PHY 712 Electrodynamics
10-10:50 AM Online

Discussion for Lecture 19:

Complete reading of Chapter 7

1. **Comments on reflectivity of plane waves**
2. **Summary of complex response functions for electromagnetic fields**
3. **Comment on spectral properties of electromagnetic waves**

Physics Colloquium – 4 PM 3/11/2021

PHYSICS COLLOQUIUM

THURSDAY
•
MARCH 11, 2021

"Metal Oxos in Chemistry and Biology"

The dianionic oxo ligand occupies a very special place in coordination chemistry, owing to its ability to donate pi electrons to stabilize high oxidation states of metals. The ligand field theory of multiple bonding in metal-oxos predicts that there must be an "oxo wall" between Fe-Ru-Os and Co-Rh-Ir in the periodic table. Metal-oxos on the left side of the wall are reactive intermediates in three of the most important chemical reactions on planet Earth: water oxidation to oxygen in green leaves; oxygen reduction to water in the respiratory chain; and hydrocarbon oxygenation catalyzed by an enzyme called cytochrome P450. I will focus on water oxidation to oxygen in my talk, as the reaction liberates the protons and electrons needed to make clean fuels and materials.

Dr. Harry Gray
Arnold O. Beckman Professor of Chemistry
Founding Director, Beckman Institute
Division of Chemistry and Chemical Engineering

Course schedule for Spring 2021

(Preliminary schedule -- subject to frequent adjustment.)

	Lecture date	JDJ Reading	Topic	HW	Due date
1	Wed: 01/27/2021	Chap. 1 & Appen.	Introduction, units and Poisson equation	#1	01/29/2021
2	Fri: 01/29/2021	Chap. 1	Electrostatic energy calculations	#2	02/01/2021
3	Mon: 02/01/2021	Chap. 1 & 2	Electrostatic potentials and fields	#3	02/03/2021
4	Wed: 02/03/2021	Chap. 1 - 3	Poisson's equation in 2 and 3 dimensions	#4	02/05/2021
5	Fri: 02/05/2021	Chap. 1 - 3	Brief introduction to numerical methods	#5	02/08/2021
6	Mon: 02/08/2021	Chap. 2 & 3	Image charge constructions	#6	02/10/2021
7	Wed: 02/10/2021	Chap. 2 & 3	Cylindrical and spherical geometries		
8	Fri: 02/12/2021	Chap. 3 & 4	Spherical geometry and multipole moments	#7	02/15/2021
9	Mon: 02/15/2021	Chap. 4	Dipoles and Dielectrics	#8	02/19/2021
10	Wed: 02/17/2021	Chap. 4	Dipoles and Dielectrics		
11	Fri: 02/19/2021	Chap. 4	Polarization and Dielectrics	#9	02/24/2021
12	Mon: 02/22/2021	Chap. 5	Magnetostatics	#10	02/26/2021
13	Wed: 02/24/2021	Chap. 5	Magnetic dipoles and hyperfine interaction	#11	03/01/2021
14	Fri: 02/26/2021	Chap. 5	Magnetic dipoles and dipolar fields		
15	Mon: 03/01/2021	Chap. 6	Maxwell's Equations	#12	03/08/2021
16	Wed: 03/03/2021	Chap. 6	Electromagnetic energy and forces		
17	Fri: 03/05/2021	Chap. 7	Electromagnetic plane waves		
18	Mon: 03/08/2021	Chap. 7	Electromagnetic plane waves	#13	03/10/2021
19	Wed: 03/10/2021	Chap. 7	Optical effects of refractive indices	#14	03/12/2021
20	Fri: 03/12/2021	Chap. 1-7	Review		
	Mon: 03/15/2021	No class	*APS March Meeting*	Take Home Exam	

Please submit topic suggestions for Friday's review –

Questions for today—

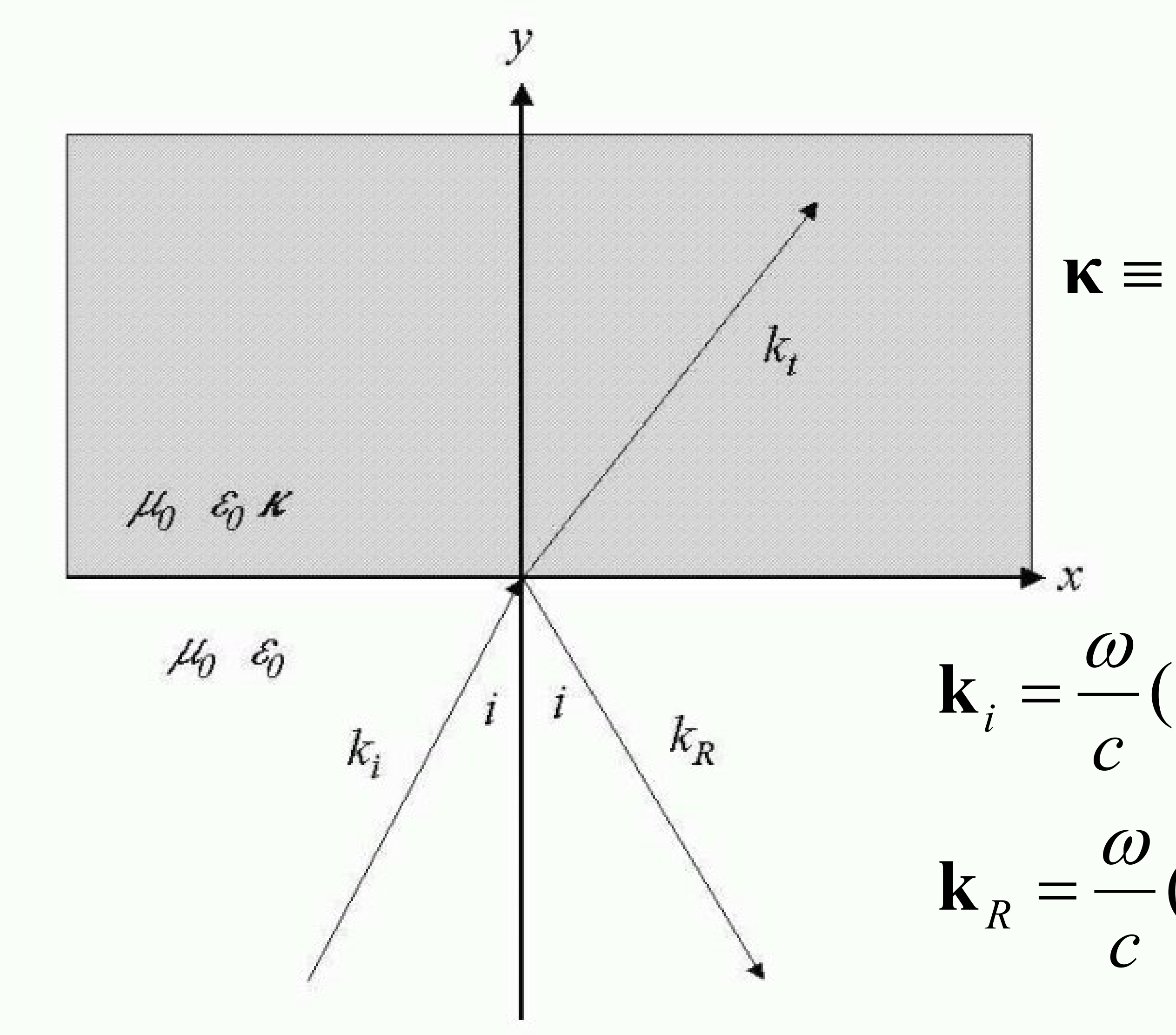
From Nick -- Can you go over how we proved the [dielectric] function was analytic?

From Derick – (Paraphrased) How do the Fresnel equations for isotropic and anisotropic materials compare?

From Tim -- In slide 20 are permittivity function and dielectric function the same function?In slide 31 how are we able to split the integral of f(z) into f(z_r) and f(z). What is the difference between z_r and z?

From Gao -- Why do we study Kramers-Kronig transform? Do you have application examples?

Extension of reflection/refractionanalysis to anisotropic media --



assume

$$\boldsymbol{\kappa} \equiv \begin{pmatrix} \kappa_{xx} & 0 & 0 \\ 0 & \kappa_{yy} & 0 \\ 0 & 0 & \kappa_{zz} \end{pmatrix}$$

$$\mathbf{k}_i = \frac{\omega}{c}(\sin i\hat{\mathbf{x}} + \cos i\hat{\mathbf{y}}),$$

$$\mathbf{k}_R = \frac{\omega}{c}(\sin i\hat{\mathbf{x}} - \cos i\hat{\mathbf{y}}).$$

For s-polarization (E fields along z-axis)

$$\frac{E_0''}{E_0} = \frac{\cos i - n_y}{\cos i + n_y}.$$

$$n_y^2 = \kappa_{zz} - \sin^2 i$$

For p-polarization (E fields in x-y plane)

$$\frac{E_0''}{E_0} = \frac{\kappa_{xx} \cos i - n_y}{\kappa_{xx} \cos i + n_y}.$$

$$n_y^2 = \frac{\kappa_{xx}}{\kappa_{yy}}(\kappa_{yy} - \sin^2 i).$$

Example of a birefringent crystal --

Some comments on the Fresnel Equations

1. Different behaviors of *s* and *p* polarization
2. Brewster's angle
3. Total internal reflection

Review: Electromagnetic plane waves in isotropic medium with linear and real permeability and permittivity: μ ε.

$$\mathbf{E}(\mathbf{r},t) = \Re\left(\mathbf{E}_0 e^{i\frac{\omega}{c}\left(n\hat{\mathbf{k}}\cdot\mathbf{r}-ct\right)}\right) \qquad n^2 = c^2\mu\varepsilon$$

$$\mathbf{B}(\mathbf{r},t) = \frac{n}{c}\hat{\mathbf{k}}\times\mathbf{E}(\mathbf{r},t) = \sqrt{\mu\varepsilon}\,\hat{\mathbf{k}}\times\mathbf{E}(\mathbf{r},t)$$

