PHY 712 Electrodynamics
10-10:50 AM Online

Discussion for Lecture 21:

Chap. 8 in Jackson – Wave Guides

1. **TEM, TE, and TM modes**
2. **Justification for boundary conditions; behavior of waves near conducting surfaces**

13	Wed: 02/24/2021	Chap. 5	Magnetic dipoles and hyperfine interaction	#11	03/01/2021
14	Fri: 02/26/2021	Chap. 5	Magnetic dipoles and dipolar fields		
15	Mon: 03/01/2021	Chap. 6	Maxwell's Equations	#12	03/08/2021
16	Wed: 03/03/2021	Chap. 6	Electromagnetic energy and forces		
17	Fri: 03/05/2021	Chap. 7	Electromagnetic plane waves		
18	Mon: 03/08/2021	Chap. 7	Electromagnetic plane waves	#13	03/10/2021
19	Wed: 03/10/2021	Chap. 7	Optical effects of refractive indices	#14	03/12/2021
20	Fri: 03/12/2021	Chap. 1-7	Review		
	Mon: 03/15/2021	No class	*APS March Meeting*	Take Home Exam	
	Wed: 03/17/2021	No class	*APS March Meeting*	Take Home Exam	
	Fri: 03/19/2021	No class	*APS March Meeting*	Take Home Exam	
21	Mon: 03/22/2021	Chap. 8	EM waves in wave guides		
22	Wed: 03/24/2021	Chap. 9	Radiation from localized oscillating sources		

Your questions –

From Gao -- How do we know what kinds of mode(TE, TM, TEM, or others) a guide will have at a first glance?

Comment -- In general TEM modes propagate in free space while the possible modes associated with media with one or more metallic surface are more complicated.

Maxwell's equations

For linear isotropic media and no sources: $\mathbf{D} = \varepsilon\mathbf{E}; \quad \mathbf{B} = \mu\mathbf{H}$

Coulomb's law: $\nabla \cdot \mathbf{E} = 0$

Ampere-Maxwell's law: $\nabla \times \mathbf{B} - \mu\varepsilon \frac{\partial \mathbf{E}}{\partial t} = 0$

Faraday's law: $\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$

No magnetic monopoles: $\nabla \cdot \mathbf{B} = 0$

Analysis of Maxwell's equations without sources -- continued:

$$\text{Coulomb's law:} \qquad \nabla \cdot \mathbf{E} = 0$$

$$\text{Ampere-Maxwell's law:} \qquad \nabla \times \mathbf{B} - \mu\varepsilon \frac{\partial \mathbf{E}}{\partial t} = 0$$

$$\text{Faraday's law:} \qquad \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$$

$$\text{No magnetic monopoles:} \qquad \nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \left(\nabla \times \mathbf{B} - \mu\varepsilon \frac{\partial \mathbf{E}}{\partial t} \right) = -\nabla^2 \mathbf{B} - \mu\varepsilon \frac{\partial (\nabla \times \mathbf{E})}{\partial t}$$

$$= -\nabla^2 \mathbf{B} + \mu\varepsilon \frac{\partial^2 \mathbf{B}}{\partial t^2} = 0$$

$$\nabla \times \left(\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} \right) = -\nabla^2 \mathbf{E} + \frac{\partial (\nabla \times \mathbf{B})}{\partial t}$$

$$= -\nabla^2 \mathbf{E} + \mu\varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0$$

Analysis of Maxwell's equations without sources -- continued:

Both E and B fields are solutions to a wave equation:

$$\nabla^2 \mathbf{B} - \frac{1}{v^2}\frac{\partial^2 \mathbf{B}}{\partial t^2} = 0$$

$$\nabla^2 \mathbf{E} - \frac{1}{v^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = 0$$

$$\text{where} \quad v^2 \equiv c^2 \frac{\mu_0 \varepsilon_0}{\mu \varepsilon} \equiv \frac{c^2}{n^2}$$

Plane wave solutions to wave equation:

$$\mathbf{B}(\mathbf{r},t) = \Re\left(\mathbf{B}_0 e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t}\right) \qquad \mathbf{E}(\mathbf{r},t) = \Re\left(\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t}\right)$$

Analysis of Maxwell's equations without sources -- continued:

Plane wave solutions to wave equation:

$$\mathbf{B}(\mathbf{r},t)=\Re\left(\mathbf{B}_0 e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\right) \qquad \mathbf{E}(\mathbf{r},t)=\Re\left(\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\right)$$

$$|\mathbf{k}|^2=\left(\frac{\omega}{v}\right)^2=\left(\frac{n\omega}{c}\right)^2 \qquad \text{where } n\equiv\sqrt{\frac{\mu\varepsilon}{\mu_0\varepsilon_0}}$$

Note: ε, μ, n, k can all be complex; for the moment we will assume that they are all real (no dissipation).

Note that $\mathbf{E}_0$ and $\mathbf{B}_0$ are not independent;

$$\left.\begin{aligned}
&\text{from Faraday's law:}\quad \nabla\times\mathbf{E}+\frac{\partial\mathbf{B}}{\partial t}=0\\
&\Rightarrow \mathbf{B}_0=\frac{\mathbf{k}\times\mathbf{E}_0}{\omega}=\frac{n\hat{\mathbf{k}}\times\mathbf{E}_0}{c}\\
&\text{also note:}\quad \hat{\mathbf{k}}\cdot\mathbf{E}_0=0 \ \text{ and } \ \hat{\mathbf{k}}\cdot\mathbf{B}_0=0
\end{aligned}\right\}\ \text{For real } \varepsilon, \mu, n, k$$

Analysis of Maxwell's equations without sources -- continued:

Summary of plane electromagnetic waves:

