PHY 712 Electrodynamics
10-10:50 AM MWF Online

Notes for Lecture 22:

Sources of radiation

Start reading Chap. 9

A. Electromagnetic waves due to specific sources

B. Dipole radiation patterns

19	Wed: 03/10/2021	Chap. 7	Optical effects of refractive indices	#14	03/12/2021
20	Fri: 03/12/2021	Chap. 1-7	Review		
	Mon: 03/15/2021	No class	*APS March Meeting*	Take Home Exam	
	Wed: 03/17/2021	No class	*APS March Meeting*	Take Home Exam	
	Fri: 03/19/2021	No class	*APS March Meeting*	Take Home Exam	
21	Mon: 03/22/2021	Chap. 8	EM waves in wave guides		
22	Wed: 03/24/2021	Chap. 9	Radiation from localized oscillating sources	#15	03/26/2021
23	Fri: 03/26/2021	Chap. 9	Radiation from oscillating sources		

Maxwell's equations

Microscopic or vacuum form $(\mathbf{P}=0;\ \mathbf{M}=0)$:

Coulomb's law: $\nabla \cdot \mathbf{E} = \rho / \varepsilon_0$

Ampere-Maxwell's law: $\nabla \times \mathbf{B} - \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{J}$

Faraday's law: $\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$

No magnetic monopoles: $\nabla \cdot \mathbf{B} = 0$

$$\Rightarrow c^2 = \frac{1}{\varepsilon_0 \mu_0}$$

Since Maxwell's equations were introduced and used in Chapters 6-8, we have focused on the properties of the fields themselves. Now we will begin to study how these fields are produced by particular sources. The sources that we will consider are harmonic in time and their spatial form (considered to be localized in space) is represented by a multiplicative factor. More generally, we are considering one component in the Fourier transform for the source function. The results are quite different from the Liénard-Wiechert potentials discussed a few weeks ago. In this slide, Maxwell's equations are presented for the case that the sources are completely represented by the charge and current densities.

scalar potentials

$$\nabla \cdot \mathbf{B} = 0 \qquad \Rightarrow \mathbf{B} = \nabla \times \mathbf{A}$$

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \quad \Rightarrow \nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} \right) = 0$$

$$\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\nabla \Phi$$

$$\text{or} \quad \mathbf{E} = -\nabla \Phi - \frac{\partial \mathbf{A}}{\partial t}$$

It is convenient to express the coupled vector fields in terms of the scalar and vector potentials as we have discussed previously.

Formulation of Maxwell's equations in terms of vector and scalar potentials -- continued

Lorentz gauge form -- require: $\nabla \cdot \mathbf{A}_L + \frac{1}{c^2}\frac{\partial \Phi_L}{\partial t} = 0$

$$-\nabla^2 \Phi_L + \frac{1}{c^2}\frac{\partial^2 \Phi_L}{\partial t^2} = \rho / \varepsilon_0$$

$$-\nabla^2 \mathbf{A}_L + \frac{1}{c^2}\frac{\partial^2 \mathbf{A}_L}{\partial t^2} = \mu_0 \mathbf{J}$$

General equation form:

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\Psi = -4\pi f$$

$$\Psi(\mathbf{r},t) = \begin{cases} \Phi(\mathbf{r},t) \\ A_x(\mathbf{r},t) \\ A_y(\mathbf{r},t) \\ A_z(\mathbf{r},t) \end{cases} \quad f(\mathbf{r},t) = \begin{cases} \rho(\mathbf{r},t)/(4\pi\varepsilon_0) \\ \mu_0 J_x(\mathbf{r},t)/(4\pi) \\ \mu_0 J_y(\mathbf{r},t)/(4\pi) \\ \mu_0 J_z(\mathbf{r},t)/(4\pi) \end{cases}$$

We will focus our attention on the Lorentz Gauge representations. In this case, the scalar potential and each of the three Cartesian components of the vector potential each have to solve an inhomogeneous differential equation of the same form.