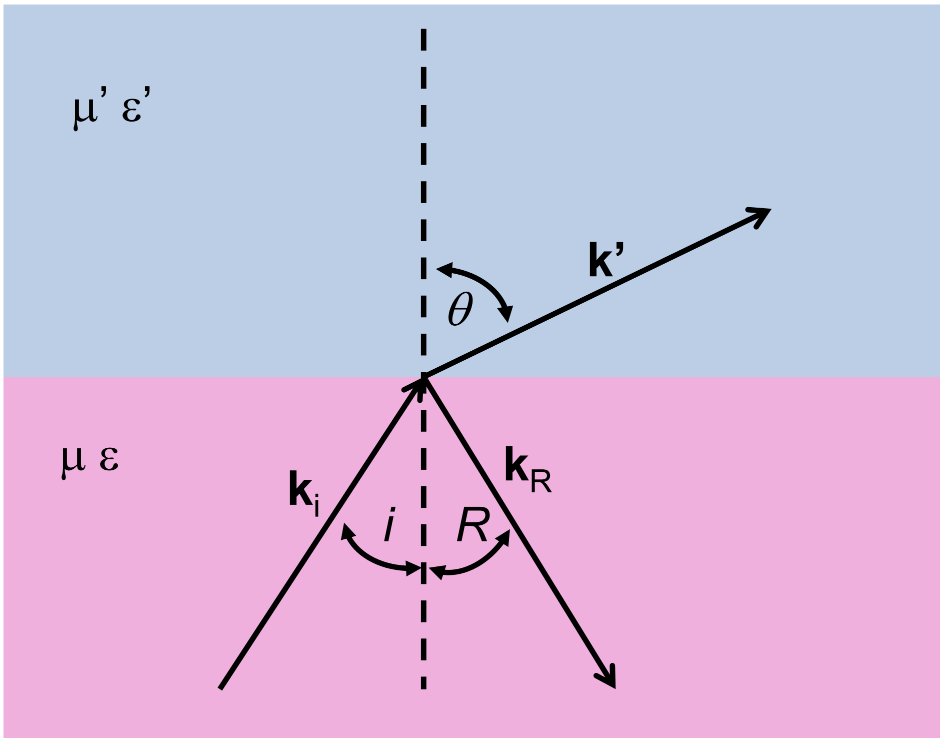
Poynting vector for plane electromagnetic waves:

$$\langle\mathbf{S}\rangle_{avg} = \frac{n|\mathbf{E}_0|^2}{2\mu c}\hat{\mathbf{k}} = \frac{1}{2}\sqrt{\frac{\varepsilon}{\mu}}|\mathbf{E}_0|^2\hat{\mathbf{k}}$$

Energy density for plane electromagnetic waves:

$$\langle u\rangle_{avg} = \frac{1}{2}\varepsilon|\mathbf{E}_0|^2$$

Review:
Reflection and refraction of plane electromagnetic waves at a plane interface between dielectrics (assumed to be lossless)



$$n' = \varepsilon' \mu'$$

$$n = \varepsilon\mu$$

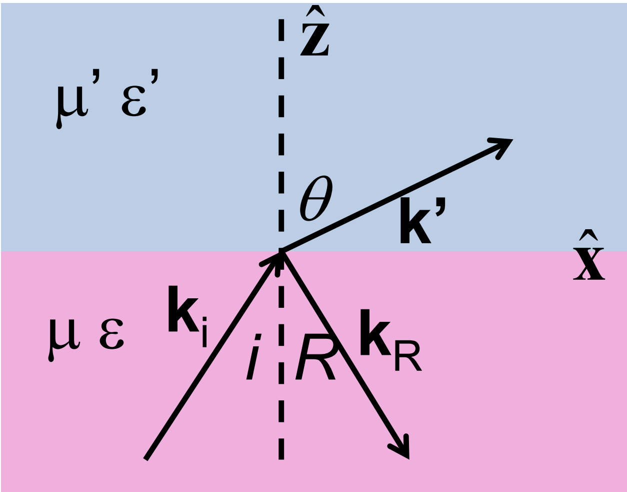
$$i = R$$

$$n \sin i = n' \sin\theta$$

$$|\mathbf{k}_i| = |\mathbf{k}_R| = n\frac{\omega}{c}$$

$$|\mathbf{k}'| = n'\frac{\omega}{c}$$

Review:
Reflection and refraction between two isotropic media



Reflectance, transmittance :

$$\mathrm{R} = \frac{\mathbf{S}_R \cdot \hat{\mathbf{z}}}{\mathbf{S}_i \cdot \hat{\mathbf{z}}} = \left| \frac{E_{0R}}{E_{0i}} \right|^2 \qquad \mathrm{T} = \frac{\mathbf{S}' \cdot \hat{\mathbf{z}}}{\mathbf{S}_i \cdot \hat{\mathbf{z}}} = \left| \frac{E'_0}{E_{0i}} \right|^2 \frac{n'}{n} \frac{\mu}{\mu'} \frac{\cos\theta}{\cos i}$$

Note that $\quad \mathrm{R} + \mathrm{T} = 1$

For s-polarization (E perpendicular to plane of incidence)

$$\frac{E_{0R}}{E_{0i}} = \frac{n\cos i - \frac{\mu}{\mu'}n'\cos\theta}{n\cos i + \frac{\mu}{\mu'}n'\cos\theta} \qquad \frac{E'_0}{E_{0i}} = \frac{2n\cos i}{n\cos i + \frac{\mu}{\mu'}n'\cos\theta}$$

Note that: $n'\cos\theta = \sqrt{n'^2 - n^2\sin^2 i}$

For p-polarization (E in plane of incidence)

$$\frac{E_{0R}}{E_{0i}} = \frac{\frac{\mu}{\mu'}n'^2\cos i - nn'\cos\theta}{\frac{\mu}{\mu'}n'^2\cos i + nn'\cos\theta} \qquad \frac{E'_0}{E_{0i}} = \frac{2nn'\cos i}{\frac{\mu}{\mu'}n'^2\cos i + nn'\cos\theta}$$

Note that: $n'\cos\theta = \sqrt{n'^2 - n^2\sin^2 i}$

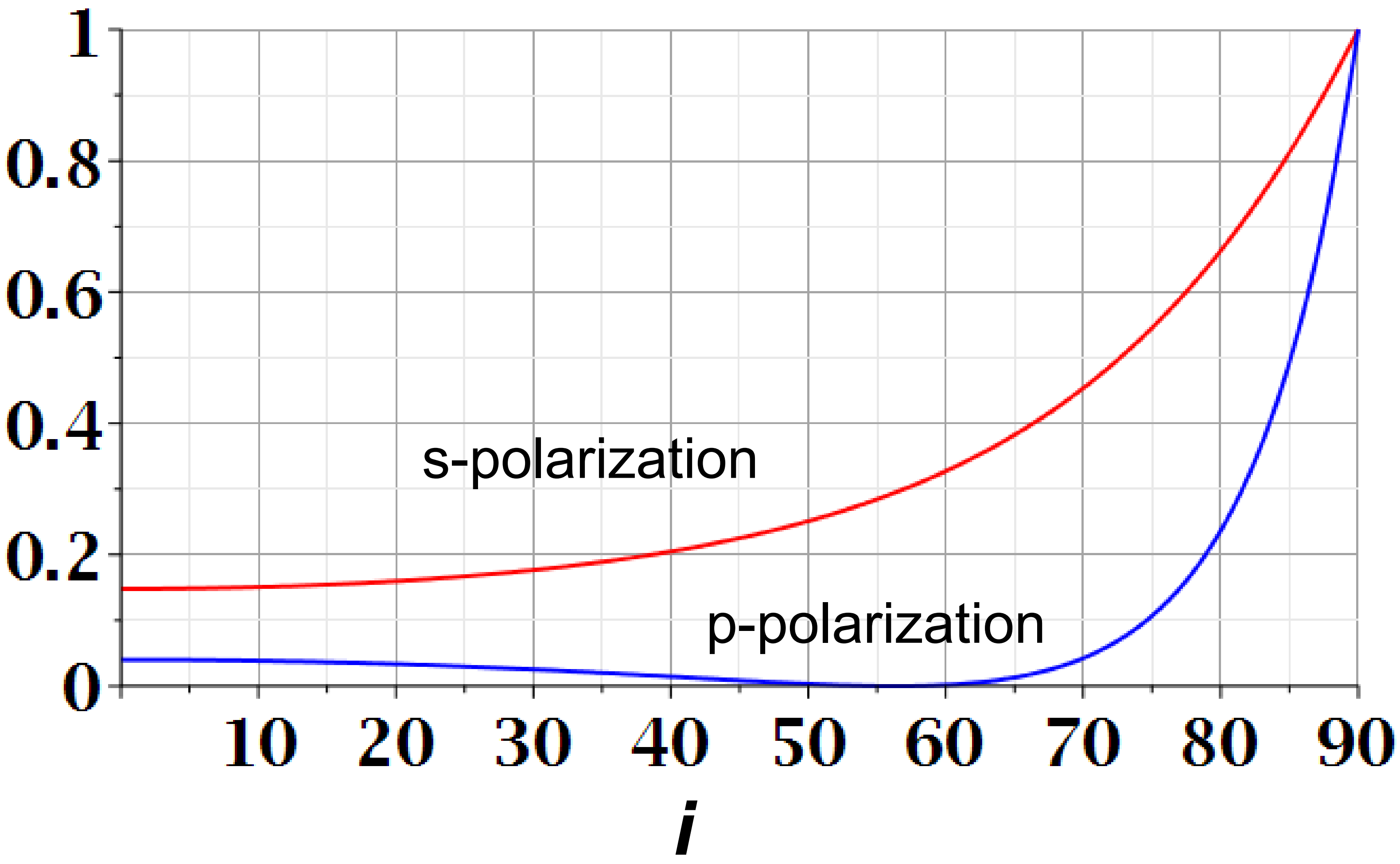
Reflectance for s-polarization

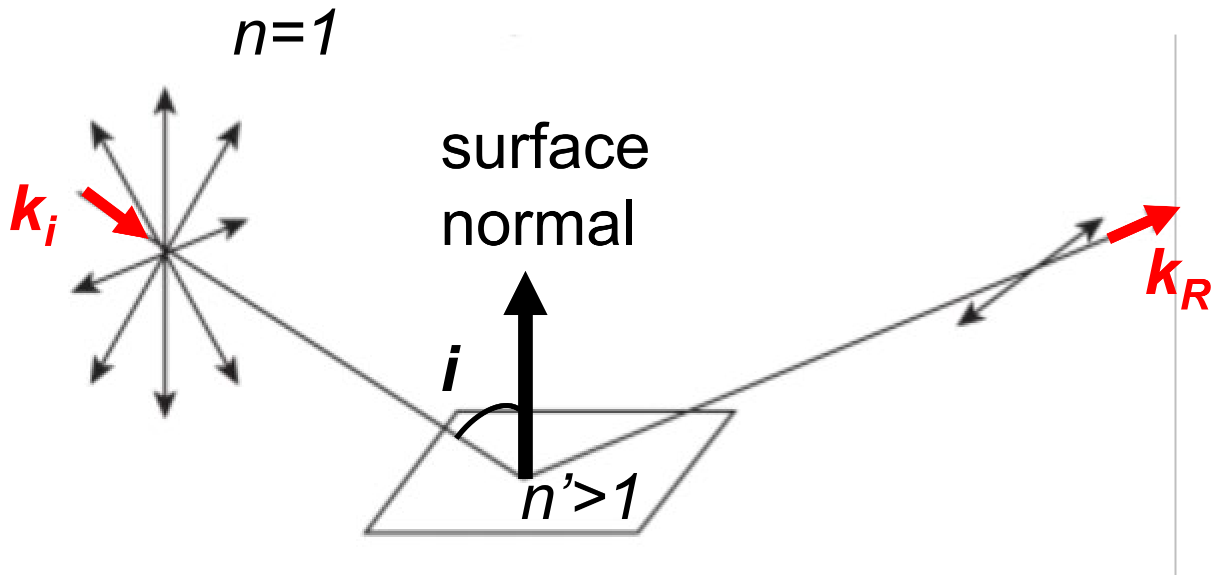
$$R_s = \left|\frac{E_{0R}}{E_{0i}}\right|^2 = \left|\frac{n\cos i - \frac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin^2 i}}{n\cos i + \frac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin^2 i}}\right|^2$$