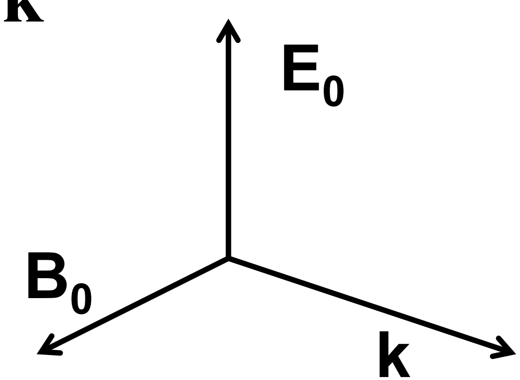
$$\mathbf{B}(\mathbf{r},t)=\Re\left(\frac{n\hat{\mathbf{k}}\times\mathbf{E}_0}{c}e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\right)\qquad \mathbf{E}(\mathbf{r},t)=\Re\left(\mathbf{E}_0e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\right)$$

$$|\mathbf{k}|^2=\left(\frac{\omega}{v}\right)^2=\left(\frac{n\omega}{c}\right)^2 \quad \text{where } n\equiv\sqrt{\frac{\mu\varepsilon}{\mu_0\varepsilon_0}} \quad \text{and } \hat{\mathbf{k}}\cdot\mathbf{E}_0=0$$

Poynting vector and energy density:

$$\langle\mathbf{S}\rangle_{avg}=\frac{n|\mathbf{E}_0|^2}{2\mu c}\hat{\mathbf{k}}=\frac{1}{2}\sqrt{\frac{\varepsilon}{\mu}}|\mathbf{E}_0|^2\,\hat{\mathbf{k}}$$

$$\langle u\rangle_{avg}=\frac{1}{2}\varepsilon|\mathbf{E}_0|^2$$

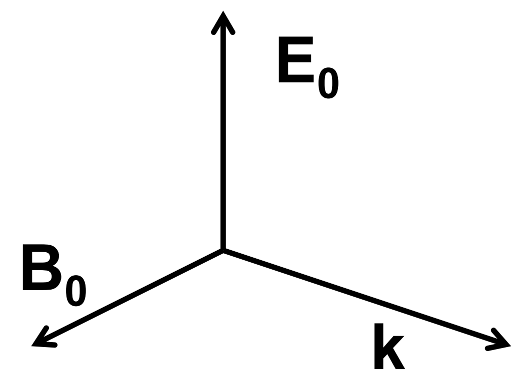


Transverse electric and magnetic waves (TEM)

$$\mathbf{B}(\mathbf{r},t) = \Re\left(\frac{n\hat{\mathbf{k}} \times \mathbf{E}_0}{c} e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t} \right) \qquad \mathbf{E}(\mathbf{r},t) = \Re\left(\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t} \right)$$

$$|\mathbf{k}|^2 = \left(\frac{\omega}{v}\right)^2 = \left(\frac{n\omega}{c}\right)^2 \quad \text{where } n \equiv \sqrt{\frac{\mu\varepsilon}{\mu_0\varepsilon_0}} \quad \text{and } \hat{\mathbf{k}} \cdot \mathbf{E}_0 = 0$$

TEM modes describe electromagnetic waves in lossless media and vacuum



For real
ε, μ, n, k

Effects of complex dielectric; fields near the surface on an ideal conductor

Suppose for an isotropic medium: $\mathbf{D} = \varepsilon_b \mathbf{E} \qquad \mathbf{J} = \sigma\mathbf{E}$

Maxwell's equations in terms of $\mathbf{H}$ and $\mathbf{E}$:

$$\nabla \cdot \mathbf{E} = 0 \qquad\qquad \nabla \cdot \mathbf{H} = 0$$

$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t} \qquad\qquad \nabla \times \mathbf{H} = \sigma\mathbf{E} + \varepsilon_b \frac{\partial \mathbf{E}}{\partial t}$$

$$\left(\nabla^2 - \mu\sigma \frac{\partial}{\partial t} - \mu\varepsilon_b \frac{\partial^2}{\partial t^2} \right) \mathbf{F} = 0 \qquad \mathbf{F} = \mathbf{E}, \mathbf{H}$$

Plane wave form for $\mathbf{E}$:

$$\mathbf{E}(\mathbf{r},t) = \Re\left(\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t}\right) \qquad \text{where } \mathbf{k} = (n_R + i n_I)\frac{\omega}{c}\hat{\mathbf{k}}$$

$$\Rightarrow \mathbf{E}(\mathbf{r},t) = e^{-\hat{\mathbf{k}}\cdot\mathbf{r}/\delta} \Re\left(\mathbf{E}_0 e^{i n_R (\omega/c)\hat{\mathbf{k}}\cdot\mathbf{r} - i\omega t}\right)$$

Some details:

Plane wave form for $\mathbf{E}$:

$$\mathbf{E}(\mathbf{r},t)=\Re\left(\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\right) \qquad \text{where } \mathbf{k}=\left(n_R+in_I\right)\frac{\omega}{c}\hat{\mathbf{k}}$$

$$\left(\nabla^2-\mu\sigma\frac{\partial}{\partial t}-\mu\varepsilon_b\frac{\partial^2}{\partial t^2}\right)\mathbf{E}=0$$

$$-\left(n_R+in_I\right)^2+i\frac{\mu\sigma c^2}{\omega}+\mu\varepsilon_b c^2=0$$

Fields near the surface on an ideal conductor -- continued

For our system:

$$\frac{\omega}{c}n_R = \omega\sqrt{\frac{\mu\varepsilon_b}{2}}\left(\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon_b}\right)^2}+1\right)^{1/2}$$

$$\frac{\omega}{c}n_I = \omega\sqrt{\frac{\mu\varepsilon_b}{2}}\left(\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon_b}\right)^2}-1\right)^{1/2}$$

$$\text{For } \frac{\sigma}{\omega} >> 1 \qquad \frac{\omega}{c}n_R \approx \frac{\omega}{c}n_I \approx \sqrt{\frac{\mu\sigma\omega}{2}} \equiv \frac{1}{\delta}$$

