


Diagram illustrating the Kramers-Kronig transform. A complex plane is shown with the real axis labeled $\text{Re}(z)$ and the imaginary axis labeled $\text{Im}(z)$. A point α is marked on the real axis. A red semi-circle is drawn in the upper half-plane, and a small red circle is drawn around α . A large blue arrow points from the upper semi-circle to the first term of the integral equation below. A smaller blue arrow points from the small circle to the second term. A red arrow points from the second term to $=0$.

$$f(\alpha) = \frac{1}{2\pi i} \oint_{\text{includes } \alpha} dz \frac{f(z)}{z - \alpha} = \frac{1}{2\pi i} \left(\int_{-\infty}^{\infty} dz_R \frac{f(z_R)}{z_R - \alpha} + \int_{\text{rest}} dz \frac{f(z)}{z - \alpha} \right) = 0$$

$$f(\alpha) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} dz_R \frac{f(z_R)}{z_R - \alpha} = \frac{1}{2\pi i} P \int_{-\infty}^{\infty} dz_R \frac{f(z_R)}{z_R - \alpha} + \frac{1}{2} f(\alpha)$$

Kramers-Kronig transform -- continued

$$f(\alpha) = \frac{1}{2\pi i} P \int_{-\infty}^{\infty} dz_R \frac{f(z_R)}{z_R - \alpha} + \frac{1}{2} f(\alpha)$$

Suppose $f(z_R) = f_R(z_R) + if_I(z_R)$:

$$\Rightarrow \frac{1}{2} (f_R(\alpha) + if_I(\alpha)) = \frac{1}{2\pi i} P \int_{-\infty}^{\infty} dz_R \frac{f_R(z_R) + if_I(z_R)}{z_R - \alpha}$$

$$\Rightarrow f_R(\alpha) = \frac{1}{\pi} P \int_{-\infty}^{\infty} dz_R \frac{f_I(z_R)}{z_R - \alpha}$$

$$f_I(\alpha) = -\frac{1}{\pi} P \int_{-\infty}^{\infty} dz_R \frac{f_R(z_R)}{z_R - \alpha}$$

Kramers-Kronig transform -- continued

$$f_R(\alpha) = \frac{1}{\pi} P \int_{-\infty}^{\infty} dz_R \frac{f_I(z_R)}{z_R - \alpha}$$

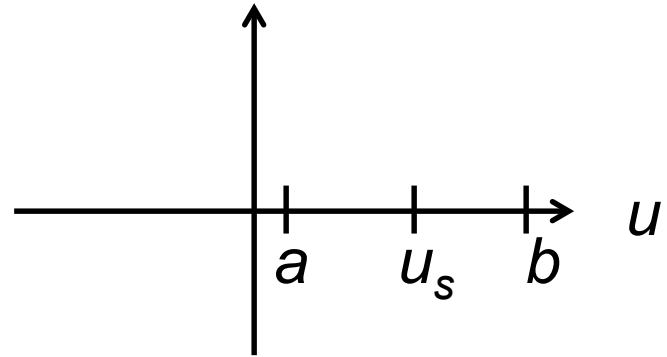
$$f_I(\alpha) = -\frac{1}{\pi} P \int_{-\infty}^{\infty} dz_R \frac{f_R(z_R)}{z_R - \alpha}$$

This Kramers-Kronig transform is useful for the dielectric function

$$\text{when } f(z_R) \Rightarrow \frac{\varepsilon(\omega)}{\varepsilon_0} - 1$$

Must show that:

1. $f(z)$ is analytic for $z_I > 0$
2. $f(z)$ vanishes for $z \rightarrow \infty$



Some practical considerations

Principal parts integration :

$$P \int_a^b du \, g(u) = \lim_{\nu \rightarrow 0} \left(\int_a^{u_s - \nu} du \, g(u) + \int_{u_s + \nu}^b du \, g(u) \right)$$

Example :

$$\begin{aligned} P \int_a^b du \, \frac{1}{u - u_s} &= \lim_{\nu \rightarrow 0} \left(\int_a^{u_s - \nu} du \, \frac{1}{u - u_s} + \int_{u_s + \nu}^b du \, \frac{1}{u - u_s} \right) \\ &= \lim_{\nu \rightarrow 0} \left(\ln \left(\frac{\nu}{u_s - a} \right) + \ln \left(\frac{b - u_s}{\nu} \right) \right) = \ln \left(\frac{b - u_s}{u_s - a} \right) \end{aligned}$$