Reflectance for p-polarization

$$R_p = \left|\frac{E_{0R}}{E_{0i}}\right|^2 = \left|\frac{\frac{\mu}{\mu'}n'\cos i - \frac{n}{n'}\sqrt{n'^2 - n^2\sin^2 i}}{\frac{\mu}{\mu'}n'\cos i + \frac{n}{n'}\sqrt{n'^2 - n^2\sin^2 i}}\right|^2$$

Example for $\mu = \mu'$; $n = 1$ and $n' = 1.5$





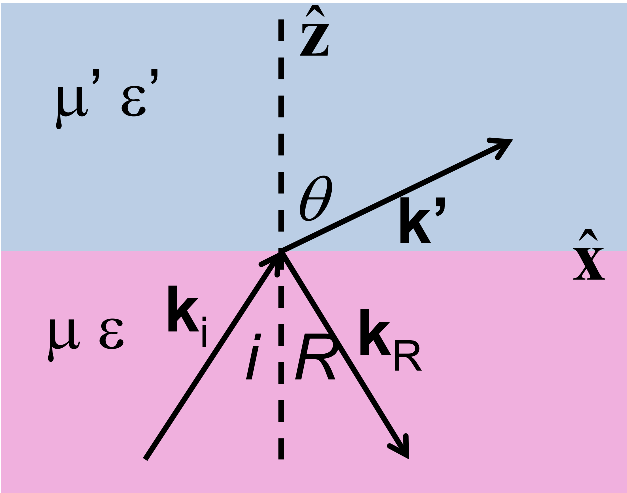
Polarization due to reflection from a refracting surface

Brewster's angle: for $i = i_B$, $\mathrm{R}_p(i_B) = 0$

$$R_p = \left|\frac{E_{0R}}{E_{0i}}\right|^2 = \left|\frac{\frac{\mu}{\mu'}n'\cos i - \frac{n}{n'}\sqrt{n'^2 - n^2\sin^2 i}}{\frac{\mu}{\mu'}n'\cos i + \frac{n}{n'}\sqrt{n'^2 - n^2\sin^2 i}}\right|^2$$

For $\mu' = \mu$, $\quad i_B = \tan^{-1}\left(\frac{n'}{n}\right)$

Reflection and refraction between two isotropic media -- continued



For each wave:

$$\mathbf{E}(\mathbf{r},t) = \Re\left(\mathbf{E}_0 e^{i\frac{\omega}{c}\left(n\hat{\mathbf{k}}\cdot\mathbf{r}-ct\right)}\right) \qquad n^2 = c^2\mu\varepsilon$$

$$\mathbf{B}(\mathbf{r},t) = \frac{n}{c}\hat{\mathbf{k}}\times\mathbf{E}(\mathbf{r},t) = \sqrt{\mu\varepsilon}\,\hat{\mathbf{k}}\times\mathbf{E}(\mathbf{r},t)$$

Matching condition at interface:

$$n'\cos\theta = \sqrt{n'^2 - n^2\sin^2 i}$$

Total internal reflection:

If $n > n'$, for $i > i_0 \equiv \sin^{-1}\left(\frac{n'}{n}\right)$,

refracted field no longer propagates in medium $\mu'\ \varepsilon'$

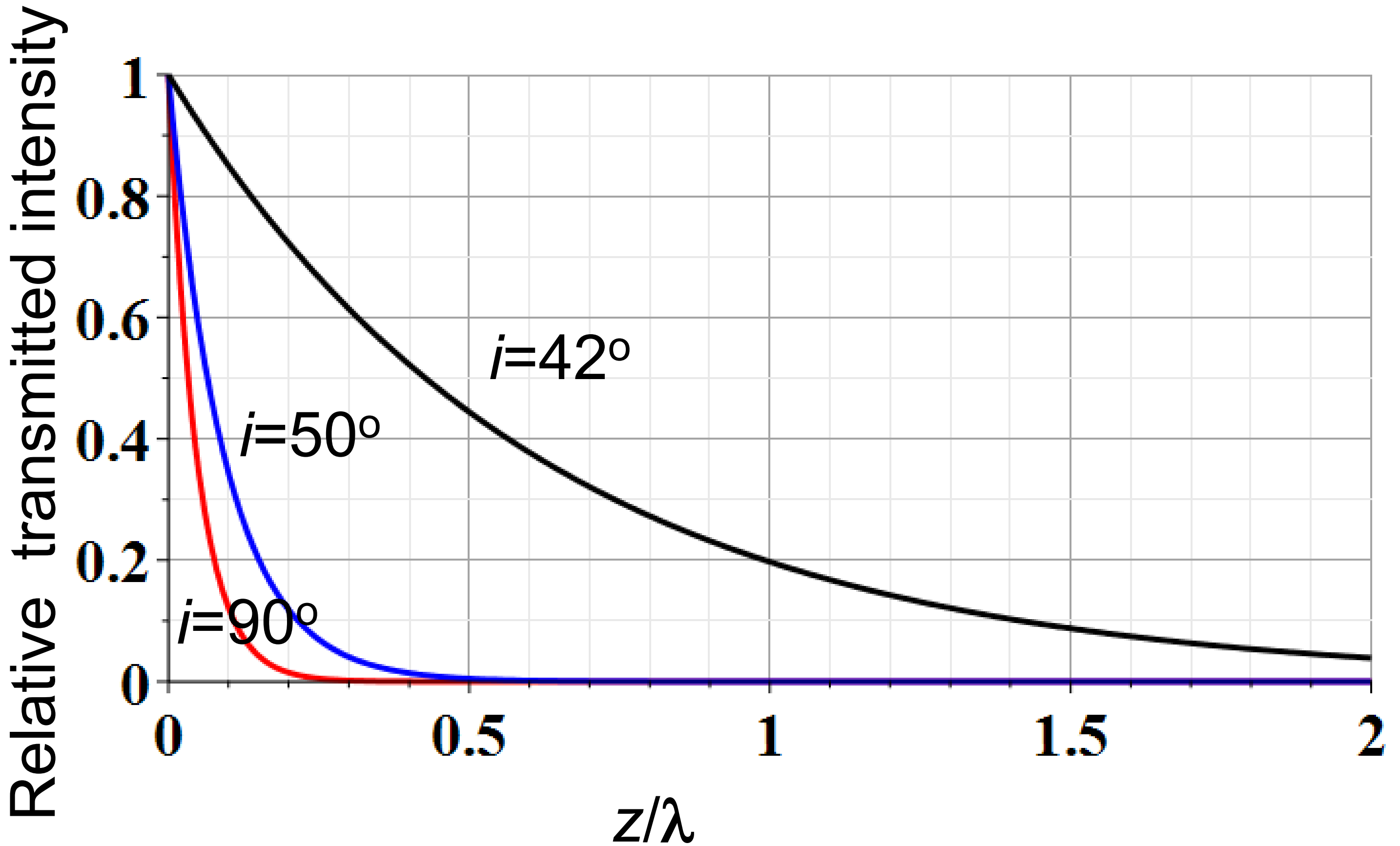
For $i > i_0$

$$n'\cos\theta = i\sqrt{n^2\sin^2 i - n'^2} = i\,n\sqrt{\frac{\sin^2 i}{\sin^2 i_0} - 1}$$

$$\mathbf{E}'(\mathbf{r},t) = e^{-\left(\frac{n\omega}{c}\sqrt{\frac{\sin^2 i}{\sin^2 i_0}-1}\right)z}\,\Re\left(\mathbf{E}'_0\, e^{i\frac{\omega}{c}\left(n\hat{\mathbf{k}}_{\parallel}\cdot\mathbf{r}-ct\right)}\right)$$

Example of total internal reflection

$n'=1$ and $n=1.5$ ➔ $i_0 = \sin^{-1}(1/1.5)=41.81^o$



Transmitted illumination confined within a few wavelengths of the surface.

TIRF (total internal reflection fluorescence)