"skin depth"

$$\Rightarrow \mathbf{E}(\mathbf{r},t) = e^{-\hat{\mathbf{k}}\cdot\mathbf{r}/\delta}\,\Re\left(\mathbf{E}_0 e^{i\hat{\mathbf{k}}\cdot\mathbf{r}/\delta - i\omega t}\right)$$

$$\Rightarrow \mathbf{H}(\mathbf{r},t) = \frac{n}{c\mu}\hat{\mathbf{k}}\times\mathbf{E}(\mathbf{r},t) = \frac{1+i}{\delta\mu\omega}\hat{\mathbf{k}}\times\mathbf{E}(\mathbf{r},t)$$

Some representative values of skin depth

Ref: Lorrain[2] and Corson

$$\frac{\omega}{c} n_R \approx \frac{\omega}{c} n_I \approx \sqrt{\frac{\mu\sigma\omega}{2}} \equiv \frac{1}{\delta}$$

Note that frequency given in units of Hz $\Rightarrow \frac{\omega}{2\pi}$

	σ (10^7 S/m)	μ/μ_0	δ (0.001m) at 60 Hz	δ (0.001m) at 1 MHz
Al	3.54	1	10.9	84.6
Cu	5.80	1	8.5	66.1
Fe	1.00	100	1.0	10.0
Mumetal	0.16	2000	0.4	3.0
Zn	1.86	1	15.1	117

Relative energies associated with field

Electric energy density: $\varepsilon_b |\mathbf{E}|^2$

Magnetic energy density: $\mu |\mathbf{H}|^2$

Ratio inside conducting media: $$\frac{\varepsilon_b |\mathbf{E}|^2}{\mu |\mathbf{H}|^2} = \frac{\varepsilon_b}{\mu \left|\frac{1+i}{\delta\mu\omega}\right|^2} = \frac{\varepsilon_b \mu \omega^2 \delta^2}{2}$$

$$\lambda = \frac{2\pi c}{\omega} = \frac{c}{f}$$

$$= 2\pi^2 \frac{\varepsilon_b}{\varepsilon_0} \frac{\mu}{\mu_0} \frac{\delta^2}{\lambda^2}$$

For $\dfrac{\varepsilon_b |\mathbf{E}|^2}{\mu |\mathbf{H}|^2} \ll 1 \quad \Rightarrow$ magnetic energy dominates

Note that in free space, $\dfrac{\varepsilon_0 |\mathbf{E}|^2}{\mu_0 |\mathbf{H}|^2} = 1$

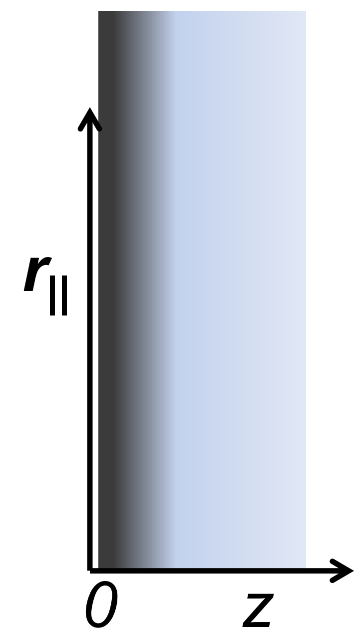
Fields near the surface on an ideal conductor -- continued

$$\text{For } \frac{\sigma}{\omega} >> 1 \qquad \frac{\omega}{c} n_R \approx \frac{\omega}{c} n_I \approx \sqrt{\frac{\mu\sigma\omega}{2}} \equiv \frac{1}{\delta}$$

$$\text{In this limit, } \sqrt{\frac{\mu\varepsilon}{\mu_0\varepsilon_0}} = c\sqrt{\mu\varepsilon} = n_R + in_I = \frac{c}{\omega}\frac{1}{\delta}(1+i)$$

$$\mathbf{E}(\mathbf{r},t) = e^{-\hat{\mathbf{k}}\cdot\mathbf{r}/\delta}\Re\left(\mathbf{E}_0 e^{i\hat{\mathbf{k}}\cdot\mathbf{r}/\delta - i\omega t}\right)$$

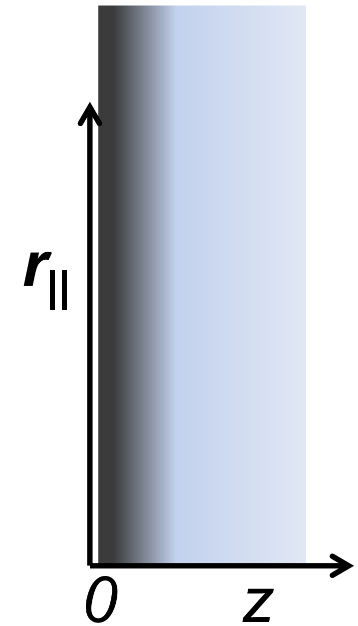
$$\mathbf{H}(\mathbf{r},t) = \frac{n}{c\mu}\hat{\mathbf{k}}\times\mathbf{E}(\mathbf{r},t) = \frac{1+i}{\delta\mu\omega}\hat{\mathbf{k}}\times\mathbf{E}(\mathbf{r},t)$$



Fields near the surface on an ideal conductor -- continued

$$\mathbf{E}(\mathbf{r},t) = e^{-\hat{\mathbf{k}}\cdot\mathbf{r}/\delta}\Re\left(\mathbf{E}_0 e^{i\hat{\mathbf{k}}\cdot\mathbf{r}/\delta - i\omega t}\right)$$

$$\mathbf{H}(\mathbf{r},t) = \frac{n}{c\mu}\hat{\mathbf{k}}\times\mathbf{E}(\mathbf{r},t) = \frac{1+i}{\delta\mu\omega}\hat{\mathbf{k}}\times\mathbf{E}(\mathbf{r},t)$$