Solution of Maxwell's equations in the Lorentz gauge -- continued

$$G(\mathbf{r},t;\mathbf{r}',t') = \frac{1}{|\mathbf{r}-\mathbf{r}'|}\,\delta\left(t' - \left(t - |\mathbf{r}-\mathbf{r}'|/c\right)\right)$$

Solution for field $\Psi(\mathbf{r},t)$:

$$\Psi(\mathbf{r},t) = \Psi_{f=0}(\mathbf{r},t) + \int d^3r' \int dt' \frac{1}{|\mathbf{r}-\mathbf{r}'|}\,\delta\left(t' - \left(t - \frac{1}{c}|\mathbf{r}-\mathbf{r}'|\right)\right) f(\mathbf{r}',t')$$

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For a spatially localized source, the physically meaningful solution can be written as an integral over the source time *t'* and space ***r'*** as discussed previously before.

Electromagnetic waves from time harmonic sources

Charge density: $\rho(\mathbf{r},t) = \Re\left(\tilde{\rho}(\mathbf{r},\omega)e^{-i\omega t}\right)$

Current density: $\mathbf{J}(\mathbf{r},t) = \Re\left(\tilde{\mathbf{J}}(\mathbf{r},\omega)e^{-i\omega t}\right)$

Note that the continuity condition applies:

$$\frac{\partial\rho(\mathbf{r},t)}{\partial t} + \nabla\cdot\mathbf{J}(\mathbf{r},t) = 0 \quad \Rightarrow -i\omega\tilde{\rho}(\mathbf{r},\omega) + \nabla\cdot\tilde{\mathbf{J}}(\mathbf{r},\omega) = 0$$

General source: $f(\mathbf{r},t) = \Re\left(\tilde{f}(\mathbf{r},\omega)e^{-i\omega t}\right)$

For $\tilde{f}(\mathbf{r},\omega) = \frac{1}{4\pi\varepsilon_0}\tilde{\rho}(\mathbf{r},\omega)$

or $\tilde{f}(\mathbf{r},\omega) = \frac{\mu_0}{4\pi}\tilde{J}_i(\mathbf{r},\omega)$

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Now we specialize to the pure harmonic time dependence. Mathematically, we will evaluate the sources with the complex function exp(*-iωt*), taking the real part at the end of the analysis. Note that because we need to conserve charge, the continuity equation must satisfied which consequently means that the current and charge densities are functionally related.

Electromagnetic waves from time harmonic sources – continued:

$$\Psi(\mathbf{r},t) = \Psi_{f=0}(\mathbf{r},t) + \int d^3r' \int dt' \frac{1}{|\mathbf{r}-\mathbf{r}'|} \delta\left(t' - \left(t - \frac{1}{c}|\mathbf{r}-\mathbf{r}'|\right)\right) f(\mathbf{r}',t')$$

$$\tilde{\Psi}(\mathbf{r},\omega)e^{-i\omega t} = \tilde{\Psi}_{f=0}(\mathbf{r},\omega)e^{-i\omega t} + \int d^3r' \int dt' \frac{1}{|\mathbf{r}-\mathbf{r}'|} \delta\left(t' - \left(t - \frac{1}{c}|\mathbf{r}-\mathbf{r}'|\right)\right) \tilde{f}(\mathbf{r}',\omega)e^{-i\omega t'}$$

$$= \tilde{\Psi}_{f=0}(\mathbf{r},\omega)e^{-i\omega t} + \int d^3r' \frac{e^{i\frac{\omega}{c}|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \tilde{f}(\mathbf{r}',\omega)e^{-i\omega t}$$

Putting the form of the source term in the integral, we can first perform the integral over the source time *t'*, resulting in the last equation of the slide. Notice that the full solution of the differential equation also may have a solution to the inhomogeneous equation as represented by the last term.