www.nikon.com/products/microscope-solutions/bioscience.../nikon_note_10_lr.pdf

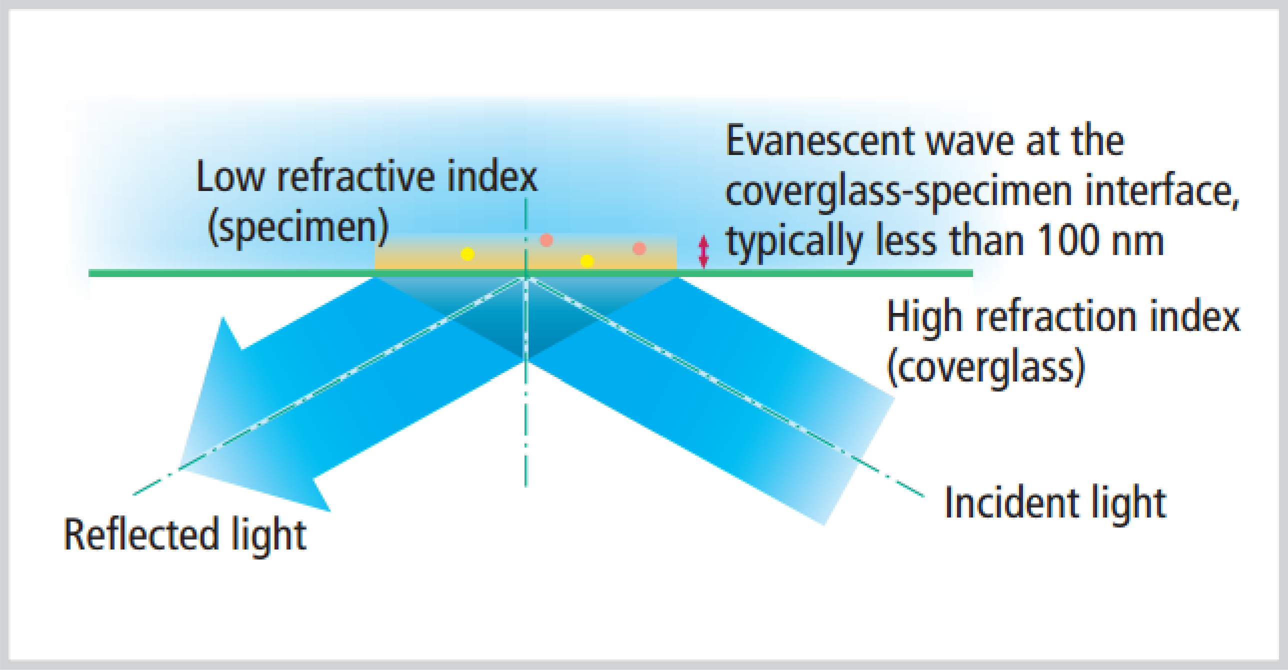


Figure 1: Creation of an evanescent wave at the coverglass-specimen interface

Design of TIRF device using laser and high power lens

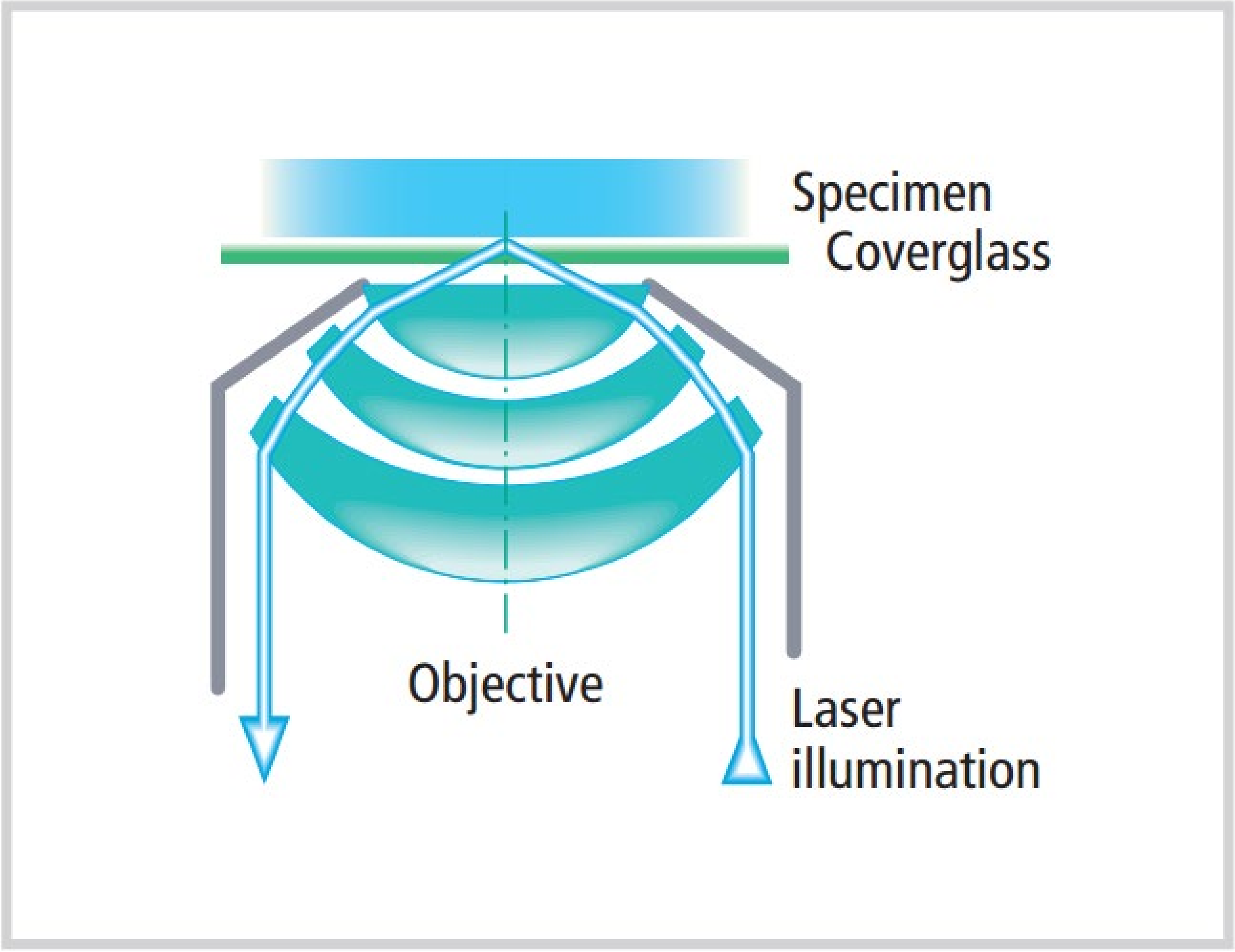
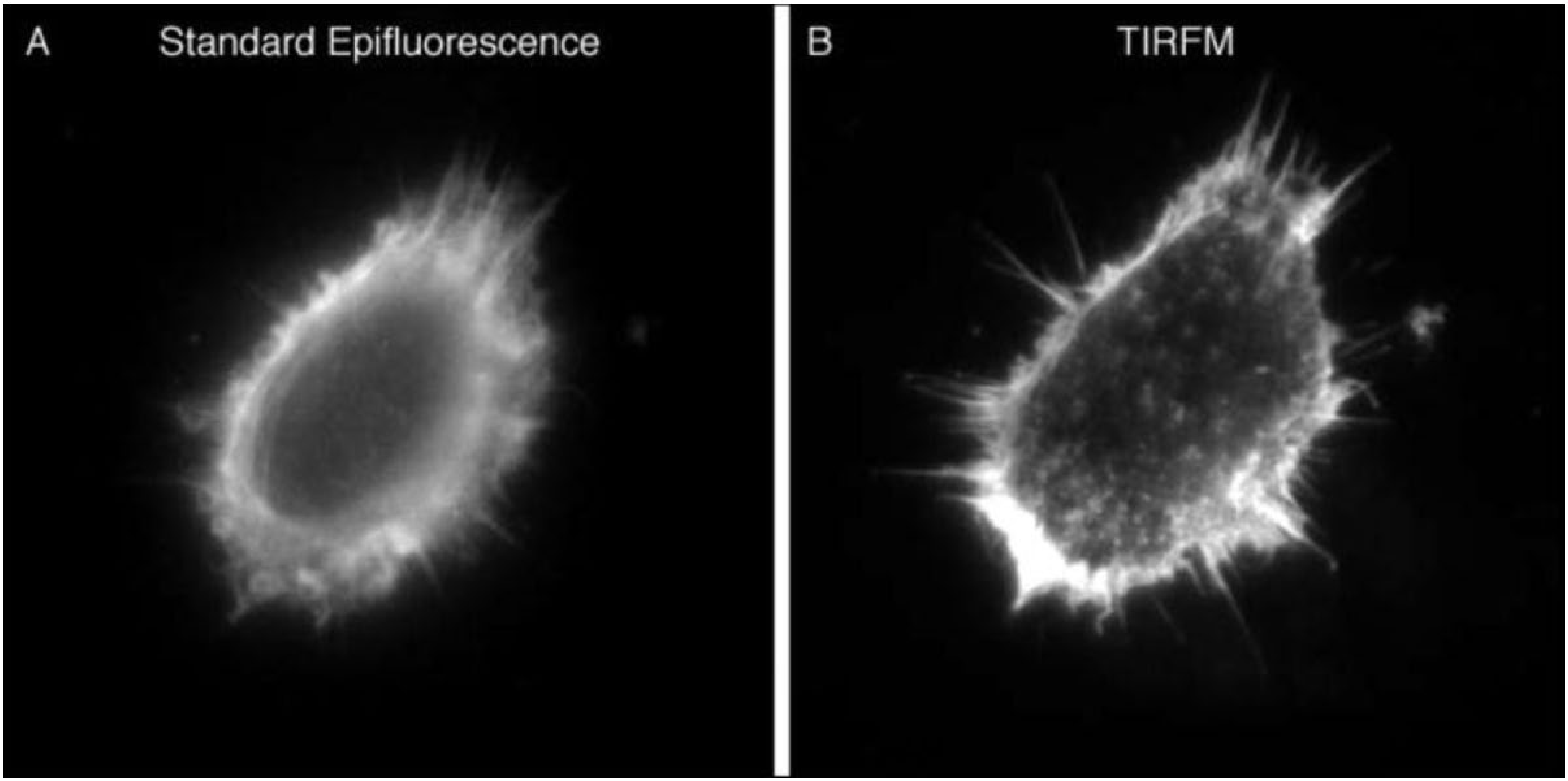


Figure 2: Through-the-lens laser TIRF.

PMC full text: Curr Protoc Cytom. Author manuscript; available in PMC 2015 Aug 18.
Published in final edited form as:
Curr Protoc Cytom. 2009 Oct; 0 12: Unit12.18.
doi: 10.1002/0471142956.cy1218s50

Figure 1



Special case: normal incidence (i=0, θ=0)

$$\frac{E_{0R}}{E_{0i}} = \frac{\frac{\mu}{\mu'}n' - n}{\frac{\mu}{\mu'}n' + n} \qquad \frac{E'_0}{E_{0i}} = \frac{2n}{\frac{\mu}{\mu'}n' + n}$$

Reflectance, transmittance:

$$\mathrm{R} = \left|\frac{E_{0R}}{E_{0i}}\right|^2 = \left|\frac{\frac{\mu}{\mu'}n' - n}{\frac{\mu}{\mu'}n' + n}\right|^2$$

$$\mathrm{T} = \left|\frac{E'_0}{E_{0i}}\right|^2 \frac{n'}{n}\frac{\mu}{\mu'} = \left|\frac{2n}{\frac{\mu}{\mu'}n' + n}\right|^2 \frac{n'}{n}\frac{\mu}{\mu'}$$

Extension to complex refractive index $n = n_R + i\, n_I$

Suppose $\mu = \mu'$, $n =$ real, $n' = n'_R + i n'_I$

Reflectance at normal incidence:

$$\mathrm{R} = \left|\frac{E_{0R}}{E_{0i}}\right|^2 = \left|\frac{\frac{\mu}{\mu'}n' - n}{\frac{\mu}{\mu'}n' + n}\right|^2 = \frac{(n'_R - n)^2 + (n'_I)^2}{(n'_R + n)^2 + (n'_I)^2}$$

Note that for $n'_I \gg |n'_R \pm n|$:

$$\mathrm{R} \approx 1$$

Origin of imaginary contributions to permittivity --
Review: Drude model dielectric function:

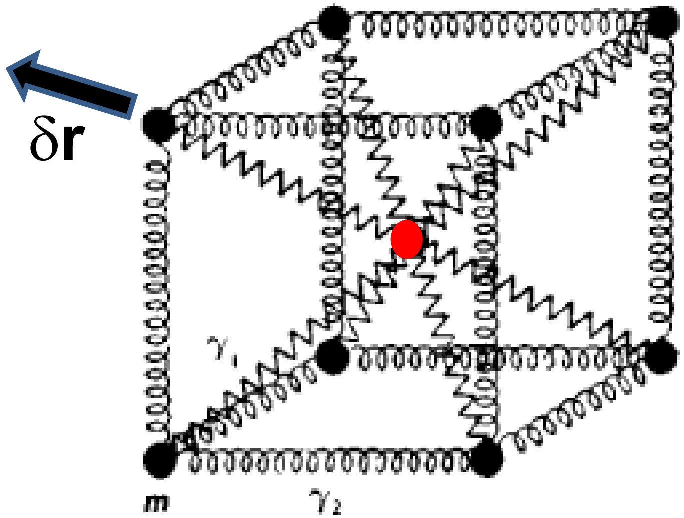
$$\frac{\varepsilon(\omega)}{\varepsilon_0} = 1 + N\sum_i f_i \frac{q_i^{\,2}}{\varepsilon_0 m_i} \frac{1}{\omega_i^2 - \omega^2 - i\omega\gamma_i}$$

$$= \frac{\varepsilon_R(\omega)}{\varepsilon_0} + i\frac{\varepsilon_I(\omega)}{\varepsilon_0}$$

$$\frac{\varepsilon_R(\omega)}{\varepsilon_0} = 1 + N\sum_i f_i \frac{q_i^{\,2}}{\varepsilon_0 m_i} \frac{\omega_i^2 - \omega^2}{\left(\omega_i^2 - \omega^2\right)^2 + \omega^2\gamma_i^{\,2}}$$

$$\frac{\varepsilon_I(\omega)}{\varepsilon_0} = N\sum_i f_i \frac{q_i^{\,2}}{\varepsilon_0 m_i} \frac{\omega\gamma_i}{\left(\omega_i^2 - \omega^2\right)^2 + \omega^2\gamma_i^{\,2}}$$

Extensions of the Drude model for lattice vibrations



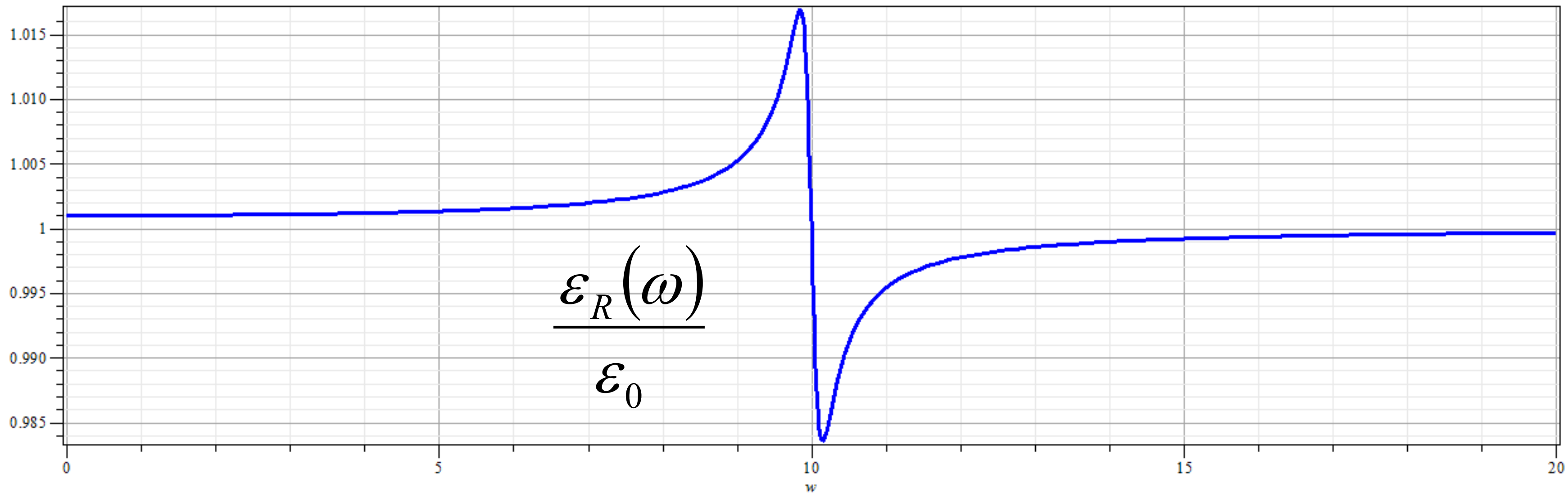
In principle, the ideas of the Drude model apply both to the ionic vibrations which occur at low frequency (~10^{12} Hz) contributing to the so called static permittivity function ε_s and to the electronic vibrations which occur at high frequency (~10^{15} Hz) contributing to the so called high frequency permittivity function ε_∞.

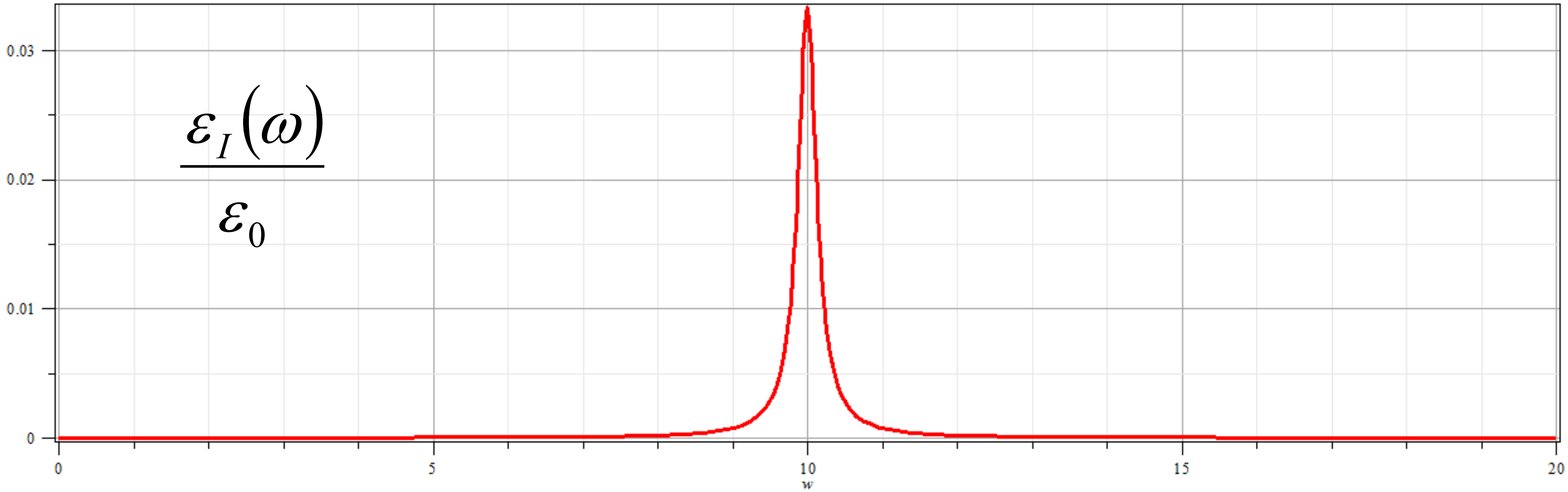
In this model at high frequencies, only the electrons contribute to the polarization: $\epsilon_\infty = \epsilon_0 + \frac{|\mathbf{P}_{electron}|}{|\mathbf{E}|}$

At low frequencies both electrons and ions contribute to the polarization: $\epsilon_s = \epsilon_0 + \frac{|\mathbf{P}_{electron}|}{|\mathbf{E}|} + \frac{|\mathbf{P}_{ion}|}{|\mathbf{E}|}$

$$\Rightarrow \frac{|\mathbf{P}_{ion}|}{|\mathbf{E}|} = \epsilon_s - \epsilon_\infty$$

Drude model dielectric function:





Drude model dielectric function – some analytic properties:

$$\frac{\varepsilon(\omega)}{\varepsilon_0} = 1 + N\sum_i f_i \frac{q_i^2}{\varepsilon_0 m_i} \frac{1}{\omega_i^2 - \omega^2 - i\omega\gamma_i}$$

$$\text{For } \omega >> \omega_i \qquad \frac{\varepsilon(\omega)}{\varepsilon_0} \approx 1 - \frac{1}{\omega^2}\left(N\sum_i f_i \frac{q_i^2}{\varepsilon_0 m_i} \right)$$

$$\equiv 1 - \frac{\omega_P^2}{\omega^2}$$

Further comments on analytic behavior of dielectric function

"Causal" relationship between **E** and **D** fields:

$$\mathbf{D}(\mathbf{r},t) = \varepsilon_0 \left\{ \mathbf{E}(\mathbf{r},t) + \int_0^\infty d\tau \; G(\tau)\mathbf{E}(\mathbf{r},t-\tau) \right\}$$

$$\frac{\varepsilon(\omega)}{\varepsilon_0} - 1 = \int_0^\infty d\tau \; G(\tau) e^{i\omega\tau}$$

Some details: Consider a convolution integral such as

$$f(t) = \int_{-\infty}^{\infty} g(t')h(t-t')dt' \quad \text{where the functions } f(t), g(t), \text{ and } h(t)$$

are all well-defined functions with Fourier transforms such as

$$\tilde{f}(\omega) = \int_{-\infty}^{\infty} f(t')e^{i\omega t'}dt' \qquad f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(\omega)e^{-i\omega t}d\omega$$

It follows that: $\tilde{f}(\omega) = \tilde{g}(\omega)\tilde{h}(\omega)$

Further comments on analytic behavior of dielectric function

"Causal" relationship between **E** and **D** fields:

$$\mathbf{D}(\mathbf{r},t) = \varepsilon_0 \left\{ \mathbf{E}(\mathbf{r},t) + \int_0^{\infty} d\tau \; G(\tau)\mathbf{E}(\mathbf{r},t-\tau) \right\}$$

$$G(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left(\frac{\varepsilon(\omega)}{\varepsilon_0} - 1 \right) e^{-i\omega\tau} d\omega \qquad \tilde{G}(\omega) = \frac{\varepsilon(\omega)}{\varepsilon_0} - 1 = \int_0^{\infty} d\tau \; G(\tau) e^{i\omega\tau}$$

For $\frac{\varepsilon(\omega)}{\varepsilon_0} - 1 = \frac{N}{\varepsilon_0}\sum_i f_i \frac{q_i^2}{m_i} \frac{1}{\omega_i^2 - \omega^2 - i\omega\gamma_i}$

$$G(\tau) = \frac{N}{\varepsilon_0}\sum_i f_i \frac{q_i^2}{m_i} e^{-\gamma_i\tau/2} \frac{\sin(\nu_i\tau)}{\nu_i}\Theta(\tau)$$

where $\nu_i \equiv \sqrt{\omega_i^2 - \gamma_i^2 / 4}$

Some details

$$G(\tau)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{\varepsilon(\omega)}{\varepsilon_0}-1\right)e^{-i\omega\tau}d\omega=\frac{1}{2\pi}\oint f(z)e^{-iz\tau}dz$$