Note that it is convenient to express the EM fields in terms of the $\mathbf{H}$ amplitude:

$$\mathbf{H}(\mathbf{r},t) = e^{-\hat{\mathbf{k}}\cdot\mathbf{r}/\delta}\Re\left(\mathbf{H}_0 e^{i\hat{\mathbf{k}}\cdot\mathbf{r}/\delta - i\omega t}\right)$$

$$\mathbf{E}(\mathbf{r},t) = \delta\mu\omega\frac{1-i}{2}\hat{\mathbf{k}}\times\mathbf{H}(\mathbf{r},t)$$

Boundary values for ideal conductor

Inside the conductor:

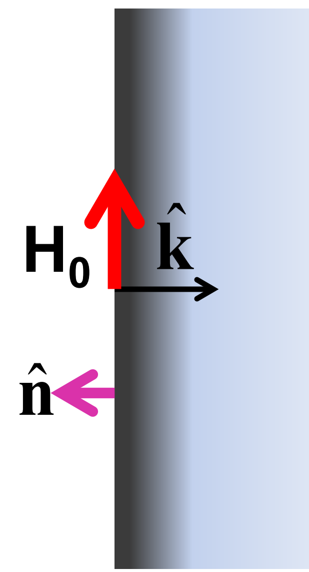
$$\mathbf{H}(\mathbf{r},t) = e^{-\hat{\mathbf{k}}\cdot\mathbf{r}/\delta}\,\Re\left(\mathbf{H}_0 e^{i\hat{\mathbf{k}}\cdot\mathbf{r}/\delta - i\omega t}\right)$$

$$\mathbf{E}(\mathbf{r},t) = \delta\mu\omega\frac{1-i}{2}\hat{\mathbf{k}}\times\mathbf{H}(\mathbf{r},t)$$

At the boundary of an ideal conductor, the **E** and **H** fields decay in the direction normal to the interface.

Ideal conductor boundary conditions:

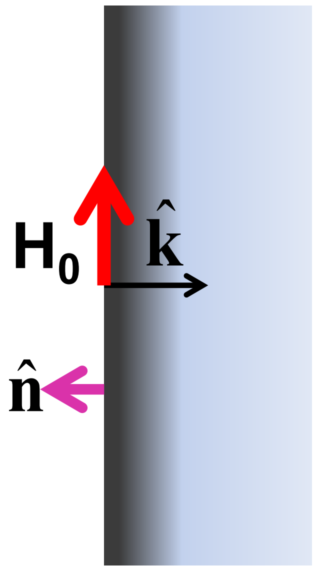
$$\hat{\mathbf{n}}\times\mathbf{E}\Big|_S = 0 \qquad\qquad \hat{\mathbf{n}}\cdot\mathbf{H}\Big|_S = 0$$



Wave guides – dielectric media with one or more metal boundary

Ideal conductor boundary conditions:

$$\hat{\mathbf{n}} \times \mathbf{E}\Big|_S = 0 \qquad \hat{\mathbf{n}} \cdot \mathbf{H}\Big|_S = 0$$



Waveguide terminology

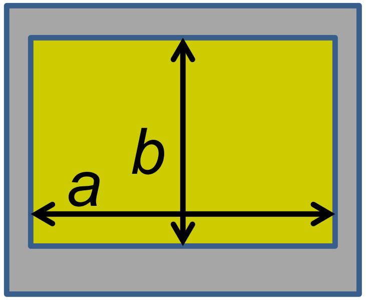
- TEM: transverse electric and magnetic (both E and H fields are perpendicular to wave propagation direction)
- TM: transverse magnetic (H field is perpendicular to wave propagation direction)
- TE: transverse electric (E field is perpendicular to wave propagation direction)

Analysis of rectangular waveguide

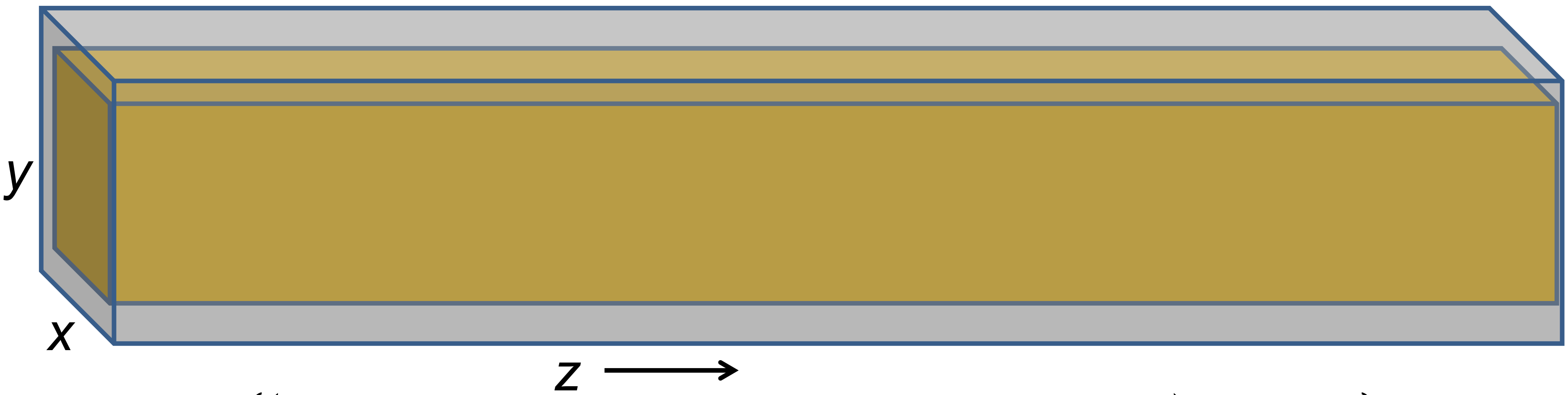
Boundary conditions at surface of waveguide:

$\mathbf{E}_{\text{tangential}}=0, \quad \mathbf{B}_{\text{normal}}=0$

Cross section view



Analysis of rectangular waveguide



$$\mathbf{B} = \Re\left\{\left(B_x(x,y)\hat{\mathbf{x}} + B_y(x,y)\hat{\mathbf{y}} + B_z(x,y)\hat{\mathbf{z}}\right)e^{ikz-i\omega t}\right\}$$

$$\mathbf{E} = \Re\left\{\left(E_x(x,y)\hat{\mathbf{x}} + E_y(x,y)\hat{\mathbf{y}} + E_z(x,y)\hat{\mathbf{z}}\right)e^{ikz-i\omega t}\right\}$$

Inside the dielectric medium: (assume ε to be real)

$$\nabla\cdot\mathbf{E} = 0 \qquad \nabla\cdot\mathbf{B} = 0$$

$$\nabla\times\mathbf{E} + \frac{\partial\mathbf{B}}{\partial t} = 0 \qquad \nabla\times\mathbf{B} - \varepsilon\mu\frac{\partial\mathbf{E}}{\partial t} = 0$$

Solution of Maxwell's equations within the pipe:

Combining Faraday's Law and Ampere's Law, we find that each field component must satisfy a two-dimensional Helmholz equation:

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}-k^2+\mu\varepsilon\omega^2\right)E_x(x,y) \;=\; 0.$$

For the rectangular wave guide discussed in Section 8.4 of your text a solution for a TE mode can have:

$$E_z(x,y)\equiv 0 \quad \text{and} \quad B_z(x,y)=B_0\cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right),$$

with $k^2 \equiv k_{mn}^2 = \mu\varepsilon\omega^2 - \left[\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right]$

Maxwell's equations within the pipe in terms of all 6 components:

$$\frac{\partial B_x}{\partial x}+\frac{\partial B_y}{\partial y}+ikB_z=0.$$

$$\frac{\partial E_x}{\partial x}+\frac{\partial E_y}{\partial y}+ikE_z=0.$$

For TE mode with $E_z \equiv 0$

$$B_x=-\frac{k}{\omega}E_y$$

$$B_y=\frac{k}{\omega}E_x$$

$$\frac{\partial E_z}{\partial y}-ikE_y=i\omega B_x.$$

$$ikE_x-\frac{\partial E_z}{\partial x}=i\omega B_y.$$

$$\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}=i\omega B_z.$$

$$\frac{\partial B_z}{\partial y}-ikB_y=-i\mu\varepsilon\omega E_x.$$

$$ikB_x-\frac{\partial B_z}{\partial x}=-i\mu\varepsilon\omega E_y.$$

$$\frac{\partial B_y}{\partial x}-\frac{\partial B_x}{\partial y}=-i\mu\varepsilon\omega E_z.$$

TE modes for rectangular wave guide continued:

$$E_z(x,y) \equiv 0 \quad \text{and} \quad B_z(x,y) = B_0 \cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right),$$

$$E_x = \frac{\omega}{k} B_y = \frac{-i\omega}{k^2 - \mu\varepsilon\omega^2}\frac{\partial B_z}{\partial y} = \frac{-i\omega}{\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]}\frac{n\pi}{b} B_0 \cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right),$$

$$E_y = -\frac{\omega}{k} B_x = \frac{i\omega}{k^2 - \mu\varepsilon\omega^2}\frac{\partial B_z}{\partial x} = \frac{i\omega}{\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]}\frac{m\pi}{a} B_0 \sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right).$$

Check boundary conditions:

$\mathbf{E}_{\text{tangential}} = 0$ because: $E_z(x,y) \equiv 0, \;\; E_x(x,0) = E_x(x,b) = 0$

and $E_y(0,y) = E_y(a,y) = 0.$

$\mathbf{B}_{\text{normal}} = 0$ because: $B_y(x,0) = B_y(x,b) = 0$

and $B_x(0,y) = B_x(a,y) = 0.$

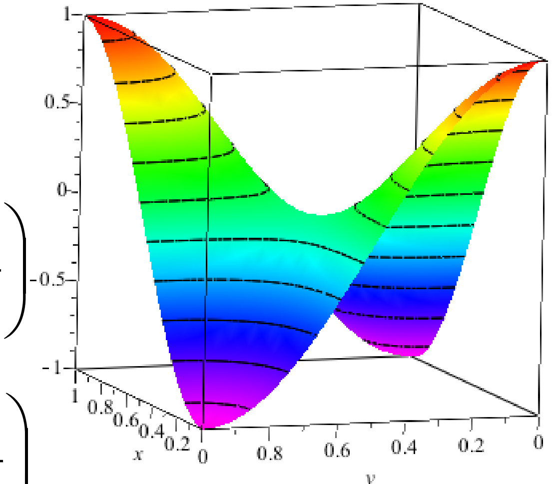
Solution for m=n=1

$$B_z(x,y) = B_0 \cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)$$

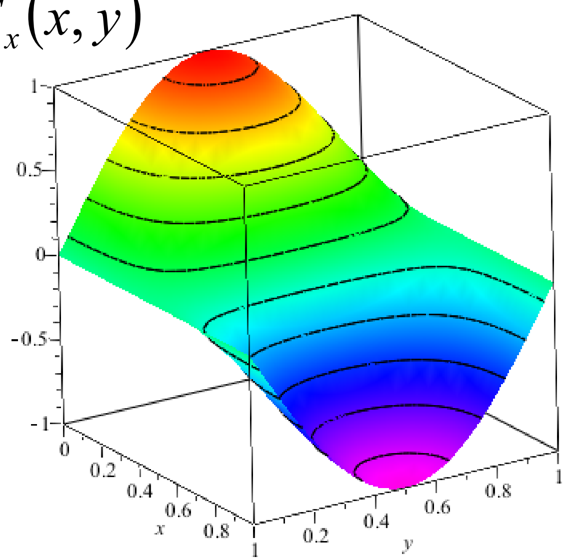
$$iE_x(x,y) = B_0\left(\frac{\omega n\pi / b}{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}\right)\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$$

$$iE_y(x,y) = B_0\left(\frac{-\omega m\pi / a}{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}\right)\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)$$