More practical considerations

For dielectric function $\varepsilon(\omega)$:

$$\varepsilon(-\omega) = \varepsilon^*(\omega)$$

$$\Rightarrow \varepsilon_R(-\omega) = \varepsilon_R(\omega)$$

$$\Rightarrow \varepsilon_I(-\omega) = -\varepsilon_I(\omega)$$

Analytic properties the dielectric function which justify

the treatment of $f(z) \Rightarrow \frac{\varepsilon(z)}{\varepsilon_0} - 1$

Must show that: 1. $f(z)$ is analytic for $z_I > 0$

2. $f(z)$ vanishes for $z \rightarrow \infty$ (for $z_I > 0$)

Analysis for Drude model dielectric function:

$$\frac{\varepsilon(\omega)}{\varepsilon_0} = 1 + N \sum_i f_i \frac{q_i^2}{\varepsilon_0 m_i} \frac{1}{\omega_i^2 - \omega^2 - i\omega\gamma_i}$$

$$\text{Let } f(z) = \frac{\varepsilon(z)}{\varepsilon_0} - 1 = N \sum_i f_i \frac{q_i^2}{\varepsilon_0 m_i} \frac{1}{\omega_i^2 - z^2 - iz\gamma_i}$$

$$\text{For } |z| \gg \omega_i$$

$$f(z) \approx -\frac{1}{z^2} \left(N \sum_i f_i \frac{q_i^2}{\varepsilon_0 m_i} \right) \Rightarrow \text{vanishes at large } z$$

Analysis for Drude model dielectric function – continued -- Analytic properties:

$$f(z) = \frac{\varepsilon(z)}{\varepsilon_0} - 1 = N \sum_i f_i \frac{q_i^2}{\varepsilon_0 m_i} \frac{1}{\omega_i^2 - z^2 - iz\gamma_i}$$

$f(z)$ has poles z_P at $\omega_i^2 - z_P^2 - iz_P\gamma_i = 0$

$$z_P = -i\frac{\gamma_i}{2} \pm \sqrt{\omega_i^2 - \left(\frac{\gamma_i}{2}\right)^2}$$

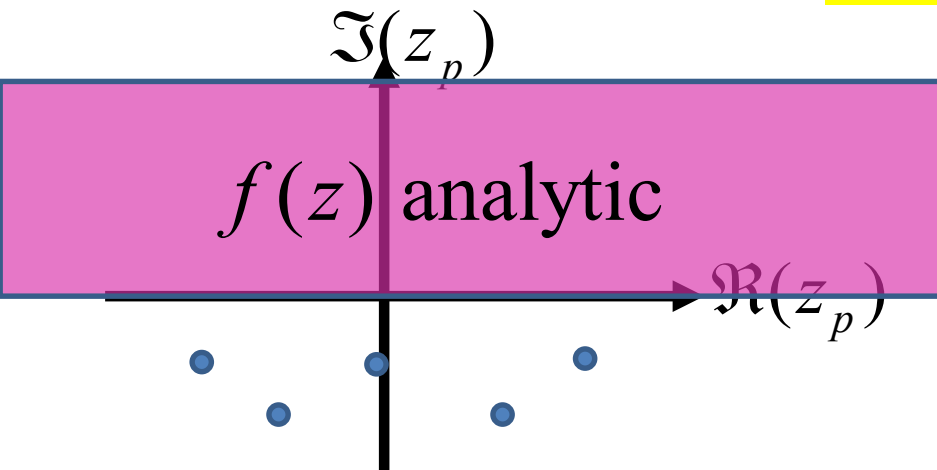
Note that $\Im(z_P) \leq 0 \Rightarrow f(z)$ is analytic for $\Im(z_P) > 0$

$$f(z) = \frac{\varepsilon(z)}{\varepsilon_0} - 1 = N \sum_i f_i \frac{q_i^2}{\varepsilon_0 m_i} \frac{1}{\omega_i^2 - z^2 - iz\gamma_i}$$

$f(z)$ has poles z_P at $\omega_i^2 - z_P^2 - iz_P\gamma_i = 0$

$$z_P = -i\frac{\gamma_i}{2} \pm \sqrt{\omega_i^2 - \left(\frac{\gamma_i}{2}\right)^2}$$

Note that $\Im(z_P) \leq 0 \Rightarrow f(z)$ is analytic for $\Im(z_P) > 0$



Because of these analytic properties, Cauchy's integral theorem results in:

Kramers-Kronig transform – for dielectric function:

$$\frac{\varepsilon_R(\omega)}{\varepsilon_0} - 1 = \frac{1}{\pi} P \int_{-\infty}^{\infty} d\omega' \frac{\varepsilon_I(\omega')}{\varepsilon_0} \frac{1}{\omega' - \omega}$$

$$\frac{\varepsilon_I(\omega)}{\varepsilon_0} = -\frac{1}{\pi} P \int_{-\infty}^{\infty} d\omega' \left(\frac{\varepsilon_R(\omega')}{\varepsilon_0} - 1 \right) \frac{1}{\omega' - \omega}$$

$$\text{with } \varepsilon_R(-\omega) = \varepsilon_R(\omega); \quad \varepsilon_I(-\omega) = -\varepsilon_I(\omega)$$

Further comments on analytic behavior of dielectric function

"Causal" relationship between **E** and **D** fields:

$$\mathbf{D}(\mathbf{r}, t) = \varepsilon_0 \left\{ \mathbf{E}(\mathbf{r}, t) + \int_0^{\infty} d\tau \, G(\tau) \mathbf{E}(\mathbf{r}, t - \tau) \right\}$$

$$\frac{\varepsilon(\omega)}{\varepsilon_0} - 1 = \int_0^{\infty} d\tau \, G(\tau) e^{i\omega\tau}$$

Some details: Consider a convolution integral such as

$$f(t) = \int_{-\infty}^{\infty} g(t') h(t - t') dt' \quad \text{where the functions } f(t), g(t), \text{ and } h(t)$$

are all well-defined functions with Fourier transforms such as

$$\tilde{f}(\omega) = \int_{-\infty}^{\infty} f(t') e^{i\omega t'} dt' \quad f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(\omega) e^{-i\omega t} d\omega$$

It follows that: $\tilde{f}(\omega) = \tilde{g}(\omega) \tilde{h}(\omega)$

Further comments on analytic behavior of dielectric function

"Causal" relationship between **E** and **D** fields:

$$\mathbf{D}(\mathbf{r}, t) = \varepsilon_0 \left\{ \mathbf{E}(\mathbf{r}, t) + \int_0^{\infty} d\tau \, G(\tau) \mathbf{E}(\mathbf{r}, t - \tau) \right\}$$

$$G(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\frac{\varepsilon(\omega)}{\varepsilon_0} - 1 \right) e^{-i\omega\tau} d\omega \quad \tilde{G}(\omega) = \frac{\varepsilon(\omega)}{\varepsilon_0} - 1 = \int_0^{\infty} d\tau \, G(\tau) e^{i\omega\tau}$$

$$\text{For } \frac{\varepsilon(\omega)}{\varepsilon_0} - 1 = \frac{N}{\varepsilon_0} \sum_i f_i \frac{q_i^2}{m_i} \frac{1}{\omega_i^2 - \omega^2 - i\omega\gamma_i}$$

$$G(\tau) = \frac{N}{\varepsilon_0} \sum_i f_i \frac{q_i^2}{m_i} e^{-\gamma_i \tau / 2} \frac{\sin(\nu_i \tau)}{\nu_i} \Theta(\tau)$$

$$\text{where } \nu_i \equiv \sqrt{\omega_i^2 - \gamma_i^2 / 4}$$

Some details

$$G(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\frac{\varepsilon(\omega)}{\varepsilon_0} - 1 \right) e^{-i\omega\tau} d\omega = \frac{1}{2\pi} \oint f(z) e^{-iz\tau} dz$$