Let $f(z)=\frac{\varepsilon(z)}{\varepsilon_0}-1=N\sum_i f_i\frac{q_i^{\ 2}}{\varepsilon_0 m_i}\frac{1}{\omega_i^2-z^2-iz\gamma_i}$

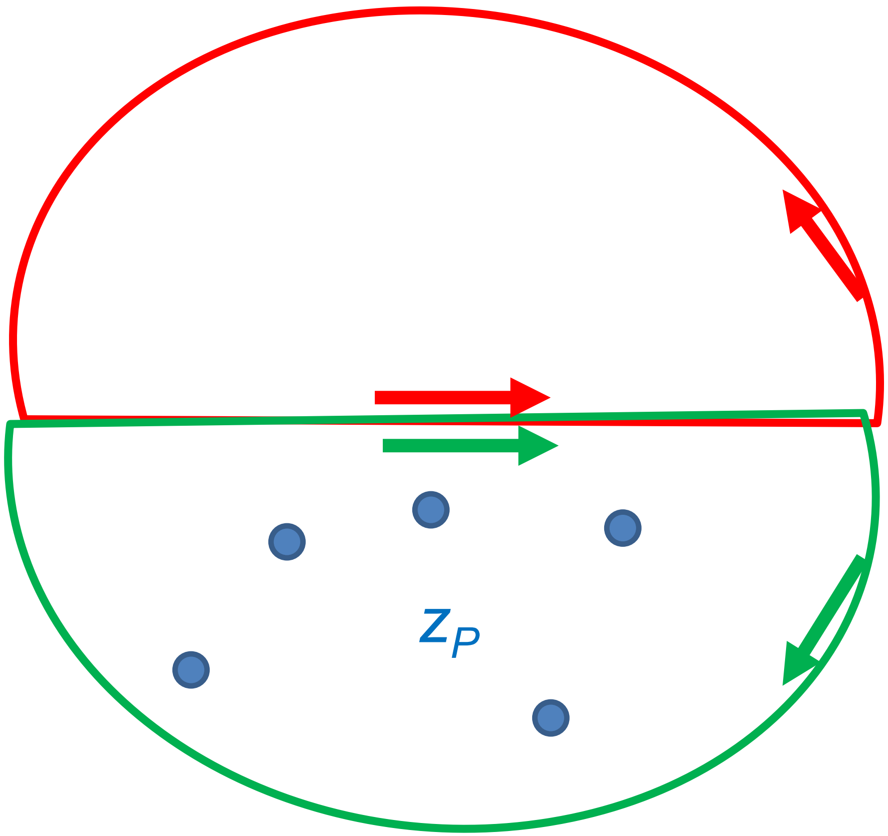
$f(z)$ has poles z_P at $\omega_i^2-z_P^{\ 2}-iz_P\gamma_i=0$

$$z_P=-i\frac{\gamma_i}{2}\pm\sqrt{\omega_i^2-\left(\frac{\gamma_i}{2}\right)^2}\quad\text{or}\quad z_P=-i\left(\frac{\gamma_i}{2}\pm\sqrt{\left(\frac{\gamma_i}{2}\right)^2-\omega_i^2}\right)$$

Note that numerically, $\mathrm{Im}(z_P)<0$

$$G(\tau) = \frac{1}{2\pi}\oint f(z)e^{-iz\tau}dz = i\sum_P \text{Res}(z_P)$$

Note that: $e^{-iz\tau} = e^{-iz_R\tau}e^{z_I\tau}$



Valid contour for $\tau < 0$

$G(\tau) = 0$ for $\tau < 0$

Valid contour for $\tau > 0$

$G(\tau) =$

$$\frac{N}{\varepsilon_0}\sum_i f_i \frac{q_i^2}{m_i} e^{-\gamma_i\tau/2} \frac{\sin(\nu_i\tau)}{\nu_i}$$

$$G(\tau) = \frac{1}{2\pi}\oint f(z) e^{-iz\tau} dz = i \sum_P \text{Res}(z_P)$$

Let $\quad f(z) = \frac{\varepsilon(z)}{\varepsilon_0} - 1 = N \sum_i f_i \frac{q_i^2}{\varepsilon_0 m_i} \frac{1}{\omega_i^2 - z^2 - iz\gamma_i}$

$f(z)$ has poles z_P at $\omega_i^2 - z_P{}^2 - iz_P\gamma_i = 0$

$$z_P = -i\frac{\gamma_i}{2} \pm \sqrt{\omega_i^2 - \left(\frac{\gamma_i}{2}\right)^2} \quad \text{or} \quad z_P = -i\left(\frac{\gamma_i}{2} \pm \sqrt{\left(\frac{\gamma_i}{2}\right)^2 - \omega_i^2}\right)$$

$$G(\tau) = \frac{N}{\varepsilon_0} \sum_i f_i \frac{q_i^2}{m_i} e^{-\gamma_i \tau/2} \frac{\sin(\nu_i \tau)}{\nu_i} \Theta(\tau)$$

where $\nu_i \equiv \sqrt{\omega_i^2 - \gamma_i^2 / 4}$ assuming $\omega_i^2 - \gamma_i^2 / 4 \geq 0$

Analysis for Drude model dielectric function – continued --

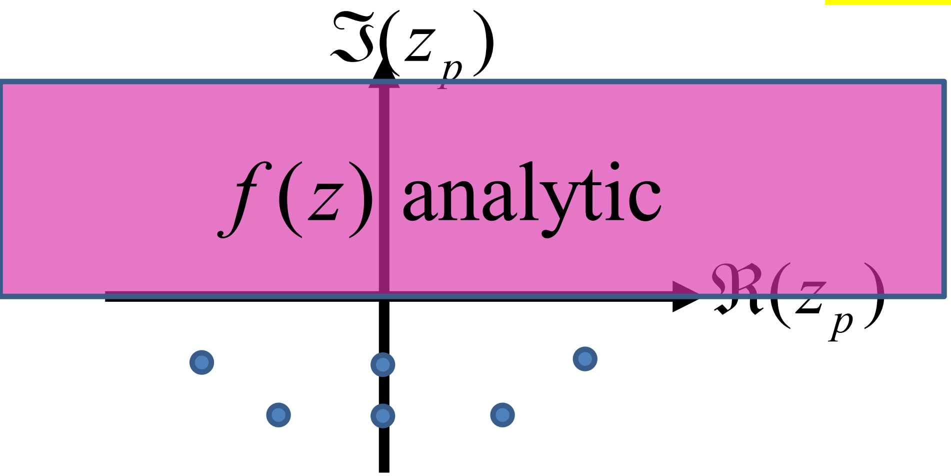
Analytic properties:

$$f(z) = \frac{\varepsilon(z)}{\varepsilon_0} - 1 = N \sum_i f_i \frac{q_i^2}{\varepsilon_0 m_i} \frac{1}{\omega_i^2 - z^2 - iz\gamma_i}$$

$f(z)$ has poles z_P at $\omega_i^2 - z_P^2 - iz_P\gamma_i = 0$

$$z_P = -i\frac{\gamma_i}{2} \pm \sqrt{\omega_i^2 - \left(\frac{\gamma_i}{2}\right)^2}$$

Note that $\Im(z_P) \leq 0 \Rightarrow f(z)$ is analytic for $\Im(z_P) > 0$



Because of these analytic properties, Cauchy's integral theorem results in:

Kramers-Kronig transform – for dielectric function:

$$\frac{\varepsilon_R(\omega)}{\varepsilon_0} - 1 = \frac{1}{\pi} P \int_{-\infty}^{\infty} d\omega' \frac{\varepsilon_I(\omega')}{\varepsilon_0} \frac{1}{\omega' - \omega}$$

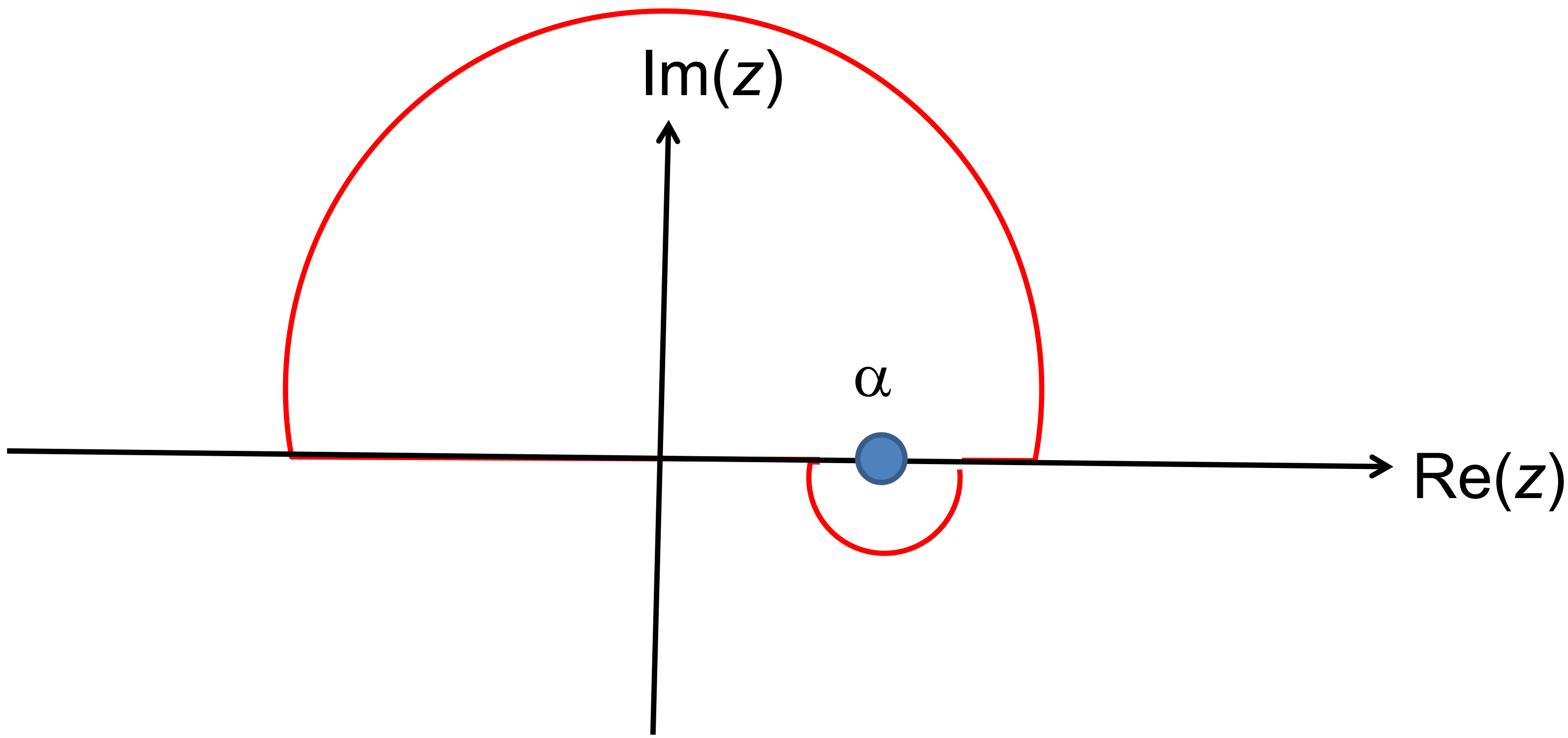
$$\frac{\varepsilon_I(\omega)}{\varepsilon_0} = -\frac{1}{\pi} P \int_{-\infty}^{\infty} d\omega' \left(\frac{\varepsilon_R(\omega')}{\varepsilon_0} - 1 \right) \frac{1}{\omega' - \omega}$$

$$\text{with} \quad \varepsilon_R(-\omega) = \varepsilon_R(\omega); \quad \varepsilon_I(-\omega) = -\varepsilon_I(\omega)$$