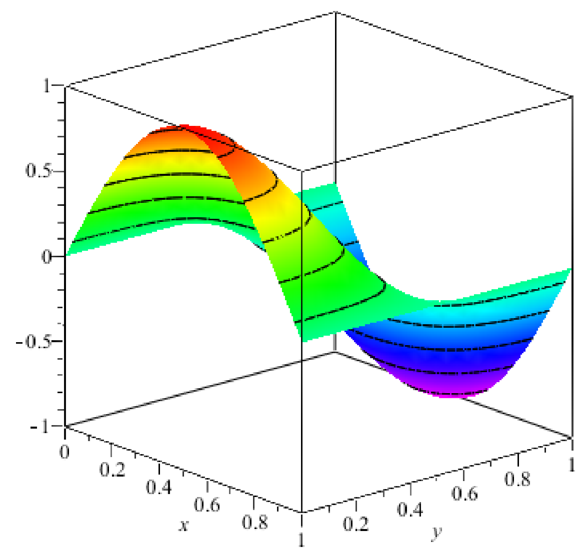
$B_z(x,y)$



$iE_x(x,y)$

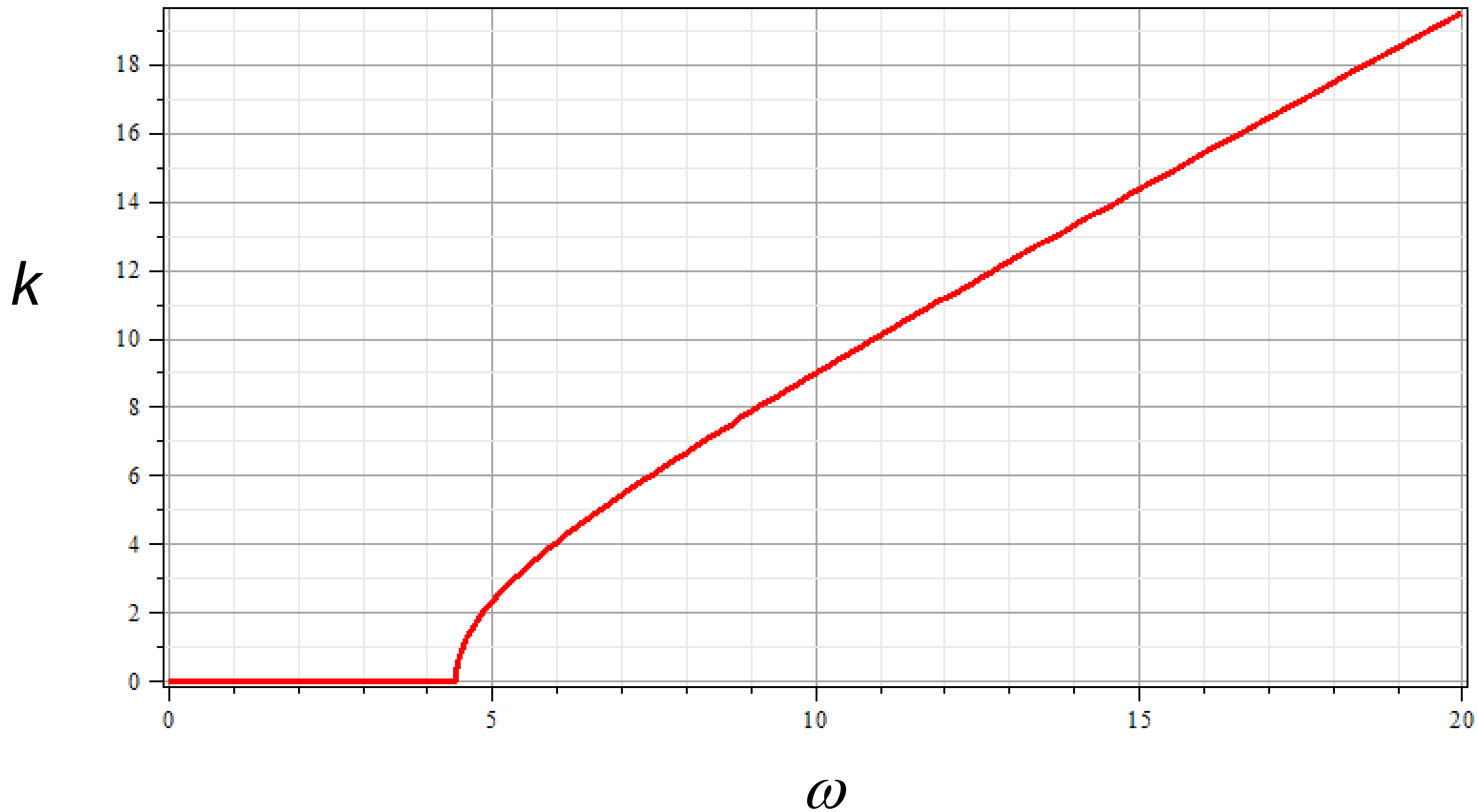


$iE_y(x,y)$

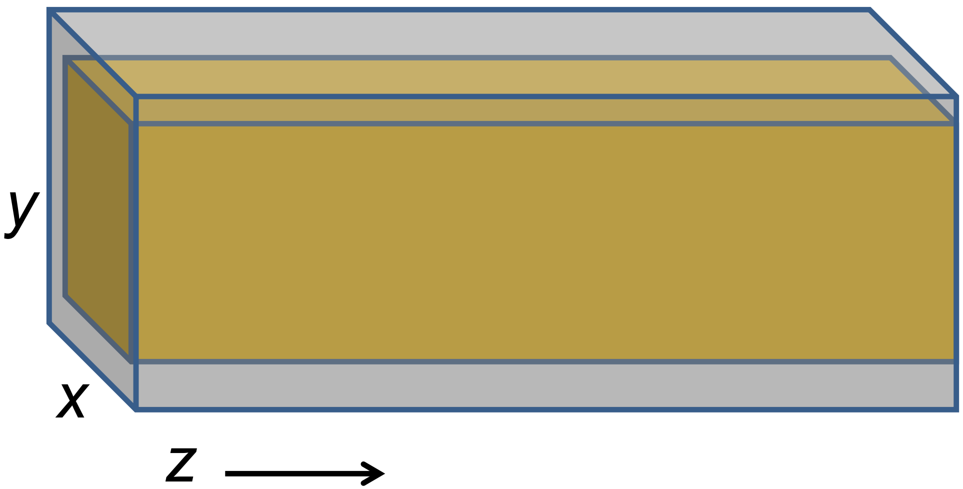


Solution for m=n=1

$$k^2 \equiv k_{mn}^2 = \mu\varepsilon\omega^2 - \left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]$$



Resonant cavity



$$0 \le x \le a$$

$$0 \le y \le b$$

$$0 \le z \le d$$

$$\mathbf{B} = \Re\left\{\left(B_x(x,y,z)\hat{\mathbf{x}} + B_y(x,y,z)\hat{\mathbf{y}} + B_z(x,y,z)\hat{\mathbf{z}}\right)e^{-i\omega t}\right\}$$

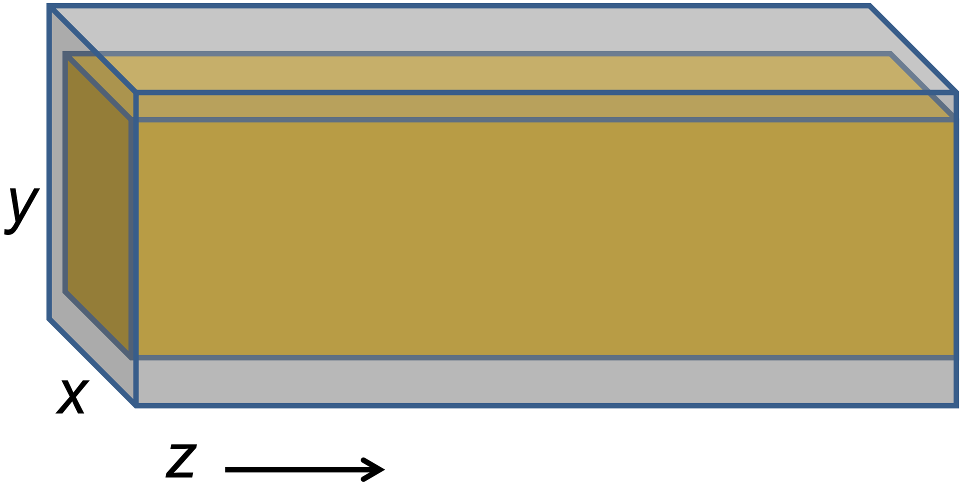
$$\mathbf{E} = \Re\left\{\left(E_x(x,y,z)\hat{\mathbf{x}} + E_y(x,y,z)\hat{\mathbf{y}} + E_z(x,y,z)\hat{\mathbf{z}}\right)e^{-i\omega t}\right\}$$