Electromagnetic waves from time harmonic sources – continued:

For scalar potential (Lorentz gauge, $k \equiv \frac{\omega}{c}$)

$$\tilde{\Phi}(\mathbf{r},\omega) = \tilde{\Phi}_0(\mathbf{r},\omega) + \frac{1}{4\pi\varepsilon_0}\int d^3r' \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\tilde{\rho}(\mathbf{r}',\omega),$$

where $\left(\nabla^2 + \frac{\omega^2}{c^2}\right)\tilde{\Phi}_0(\mathbf{r},\omega) = 0$

For vector potential (Lorentz gauge, $k \equiv \frac{\omega}{c}$)

$$\tilde{\mathbf{A}}(\mathbf{r},\omega) = \tilde{\mathbf{A}}_0(\mathbf{r},\omega) + \frac{\mu_0}{4\pi}\int d^3r' \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\tilde{\mathbf{J}}(\mathbf{r}',\omega),$$

where $\left(\nabla^2 + \frac{\omega^2}{c^2}\right)\tilde{\mathbf{A}}_0(\mathbf{r},\omega) = 0$

From the results on the previous slide, we can explicitly write out the solutions for the scalar and vector potentials in terms of the charge and current densities.

Electromagnetic waves from time harmonic sources – continued:

Useful expansion :

$$\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} = ik\sum_{lm} j_l(kr_<)h_l(kr_>)Y_{lm}(\hat{\mathbf{r}})Y^*_{lm}(\hat{\mathbf{r}}')$$

Spherical Bessel function : $j_l(kr)$

Spherical Hankel function : $h_l(kr) = j_l(kr) + in_l(kr)$

$$\tilde{\Phi}(\mathbf{r},\omega) = \tilde{\Phi}_0(\mathbf{r},\omega) + \sum_{lm}\tilde{\phi}_{lm}(r,\omega)Y_{lm}(\hat{\mathbf{r}})$$

$$\tilde{\phi}_{lm}(r,\omega) = \frac{ik}{\varepsilon_0}\int d^3r'\,\tilde{\rho}(\mathbf{r}',\omega)\,j_l(kr_<)h_l(kr_>)Y^*_{lm}(\hat{\mathbf{r}}')$$

In order to evaluate the equations on the previous slide, we can make use an exact expansion in terms of spherical harmonic functions and spherical Bessel and Hankel functions. The proof of this expansion is not trivial, but some details are available in Jackson (near Eq. 9.98) and from the NIST website https://dlmf.nist.gov/10.60. It naturally follows that the scalar potential can be expressed as a sum of spherical harmonic functions time corresponding radial forms.

Electromagnetic waves from time harmonic sources – continued:

Useful expansion :

$$\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} = ik\sum_{lm} j_l(kr_<)h_l(kr_>)Y_{lm}(\hat{\mathbf{r}})Y^*{}_{lm}(\hat{\mathbf{r}}')$$

Spherical Bessel function : $j_l(kr)$

Spherical Hankel function : $h_l(kr) = j_l(kr) + in_l(kr)$

$$\widetilde{\mathbf{A}}(\mathbf{r},\omega) = \widetilde{\mathbf{A}}_0(\mathbf{r},\omega) + \sum_{lm} \widetilde{\mathbf{a}}_{lm}(r,\omega)Y_{lm}(\hat{\mathbf{r}})$$

$$\widetilde{\mathbf{a}}_{lm}(r,\omega) = ik\mu_0\int d^3r'\,\widetilde{\mathbf{J}}(\mathbf{r}',\omega)j_l(kr_<)h_l(kr_>)Y^*{}_{lm}(\hat{\mathbf{r}}')$$

It naturally follows that the vector potential can be expressed as a sum of spherical harmonic functions time corresponding radial forms.

Forms of spherical Bessel and Hankel functions:

$$j_0(x)=\frac{\sin(x)}{x} \qquad h_0(x)=\frac{e^{ix}}{ix}$$

$$j_1(x)=\frac{\sin(x)}{x^2}-\frac{\cos(x)}{x} \qquad h_1(x)=-\left(1+\frac{i}{x}\right)\frac{e^{ix}}{x}$$

$$j_2(x)=\left(\frac{3}{x^3}-\frac{1}{x}\right)\sin(x)-\frac{3\cos(x)}{x^2} \qquad h_2(x)=i\left(1+\frac{3i}{x}-\frac{3}{x^2}\right)\frac{e^{ix}}{x}$$

Asymptotic behavior:

$$x<<1 \qquad \Rightarrow j_l(x)\approx\frac{(x)^l}{(2l+1)!!}$$

$$x>>1 \qquad \Rightarrow h_l(x)\approx(-i)^{l+1}\frac{e^{ix}}{x}$$

These relationships of spherical Bessel functions are given on page 426 of Jackson.