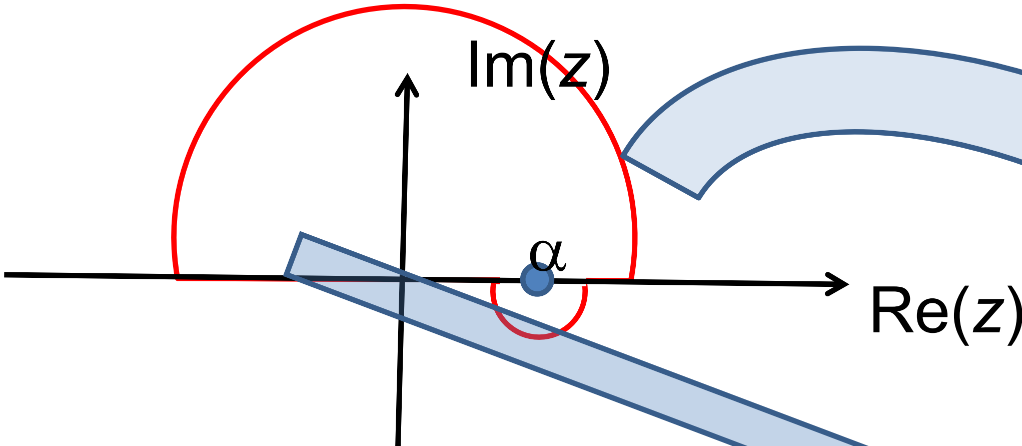
Analytic properties of the dielectric function (in the Drude model or from "first principles" -- Kramers-Kronig transform

Consider Cauchy's integral formula for an analytic function $f(z)$:

$$\oint dz\, f(z) = 0 \qquad\qquad f(\alpha) = \frac{1}{2\pi i} \oint_{includes\ \alpha} dz\, \frac{f(z)}{z - \alpha}$$



Kramers-Kronig transform -- continued

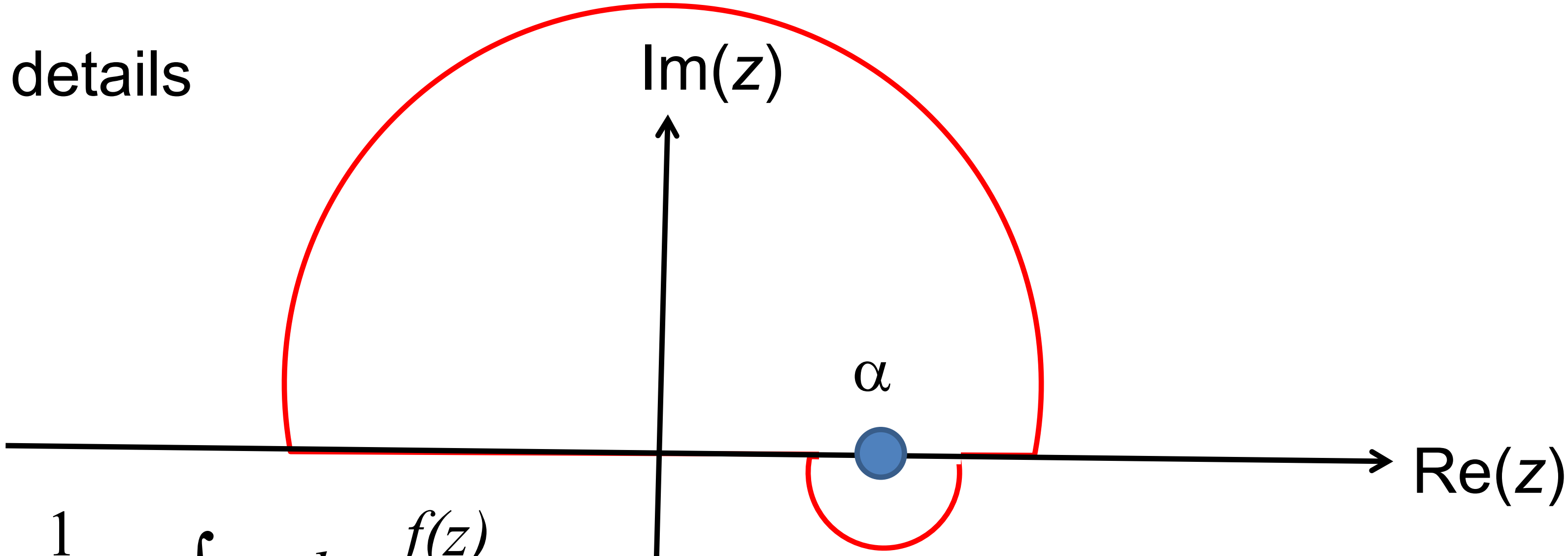


=0

$$f(\alpha)=\frac{1}{2\pi i}\oint_{includes\ \alpha} dz\ \frac{f(z)}{z-\alpha}=\frac{1}{2\pi i}\left(\int_{-\infty}^{\infty} dz_R\ \frac{f(z_R)}{z_R-\alpha}+\int_{rest} dz\ \frac{f(z)}{z-\alpha}\right)$$

$$f(\alpha)=\frac{1}{2\pi i}\int_{-\infty}^{\infty} dz_R\,\frac{f(z_R)}{z_R-\alpha}=\frac{1}{2\pi i}P\int_{-\infty}^{\infty} dz_R\,\frac{f(z_R)}{z_R-\alpha}+\frac{1}{2}f(\alpha)$$

Some details



$$f(\alpha) = \frac{1}{2\pi i} \oint_{includes\ \alpha} dz \, \frac{f(z)}{z - \alpha}$$

$$= \frac{1}{2\pi i} \left(P \int_{-\infty}^{\infty} dz_R \, \frac{f(z_R)}{z_R - \alpha} + \int_{\text{small hemisphere}} dz \, \frac{f(z)}{z - \alpha} + \int_{rest} dz \, \frac{f(z)}{z - \alpha} \right)$$

$$\int_{\text{small hemisphere}} dz \, \frac{f(z)}{z - \alpha} = \frac{1}{2} 2\pi i \, f(\alpha)$$

$$\Rightarrow f(\alpha) = \frac{1}{2\pi i} P \int_{-\infty}^{\infty} dz_R \, \frac{f(z_R)}{z_R - \alpha} + \frac{1}{2} f(\alpha)$$

Kramers-Kronig transform -- continued

$$f(\alpha) = \frac{1}{2\pi i} P \int_{-\infty}^{\infty} dz_R \; \frac{f(z_R)}{z_R - \alpha} + \frac{1}{2} f(\alpha)$$

$$\Rightarrow f(\alpha) = \frac{1}{\pi i} P \int_{-\infty}^{\infty} dz_R \; \frac{f(z_R)}{z_R - \alpha}$$

Suppose $\quad f(z_R) = f_R(z_R) + i f_I(z_R)$:

$$\Rightarrow \left(f_R(\alpha) + i f_I(\alpha) \right) = \frac{1}{\pi i} P \int_{-\infty}^{\infty} dz_R \; \frac{f_R(z_R) + i f_I(z_R)}{z_R - \alpha}$$

$$\Rightarrow f_R(\alpha) = \frac{1}{\pi} P \int_{-\infty}^{\infty} dz_R \; \frac{f_I(z_R)}{z_R - \alpha}$$

$$f_I(\alpha) = -\frac{1}{\pi} P \int_{-\infty}^{\infty} dz_R \; \frac{f_R(z_R)}{z_R - \alpha}$$

Kramers-Kronig transform -- continued

$$f_R(\alpha) = \frac{1}{\pi} P \int_{-\infty}^{\infty} dz_R \, \frac{f_I(z_R)}{z_R - \alpha}$$

$$f_I(\alpha) = -\frac{1}{\pi} P \int_{-\infty}^{\infty} dz_R \, \frac{f_R(z_R)}{z_R - \alpha}$$

For dielectric function $\varepsilon(\omega)$:

$$\varepsilon(-\omega) = \varepsilon^*(\omega)$$

$$\Rightarrow \varepsilon_R(-\omega) = \varepsilon_R(\omega)$$

$$\Rightarrow \varepsilon_I(-\omega) = -\varepsilon_I(\omega)$$

This Kramers-Kronig transform is useful for the dielectric function

when $f(z_R) \Rightarrow \frac{\varepsilon(\omega)}{\varepsilon_0} - 1$

Must show that: 1. $f(z)$ is analytic for $z_I > 0$

2. $f(z)$ vanishes for $z \to \infty$ for $z_I > 0$

Analysis for Drude model dielectric function:

$$\frac{\varepsilon(\omega)}{\varepsilon_0} = 1 + N\sum_i f_i \frac{q_i^2}{\varepsilon_0 m_i} \frac{1}{\omega_i^2 - \omega^2 - i\omega\gamma_i}$$

Let $\quad f(z) = \dfrac{\varepsilon(z)}{\varepsilon_0} - 1 = N\sum_i f_i \dfrac{q_i^2}{\varepsilon_0 m_i} \dfrac{1}{\omega_i^2 - z^2 - iz\gamma_i}$

For $\quad |z| >> \omega_i$

$$f(z) \approx -\frac{1}{z^2}\left(N\sum_i f_i \frac{q_i^2}{\varepsilon_0 m_i} \right) \quad \Rightarrow \text{vanishes at large } z$$

Analysis for Drude model dielectric function – continued -- Analytic properties:

$$f(z) = \frac{\varepsilon(z)}{\varepsilon_0} - 1 = N \sum_i f_i \frac{q_i^{\,2}}{\varepsilon_0 m_i} \frac{1}{\omega_i^2 - z^2 - iz\gamma_i}$$

$f(z)$ has poles z_P at $\omega_i^2 - z_P^{\,2} - iz_P\gamma_i = 0$

$$z_P = -i\frac{\gamma_i}{2} \pm \sqrt{\omega_i^2 - \left(\frac{\gamma_i}{2}\right)^2}$$