In general: $E_i(x,y,z) = E_i(x,y)\sin(kz)$ or $E_i(x,y)\cos(kz)$

$B_i(x,y,z) = B_i(x,y)\sin(kz)$ or $B_i(x,y)\cos(kz)$

$$\Rightarrow k = \frac{p\pi}{d}$$

Resonant cavity



$$0 \le x \le a$$

$$0 \le y \le b$$

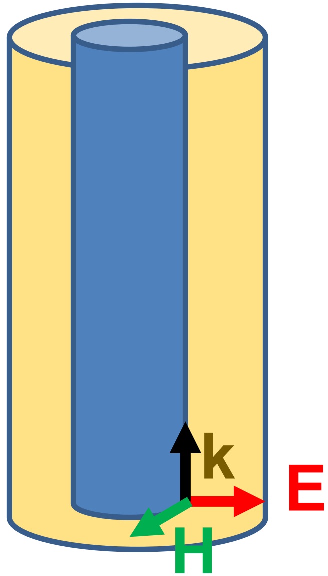
$$0 \le z \le d$$

$$k^2 = \left(\frac{p\pi}{d}\right)^2 = \mu\varepsilon\omega^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2$$

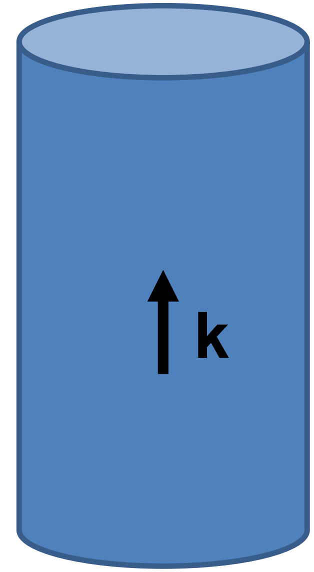
$$\Rightarrow \omega^2 = \frac{1}{\mu\varepsilon}\left(\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{p\pi}{d}\right)^2\right)$$