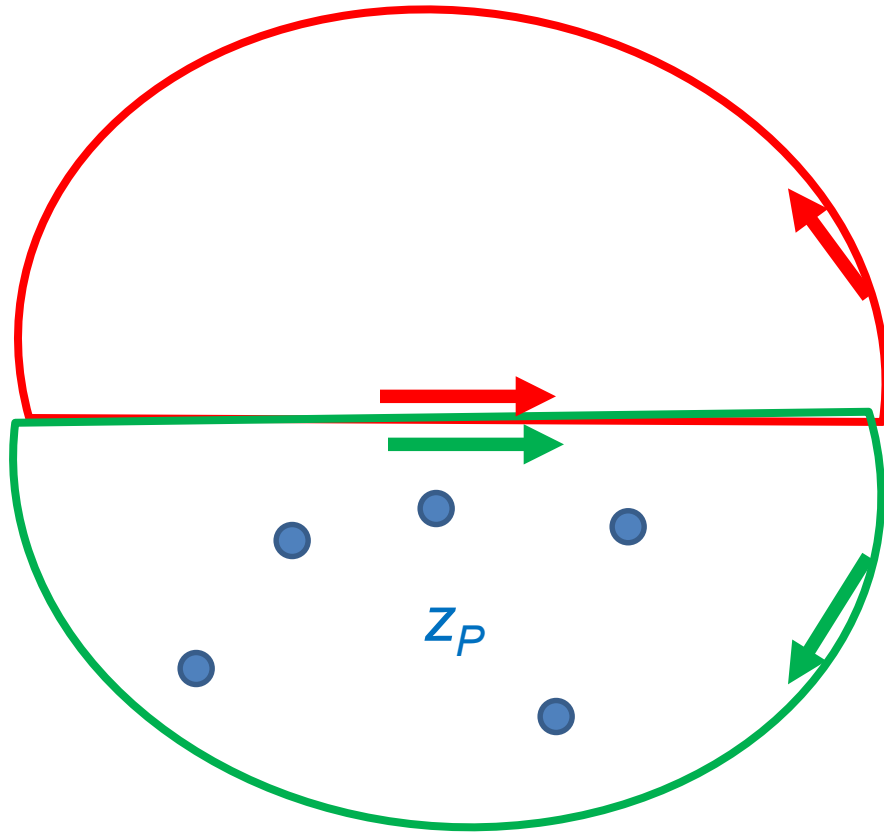
$$\text{Let } f(z) = \frac{\varepsilon(z)}{\varepsilon_0} - 1 = N \sum_i f_i \frac{q_i^2}{\varepsilon_0 m_i} \frac{1}{\omega_i^2 - z^2 - iz\gamma_i}$$

$f(z)$ has poles z_P at $\omega_i^2 - z_P^2 - iz_P\gamma_i = 0$

$$z_P = -i\frac{\gamma_i}{2} \pm \sqrt{\omega_i^2 - \left(\frac{\gamma_i}{2}\right)^2} \quad \text{or} \quad z_P = -i\left(\frac{\gamma_i}{2} \pm \sqrt{\left(\frac{\gamma_i}{2}\right)^2 - \omega_i^2}\right)$$

$$G(\tau) = \frac{1}{2\pi} \oint f(z) e^{-iz\tau} dz = i \sum_P \text{Res}(z_P)$$

Note that: $e^{-iz\tau} = e^{-iz_R\tau} e^{z_I\tau}$



Valid contour for $\tau < 0$

$G(\tau) = 0$ for $\tau < 0$

Valid contour for $\tau > 0$

$G(\tau) =$

$$\frac{N}{\mathcal{E}_0} \sum_i f_i \frac{q_i^2}{m_i} e^{-\gamma_i \tau / 2} \frac{\sin(\nu_i \tau)}{\nu_i}$$

$$G(\tau) = \frac{1}{2\pi} \oint f(z) e^{-iz\tau} dz = i \sum_P \text{Res}(z_P)$$

$$\text{Let } f(z) = \frac{\varepsilon(z)}{\varepsilon_0} - 1 = N \sum_i f_i \frac{q_i^2}{\varepsilon_0 m_i} \frac{1}{\omega_i^2 - z^2 - iz\gamma_i}$$

$$f(z) \text{ has poles } z_P \text{ at } \omega_i^2 - z_P^2 - iz_P\gamma_i = 0$$

$$z_P = -i\frac{\gamma_i}{2} \pm \sqrt{\omega_i^2 - \left(\frac{\gamma_i}{2}\right)^2} \quad \text{or} \quad z_P = -i\left(\frac{\gamma_i}{2} \pm \sqrt{\left(\frac{\gamma_i}{2}\right)^2 - \omega_i^2}\right)$$

$$G(\tau) = \frac{N}{\varepsilon_0} \sum_i f_i \frac{q_i^2}{m_i} e^{-\gamma_i\tau/2} \frac{\sin(\nu_i\tau)}{\nu_i} \Theta(\tau)$$

$$\text{where } \nu_i \equiv \sqrt{\omega_i^2 - \gamma_i^2 / 4} \quad \text{assuming } \omega_i^2 - \gamma_i^2 / 4 \geq 0$$