Note that $\Im(z_P) \leq 0 \quad \Rightarrow f(z)$ is analytic for $\Im(z_P) > 0$

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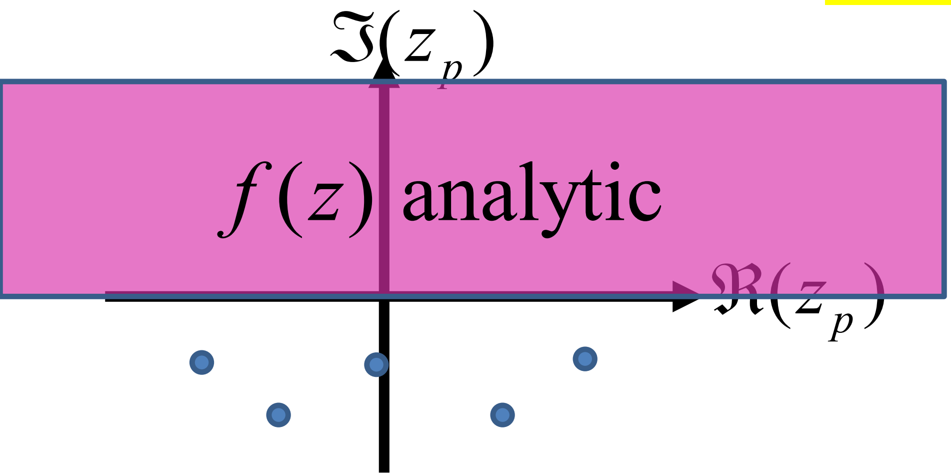


Summary –

Kramers-Kronig transform – for dielectric function:

$$\frac{\varepsilon_R(\omega)}{\varepsilon_0} - 1 = \frac{1}{\pi} P \int_{-\infty}^{\infty} d\omega' \frac{\varepsilon_I(\omega')}{\varepsilon_0} \frac{1}{\omega' - \omega}$$

$$\frac{\varepsilon_I(\omega)}{\varepsilon_0} = -\frac{1}{\pi} P \int_{-\infty}^{\infty} d\omega' \left(\frac{\varepsilon_R(\omega')}{\varepsilon_0} - 1 \right) \frac{1}{\omega' - \omega}$$

$$\text{with} \quad \varepsilon_R(-\omega) = \varepsilon_R(\omega); \quad \varepsilon_I(-\omega) = -\varepsilon_I(\omega)$$

Practical applications -- It is often possible/more convenient to calculate the imaginary response and use KK to deduce the real response or visa versa.

Analysis of Maxwell's equations without sources -- continued:

Summary of plane electromagnetic waves:

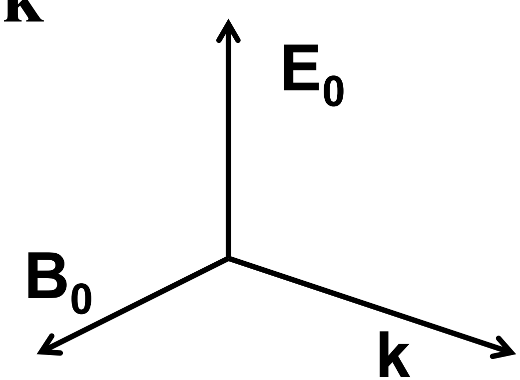
$$\mathbf{B}(\mathbf{r},t)=\Re\left(\frac{n\hat{\mathbf{k}}\times\mathbf{E}_0}{c}e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\right)\qquad \mathbf{E}(\mathbf{r},t)=\Re\left(\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\right)$$

$$|\mathbf{k}|^2=\left(\frac{\omega}{v}\right)^2=\left(\frac{n\omega}{c}\right)^2 \quad \text{where } n\equiv\sqrt{\frac{\mu\varepsilon}{\mu_0\varepsilon_0}} \quad \text{and } \hat{\mathbf{k}}\cdot\mathbf{E}_0=0$$

Poynting vector and energy density:

$$\langle\mathbf{S}\rangle_{avg}=\frac{n|\mathbf{E}_0|^2}{2\mu c}\hat{\mathbf{k}}=\frac{1}{2}\sqrt{\frac{\varepsilon}{\mu}}|\mathbf{E}_0|^2\,\hat{\mathbf{k}}$$

$$\langle u\rangle_{avg}=\frac{1}{2}\varepsilon|\mathbf{E}_0|^2$$

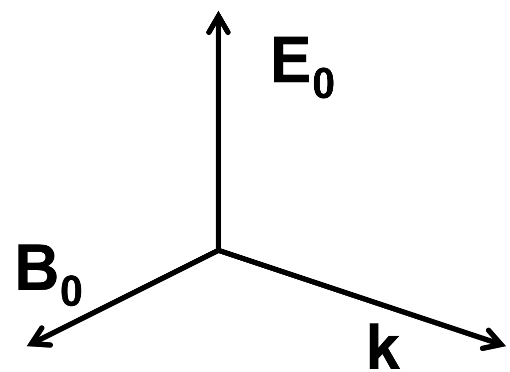


Transverse electric and magnetic waves (TEM)

$$\mathbf{B}(\mathbf{r},t) = \Re\left(\frac{n\hat{\mathbf{k}}\times\mathbf{E}_0}{c}e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\right) \qquad \mathbf{E}(\mathbf{r},t) = \Re\left(\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\right)$$

$$|\mathbf{k}|^2 = \left(\frac{\omega}{v}\right)^2 = \left(\frac{n\omega}{c}\right)^2 \quad \text{where } n \equiv \sqrt{\frac{\mu\varepsilon}{\mu_0\varepsilon_0}} \quad \text{and } \hat{\mathbf{k}}\cdot\mathbf{E}_0 = 0$$

TEM modes describe electromagnetic waves in lossless media and vacuum



For real
ε, μ, n, k

Effects of complex dielectric; fields near the surface on an ideal conductor

Suppose for an isotropic medium: $\mathbf{D} = \varepsilon_b \mathbf{E} \qquad \mathbf{J} = \sigma \mathbf{E}$

Maxwell's equations in terms of $\mathbf{H}$ and $\mathbf{E}$:

$$\nabla \cdot \mathbf{E} = 0 \qquad \nabla \cdot \mathbf{H} = 0$$

$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t} \qquad \nabla \times \mathbf{H} = \sigma \mathbf{E} + \varepsilon_b \frac{\partial \mathbf{E}}{\partial t}$$

$$\left(\nabla^2 - \mu\sigma \frac{\partial}{\partial t} - \mu\varepsilon_b \frac{\partial^2}{\partial t^2} \right) \mathbf{F} = 0 \qquad \mathbf{F} = \mathbf{E}, \mathbf{H}$$

Plane wave form for $\mathbf{E}$:

$$\mathbf{E}(\mathbf{r},t) = \Re\left(\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t} \right) \qquad \text{where } \mathbf{k} = (n_R + i n_I)\frac{\omega}{c}\hat{\mathbf{k}}$$

$$\Rightarrow \mathbf{E}(\mathbf{r},t) = e^{-\hat{\mathbf{k}}\cdot\mathbf{r}/\delta} \Re\left(\mathbf{E}_0 e^{i n_R (\omega/c) \hat{\mathbf{k}}\cdot\mathbf{r} - i\omega t} \right)$$

Some details:

Plane wave form for $\mathbf{E}$:

$$\mathbf{E}(\mathbf{r},t) = \Re\left(\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\right) \qquad \text{where } \mathbf{k} = \left(n_R + i n_I\right)\frac{\omega}{c}\hat{\mathbf{k}}$$

$$\left(\nabla^2 - \mu\sigma\frac{\partial}{\partial t} - \mu\varepsilon_b\frac{\partial^2}{\partial t^2}\right)\mathbf{E} = 0$$

$$-\left(n_R + i n_I\right)^2 + i\frac{\mu\sigma c^2}{\omega} + \mu\varepsilon_b c^2 = 0$$

Fields near the surface on an ideal conductor -- continued

For our system:

$$\frac{\omega}{c}n_R = \omega\sqrt{\frac{\mu\varepsilon_b}{2}}\left(\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon_b}\right)^2}+1\right)^{1/2}$$

$$\frac{\omega}{c}n_I = \omega\sqrt{\frac{\mu\varepsilon_b}{2}}\left(\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon_b}\right)^2}-1\right)^{1/2}$$

$$\text{For } \frac{\sigma}{\omega} >> 1 \qquad \frac{\omega}{c}n_R \approx \frac{\omega}{c}n_I \approx \sqrt{\frac{\mu\sigma\omega}{2}} \equiv \frac{1}{\delta}$$

$$\Rightarrow \mathbf{E}(\mathbf{r},t) = e^{-\hat{\mathbf{k}}\cdot\mathbf{r}/\delta}\Re\left(\mathbf{E}_0 e^{i\hat{\mathbf{k}}\cdot\mathbf{r}/\delta - i\omega t}\right)$$

$$\Rightarrow \mathbf{H}(\mathbf{r},t) = \frac{n}{c\mu}\hat{\mathbf{k}}\times\mathbf{E}(\mathbf{r},t) = \frac{1+i}{\delta\mu\omega}\hat{\mathbf{k}}\times\mathbf{E}(\mathbf{r},t)$$

Some representative values of skin depth

Ref: Lorrain[2] and Corson

$$\frac{\omega}{c}n_R \approx \frac{\omega}{c}n_I \approx \sqrt{\frac{\mu\sigma\omega}{2}} \equiv \frac{1}{\delta}$$

	σ (10^7 S/m)	μ/μ_0	δ (0.001m) at 60 Hz	δ (0.001m) at 1 MHz
Al	3.54	1	10.9	84.6
Cu	5.80	1	8.5	66.1
Fe	1.00	100	1.0	10.0
Mumetal	0.16	2000	0.4	3.0
Zn	1.86	1	15.1	117

Relative energies associated with field

Electric energy density: $\varepsilon_b |\mathbf{E}|^2$

Magnetic energy density: $\mu |\mathbf{H}|^2$

Ratio inside conducting media: $\frac{\varepsilon_b |\mathbf{E}|^2}{\mu |\mathbf{H}|^2} = \frac{\varepsilon_b}{\mu \left| \frac{1+i}{\delta \mu \omega} \right|^2} = \frac{\varepsilon_b \mu \omega^2 \delta^2}{2}$

Here wavelength is defined:

$\lambda = \frac{2\pi c}{\omega}$

$= 2\pi^2 \frac{\varepsilon_b}{\varepsilon_0} \frac{\mu}{\mu_0} \frac{\delta^2}{\lambda^2}$

For $\frac{\varepsilon_b |\mathbf{E}|^2}{\mu |\mathbf{H}|^2} \ll 1 \quad \Rightarrow$ magnetic energy dominates

Note that in free space, $\frac{\varepsilon_0 |\mathbf{E}|^2}{\mu_0 |\mathbf{H}|^2} = 1$

Various wavelengths λ --