Wave guides – dielectric media with one or more metal boundary

Coaxial cable
TEM modes



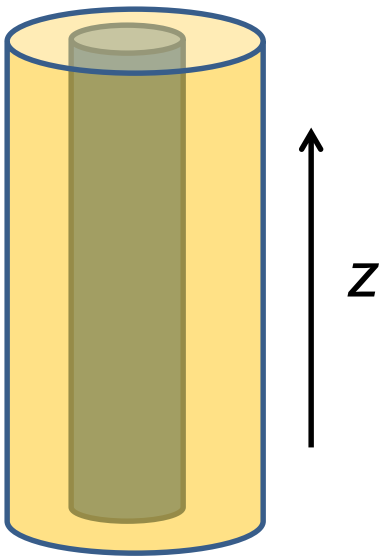
Simple optical pipe
TE or TM modes



Waveguide terminology

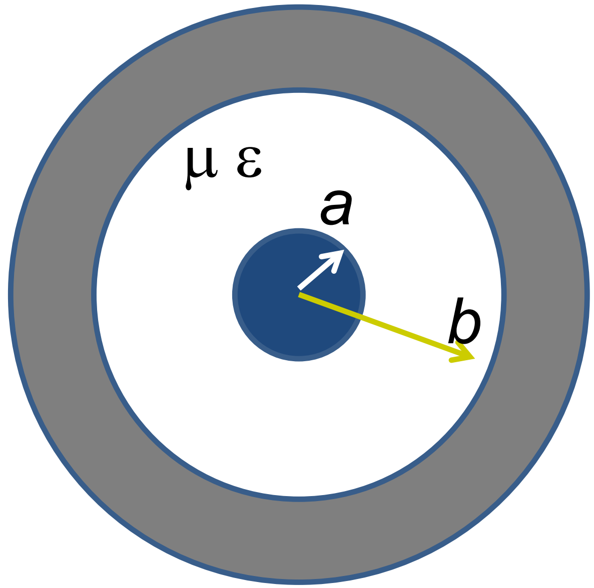
- TEM: transverse electric and magnetic (both E and H fields are perpendicular to wave propagation direction)
- TM: transverse magnetic (H field is perpendicular to wave propagation direction)
- TE: transverse electric (E field is perpendicular to wave propagation direction)

Wave guides



Coaxial cable
TEM modes

Top view:



Inside medium, μ ε assumed to be real

(following problem 8.2 in Jackson's text)

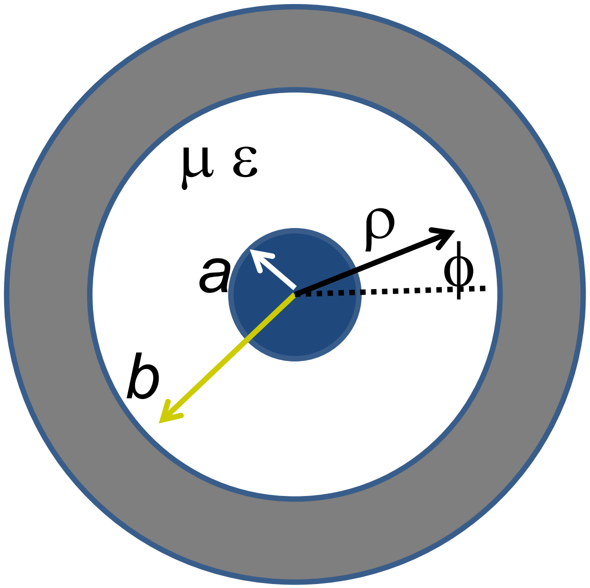
Maxwell's equations inside medium: for $a \leq \rho \leq b$

$$\nabla \times \mathbf{E} = i\omega\mathbf{B} \qquad \nabla \cdot \mathbf{E} = 0$$

$$\nabla \times \mathbf{B} = -i\omega\mu\varepsilon\mathbf{E} \qquad \nabla \cdot \mathbf{B} = 0$$

Electromagnetic waves in a coaxial cable -- continued

Top view:



Example solution for $a \leq \rho \leq b$

$$\mathbf{E} = \hat{\boldsymbol{\rho}}\Re\left(\frac{E_0 a}{\rho}e^{ikz-i\omega t}\right)$$

$$\mathbf{B} = \hat{\boldsymbol{\varphi}}\Re\left(\frac{B_0 a}{\rho}e^{ikz-i\omega t}\right)$$

$$\hat{\boldsymbol{\rho}} = \cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}}$$

$$\hat{\boldsymbol{\varphi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}}$$

Find:

$$k = \omega\sqrt{\mu\varepsilon}$$

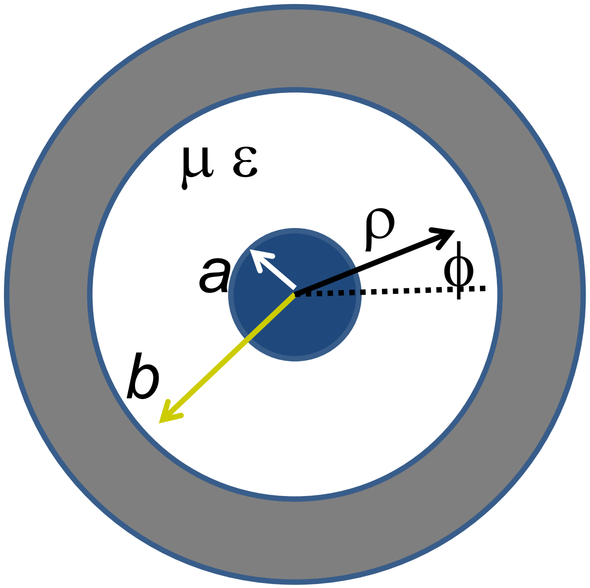
$$E_0 = \frac{B_0}{\sqrt{\mu\varepsilon}}$$

Poynting vector within cable medium (with μ, ε):

$$\langle \mathbf{S} \rangle_{avg} = \frac{1}{2\mu}\Re\left(\mathbf{E}\times\mathbf{B}^*\right) = \frac{|B_0|^2}{2\mu\sqrt{\mu\varepsilon}}\left(\frac{a}{\rho}\right)^2\hat{\mathbf{z}}$$

Electromagnetic waves in a coaxial cable -- continued

Top view:



Time averaged power in cable material:

$$\int_0^{2\pi} d\phi \int_a^b \rho d\rho \left(\langle \mathbf{S} \rangle_{avg} \cdot \hat{\mathbf{z}} \right) = \frac{|B_0|^2 \pi a^2}{\mu \sqrt{\mu \varepsilon}} \ln\left(\frac{b}{a}\right)$$