Digression on spherical Bessel functions --

Consider the homogeneous wave equation

$$\left(\nabla^2 + \frac{\omega^2}{c^2}\right)\tilde{\Phi}_0(\mathbf{r},\omega) = 0$$

Suppose $\tilde{\Phi}_0(\mathbf{r},\omega) = \psi_{lm}(r)Y_{lm}(\hat{\mathbf{r}})$

$\Rightarrow \psi_{lm}(r)$ must satisfy the following for $k = \omega / c$:

$$\left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{l(l+1)}{r^2} + k^2\right)\psi_{lm}(r) = 0$$

General Bessel function equation:

$$\left(\frac{d^2}{dx^2} + \frac{2}{x}\frac{d}{dx} - \frac{l(l+1)}{x^2} + 1\right)w_l(x) = 0 \qquad \Rightarrow \psi_{lm}(r) = w_l(kr)$$

This material summarizes some of the results from Seciont 9.6 of Jackson

Electromagnetic waves from time harmonic sources – continued:

$$\tilde{\Phi}(\mathbf{r},\omega)=\tilde{\Phi}_0(\mathbf{r},\omega)+\sum_{lm}\tilde{\phi}_{lm}(r,\omega)Y_{lm}(\hat{\mathbf{r}})$$

$$\tilde{\phi}_{lm}(r,\omega)=\frac{ik}{\varepsilon_0}\int d^3r'\,\tilde{\rho}(\mathbf{r}',\omega)\,j_l(kr_<)h_l(kr_>)Y^*{}_{lm}(\hat{\mathbf{r}}')$$

$$\tilde{\mathbf{A}}(\mathbf{r},\omega)=\tilde{\mathbf{A}}_0(\mathbf{r},\omega)+\sum_{lm}\tilde{\mathbf{a}}_{lm}(r,\omega)Y_{lm}(\hat{\mathbf{r}})$$

$$\tilde{\mathbf{a}}_{lm}(r,\omega)=ik\mu_0\int d^3r'\,\tilde{\mathbf{J}}(\mathbf{r}',\omega)\,j_l(kr_<)h_l(kr_>)Y^*{}_{lm}(\hat{\mathbf{r}}')$$

For $r >>$ (extent of source)

$$\tilde{\phi}_{lm}(r,\omega)\approx\frac{ik}{\varepsilon_0}h_l(kr)\int d^3r'\,\tilde{\rho}(\mathbf{r}',\omega)\,j_l(kr')Y^*{}_{lm}(\hat{\mathbf{r}}')$$

$$\tilde{\mathbf{a}}_{lm}(r,\omega)\approx ik\mu_0 h_l(kr)\int d^3r'\,\tilde{\mathbf{J}}(\mathbf{r}',\omega)\,j_l(kr')Y^*{}_{lm}(\hat{\mathbf{r}}')$$

What is the rational/significance of the last two equations?

Some details:

$$\tilde{\Phi}(\mathbf{r},\omega) = \tilde{\Phi}_0(\mathbf{r},\omega) + \sum_{lm} \tilde{\varphi}_{lm}(r,\omega) Y_{lm}(\hat{\mathbf{r}})$$

$$\tilde{\varphi}_{lm}(r,\omega) = \frac{ik}{\varepsilon_0} \int d^3r' \, \tilde{\rho}(\mathbf{r}',\omega) \, j_l(kr_<) h_l(kr_>) Y^*_{lm}(\hat{\mathbf{r}}')$$

$$= \frac{ik}{\varepsilon_0} \int d\Omega' Y^*_{lm}(\hat{\mathbf{r}}') \left(h_l(kr) \int_0^r r'^2 \, dr' j_l(kr') \tilde{\rho}(\mathbf{r}',\omega) + j_l(kr) \int_r^\infty r'^2 \, dr' h_l(kr') \tilde{\rho}(\mathbf{r}',\omega) \right)$$

For $r >>$ (extent of source)

$$\tilde{\phi}_{lm}(r,\omega) \approx \frac{ik}{\varepsilon_0} h_l(kr) \int d^3r' \, \tilde{\rho}(\mathbf{r}',\omega) j_l(kr') Y^*_{lm}(\hat{\mathbf{r}}')$$

$$\tilde{\mathbf{a}}_{lm}(r,\omega) \approx ik\mu_0 h_l(kr) \int d^3r' \, \tilde{\mathbf{J}}(\mathbf{r}',\omega) j_l(kr') Y^*_{lm}(\hat{\mathbf{r}}')$$

Do you agree with these results?

Electromagnetic waves from time harmonic sources – continued -- some details:

$$\tilde{\varphi}_{lm}(r,\omega)=\frac{ik}{\varepsilon_0}\int d^3r'\,\tilde{\rho}(\mathbf{r}',\omega)\,j_l(kr_<)\,h_l(kr_>)\,Y^*_{lm}(\hat{\mathbf{r}}')$$

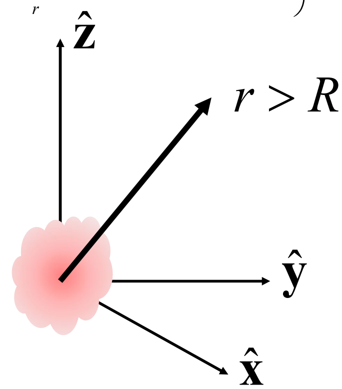
$$=\frac{ik}{\varepsilon_0}\left(h_l(kr)\int_0^r r'^2\,dr'\rho_{lm}(\mathbf{r}',\omega)j_l(kr')+j_l(kr)\int_r^\infty r'^2\,dr'\rho_{lm}(\mathbf{r}',\omega)h_l(kr')\right)$$

where $\rho_{lm}(\mathbf{r}',\omega)\equiv\int d\Omega'\rho_{lm}(\mathbf{r}',\omega)Y^*_{lm}(\hat{\mathbf{r}}')$

note that for $r>R$, where $\rho_{lm}(\mathbf{r},\omega)\approx 0$,

$$\tilde{\varphi}_{lm}(r,\omega)\approx\frac{ik}{\varepsilon_0}h_l(kr)\int_0^\infty r'^2\,dr'\rho_{lm}(\mathbf{r}',\omega)j_l(kr')$$

Similar relationships can be written for $\tilde{\mathbf{a}}_{lm}(r,\omega)$ and $\tilde{\mathbf{J}}(\mathbf{r}',\omega)$.



From this analysis, for a source confined within a sphere of radius *R*, the radiation field for the *lm* component of the field has a radial form proportional to a spherical Hankel function.

Electromagnetic waves from time harmonic sources – continued:

For r >> (extent of source)

$$\tilde{\phi}_{lm}(r,\omega) \approx \frac{ik}{\varepsilon_0} h_l(kr) \int d^3r' \, \tilde{\rho}(\mathbf{r}',\omega) j_l(kr') Y^*{}_{lm}(\hat{\mathbf{r}}')$$

$$\tilde{\mathbf{a}}_{lm}(r,\omega) \approx ik\mu_0 h_l(kr) \int d^3r' \, \tilde{\mathbf{J}}(\mathbf{r}',\omega) j_l(kr') Y^*{}_{lm}(\hat{\mathbf{r}}')$$

Note that $\tilde{\rho}(\mathbf{r}',\omega)$ and $\tilde{\mathbf{J}}(\mathbf{r}',\omega)$ are connected via the continuity condition : $-i\omega\, \tilde{\rho}(\mathbf{r},\omega) + \nabla \cdot \tilde{\mathbf{J}}(\mathbf{r},\omega) = 0$

$$\tilde{\phi}_{lm}(r,\omega) \approx \frac{ik}{\varepsilon_0} h_l(kr) \int d^3r' \, \tilde{\rho}(\mathbf{r}',\omega) j_l(kr') Y^*{}_{lm}(\hat{\mathbf{r}}')$$

$$= -\frac{k}{\omega\varepsilon_0} h_l(kr) \int d^3r' \, \tilde{\mathbf{J}}(\mathbf{r}',\omega) \cdot \nabla' \left(j_l(kr') Y^*{}_{lm}(\hat{\mathbf{r}}') \right)$$

Some further relations can be derived due to the continuity equation for the current density and the charge density.

Electromagnetic waves from time harmonic sources – continued:

Various approximations:

$$kr >> 1 \qquad \Rightarrow h_l(kr) \approx (-i)^{l+1} \frac{e^{ikr}}{kr}$$

$$kr' << 1 \qquad \Rightarrow j_l(kr') \approx \frac{(kr')^l}{(2l+1)!!}$$

Lowest (non-trivial) contributions in l expansions:

$$\tilde{\varphi}_{1m}(r,\omega) \approx \frac{ik}{\varepsilon_0} h_1(kr) \int d^3r' \tilde{\rho}(\mathbf{r}',\omega) \frac{kr'}{3} Y^*_{1m}(\hat{\mathbf{r}}')$$

$$\tilde{\mathbf{a}}_{00}(r,\omega) \approx ik\mu_0 h_0(kr) \int d^3r' \tilde{\mathbf{J}}(\mathbf{r}',\omega) Y^*_{00}(\hat{\mathbf{r}}')$$

The previous slides gave rigorous results far from the source. In this slide we consider further approximations. The *kr*'<<1 case is also referenced as the long wavelength approximation.

Some details -- continued: (assuming confined source)

Recall continuity condition: $-i\omega\,\tilde{\rho}(\mathbf{r},\omega)+\nabla\cdot\tilde{\mathbf{J}}(\mathbf{r},\omega)=0$

$$-i\omega\mathbf{r}\,\tilde{\rho}(\mathbf{r},\omega)+\mathbf{r}\nabla\cdot\tilde{\mathbf{J}}(\mathbf{r},\omega)$$

$$\int d^3r\;\mathbf{r}\;\tilde{\rho}(\mathbf{r},\omega)=\frac{1}{i\omega}\int d^3r\;\mathbf{r}\nabla\cdot\tilde{\mathbf{J}}(\mathbf{r},\omega)$$

$$=-\frac{1}{i\omega}\int d^3r\;\tilde{\mathbf{J}}(\mathbf{r},\omega)=\mathbf{p}(\omega)$$

Here we have used the identity:

$$\nabla\cdot(\psi\mathbf{V})=\nabla\psi\cdot\mathbf{V}+\psi(\nabla\cdot\mathbf{V})$$

We have also assumed that

$$\lim_{r\to\infty}\left(x\tilde{\mathbf{J}}(\mathbf{r},\omega)\right)=0$$

Dipole approximation continued.

Electromagnetic waves from time harmonic sources – continued:

Lowest order contribution; dipole radiation:

Define dipole moment at frequency ω:

$$\mathbf{p}(\omega) \equiv \int d^3r\ \mathbf{r}\tilde{\rho}(\mathbf{r},\omega) = -\frac{1}{i\omega}\int d^3r\ \tilde{\mathbf{J}}(\mathbf{r},\omega)$$

$$\tilde{\mathbf{A}}(\mathbf{r},\omega) = -\frac{i\mu_0\omega}{4\pi}\mathbf{p}(\omega)\frac{e^{ikr}}{r}$$

$$\tilde{\Phi}(\mathbf{r},\omega) = -\frac{ik}{4\pi\varepsilon_0}\mathbf{p}(\omega)\cdot\hat{\mathbf{r}}\left(1+\frac{i}{kr}\right)\frac{e^{ikr}}{r}$$

Note: in this case we have assumed a restricted extent of the source such that *kr'<<1.*

Dipole approximation continued.

Electromagnetic waves from time harmonic sources – continued:

$$\tilde{\mathbf{E}}(\mathbf{r},\omega) = -\nabla\tilde{\Phi}(\mathbf{r},\omega) + i\omega\tilde{\mathbf{A}}(\mathbf{r},\omega)$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{e^{ikr}}{r}\left(k^2\left((\hat{\mathbf{r}}\times\mathbf{p}(\omega))\times\hat{\mathbf{r}}\right) + \left(\frac{3\hat{\mathbf{r}}(\hat{\mathbf{r}}\cdot\mathbf{p}(\omega)) - \mathbf{p}(\omega)}{r^2}\right)(1-ikr)\right)$$

$$\tilde{\mathbf{B}}(\mathbf{r},\omega) = \nabla\times\tilde{\mathbf{A}}(\mathbf{r},\omega)$$

$$= \frac{1}{4\pi\varepsilon_0 c^2}\frac{e^{ikr}}{r}k^2(\hat{\mathbf{r}}\times\mathbf{p}(\omega))\left(1-\frac{1}{ikr}\right)$$

Power radiated for $kr >> 1$:

$$\frac{dP}{d\Omega} = r^2\hat{\mathbf{r}}\cdot\langle\mathbf{S}\rangle_{avg} = \frac{r^2\hat{\mathbf{r}}}{2\mu_0}\hat{\mathbf{r}}\cdot\Re\left(\tilde{\mathbf{E}}(\mathbf{r},\omega)\times\tilde{\mathbf{B}}^*(\mathbf{r},\omega)\right)$$

$$= \frac{c^2k^4}{32\pi^2}\sqrt{\frac{\mu_0}{\varepsilon_0}}\left|(\hat{\mathbf{r}}\times\mathbf{p}(\omega))\times\hat{\mathbf{r}}\right|^2$$

Dipole approximation continued.

Example of dipole radiation source

$$\tilde{\mathbf{J}}(\mathbf{r},\omega)=\hat{\mathbf{z}}J_0 e^{-r/R} \qquad \tilde{\rho}(\mathbf{r},\omega)=\frac{J_0}{-i\omega R}\cos\theta e^{-r/R}$$

$$\tilde{\mathbf{A}}(\mathbf{r},\omega)=\hat{\mathbf{z}}J_0\,(ik\mu_0)\int_0^\infty r'^2\,dr'\,e^{-r'/R}h_0(kr_>)j_0(kr_<)$$

$$\tilde{\Phi}(\mathbf{r},\omega)=-\frac{J_0 k}{\varepsilon_0\omega R}\cos\theta\int_0^\infty r'^2\,dr'\,e^{-r'/R}h_1(kr_>)j_1(kr_<)$$

Evaluation for $r >> R$:

$$\tilde{\mathbf{A}}(\mathbf{r},\omega)=\hat{\mathbf{z}}J_0\mu_0\frac{e^{ikr}}{r}\;\frac{2R^3}{(1+k^2R^2)^2}$$

$$\tilde{\Phi}(\mathbf{r},\omega)=\frac{J_0 k}{\varepsilon_0\omega}\cos\theta\,\frac{e^{ikr}}{r}\left(1+\frac{i}{kr}\right)\;\frac{2R^3}{(1+k^2R^2)^2}$$

Comparison of exact asymptotic results with dipole approximation.

Example of dipole radiation source -- continued

Evaluation for $r >> R$:

$$\tilde{\mathbf{A}}(\mathbf{r},\omega) = \hat{\mathbf{z}} J_0 \mu_0 \frac{e^{ikr}}{r} \; \frac{2R^3}{\left(1+k^2R^2\right)^2}$$

$$\tilde{\Phi}(\mathbf{r},\omega) = \frac{J_0 k}{\varepsilon_0 \omega} \cos\theta \frac{e^{ikr}}{r}\left(1+\frac{i}{kr}\right) \frac{2R^3}{\left(1+k^2R^2\right)^2}$$

Relationship to pure dipole approximation (exact when $kR \rightarrow 0$)

$$\mathbf{p}(\omega) \equiv \int d^3r \; \mathbf{r}\tilde{\rho}(\mathbf{r},\omega) = -\frac{1}{i\omega}\int d^3r \; \tilde{\mathbf{J}}(\mathbf{r},\omega) = -\frac{8\pi R^3 J_0}{i\omega}\hat{\mathbf{z}}$$

Corresponding dipole fields: $\tilde{\mathbf{A}}(\mathbf{r},\omega) = -\frac{i\mu_0\omega}{4\pi}\mathbf{p}(\omega)\frac{e^{ikr}}{r}$

$$\tilde{\Phi}(\mathbf{r},\omega) = -\frac{ik}{4\pi\varepsilon_0}\mathbf{p}(\omega)\cdot\hat{\mathbf{r}}\left(1+\frac{i}{kr}\right)\frac{e^{ikr}}{r}$$

Comparison of exact asymptotic results with dipole approximation – continued.