PHY 712 Electrodynamics
10-10:50 AM MWF Online

Discussion on Lecture 22:

Sources of radiation

Start reading Chap. 9

A. Electromagnetic waves due to specific sources

B. Dipole radiation patterns

19	Wed: 03/10/2021	Chap. 7	Optical effects of refractive indices	#14	03/12/2021
20	Fri: 03/12/2021	Chap. 1-7	Review		
	Mon: 03/15/2021	No class	*APS March Meeting*	Take Home Exam	
	Wed: 03/17/2021	No class	*APS March Meeting*	Take Home Exam	
	Fri: 03/19/2021	No class	*APS March Meeting*	Take Home Exam	
21	Mon: 03/22/2021	Chap. 8	EM waves in wave guides		
22	Wed: 03/24/2021	Chap. 9	Radiation from localized oscillating sources	#15	03/26/2021
23	Fri: 03/26/2021	Chap. 9	Radiation from oscillating sources		

PHY 712 -- Assignment #15

March 24, 2021

Start reading Chapters 9 in **Jackson** .

1. Work problem 9.10(b) in **Jackson**.

PHYSICS COLLOQUIUM

THURSDAY

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MARCH 25, 2021

4 PM online

"You Have Your Physics Results. Now What"

Samindranath Mitra, PhD

Editor

Physical Review Letters

American Physical Society

In a talk that I am hoping will quickly morph into a free-flowing Q and A session, I will discuss the roles of journals in general and PRL in particular in disseminating physics results through a cascading sequence involving journal editors, referees, conference chairs, journalists, department chairs, deans, funding agencies, and others. While some of the essential tools of physics dissemination are in essence unchanged, the arrival of social media, search engines, and electronic repositories have us in a state of flux.

Your questions –

From Tim -- So is it a good approach to first figure out if the charge is moving, hence we would use the vector potential because of the current density term in the integrand. If the charge is stationary then we should use the scalar potential. Can you have a combination of both? Or are these just two different ways of looking at the same source. Do the spherical Bessel functions arise because the potential of a point charge is spherically symmetric?

From Nick -- I was following okay until slide 15/16. \tilde{\phi} is defined with \tilde{\rho} on 15 but no tilde on 16. Why the change? And can you elaborate more on the integral form of \rho?

From Gao -- Why do you put scalar potential and vector potential in the same form differential equation not in two separate equations? What conclusion will it lead to or what benefits will it bring?

Maxwell's equations

Microscopic or vacuum form $(\mathbf{P}=0;\ \mathbf{M}=0)$:

Coulomb's law: $\nabla \cdot \mathbf{E} = \rho / \varepsilon_0$

Ampere-Maxwell's law: $\nabla \times \mathbf{B} - \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{J}$

Faraday's law: $\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$

No magnetic monopoles: $\nabla \cdot \mathbf{B} = 0$

$$\Rightarrow c^2 = \frac{1}{\varepsilon_0 \mu_0}$$

Formulation of Maxwell's equations in terms of vector and scalar potentials

$$\nabla \cdot \mathbf{B} = 0 \qquad \Rightarrow \mathbf{B} = \nabla \times \mathbf{A}$$

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \qquad \Rightarrow \nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} \right) = 0$$

$$\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\nabla \Phi$$

$$\text{or} \quad \mathbf{E} = -\nabla \Phi - \frac{\partial \mathbf{A}}{\partial t}$$

Formulation of Maxwell's equations in terms of vector and scalar potentials -- continued

$$\nabla \cdot \mathbf{E} = \rho / \varepsilon_0 :$$

$$-\nabla^2 \Phi - \frac{\partial (\nabla \cdot \mathbf{A})}{\partial t} = \rho / \varepsilon_0$$

$$\nabla \times \mathbf{B} - \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{J}$$

$$\nabla \times (\nabla \times \mathbf{A}) + \frac{1}{c^2} \left(\frac{\partial (\nabla \Phi)}{\partial t} + \frac{\partial^2 \mathbf{A}}{\partial t^2} \right) = \mu_0 \mathbf{J}$$

Complicated coupled mess!

Formulation of Maxwell's equations in terms of vector and scalar potentials -- continued

$$\text{Lorentz gauge form -- require:}\quad \nabla\cdot\mathbf{A}_L+\frac{1}{c^2}\frac{\partial\Phi_L}{\partial t}=0$$

$$-\nabla^2\Phi_L+\frac{1}{c^2}\frac{\partial^2\Phi_L}{\partial t^2}=\rho/\varepsilon_0$$

$$-\nabla^2\mathbf{A}_L+\frac{1}{c^2}\frac{\partial^2\mathbf{A}_L}{\partial t^2}=\mu_0\mathbf{J}$$

This choice decouples the equations for the scalar and vector potentials.

General equation form:

$$\left(\nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\Psi=-4\pi f$$

$$\Psi(\mathbf{r},t)=\begin{cases}\Phi(\mathbf{r},t)\\ A_x(\mathbf{r},t)\\ A_y(\mathbf{r},t)\\ A_z(\mathbf{r},t)\end{cases}\qquad f(\mathbf{r},t)=\begin{cases}\rho(\mathbf{r},t)/(4\pi\varepsilon_0)\\ \mu_0 J_x(\mathbf{r},t)/(4\pi)\\ \mu_0 J_y(\mathbf{r},t)/(4\pi)\\ \mu_0 J_z(\mathbf{r},t)/(4\pi)\end{cases}$$

Solution of Maxwell's equations in the Lorentz gauge -- continued

$$G(\mathbf{r},t;\mathbf{r}',t') = \frac{1}{|\mathbf{r}-\mathbf{r}'|}\delta\left(t' - \left(t - |\mathbf{r}-\mathbf{r}'|/c\right)\right)$$

Solution for field $\Psi(\mathbf{r},t)$:

$$\Psi(\mathbf{r},t) = \Psi_{f=0}(\mathbf{r},t) +$$

$$\int d^3r' \int dt' \frac{1}{|\mathbf{r}-\mathbf{r}'|}\delta\left(t' - \left(t - \frac{1}{c}|\mathbf{r}-\mathbf{r}'|\right)\right) f(\mathbf{r}',t')$$

Electromagnetic waves from time harmonic sources

Charge density: $\rho(\mathbf{r},t) = \Re\left(\tilde{\rho}(\mathbf{r},\omega)e^{-i\omega t}\right)$

Current density: $\mathbf{J}(\mathbf{r},t) = \Re\left(\tilde{\mathbf{J}}(\mathbf{r},\omega)e^{-i\omega t}\right)$

Note that the continuity condition applies:

$$\frac{\partial\rho(\mathbf{r},t)}{\partial t} + \nabla\cdot\mathbf{J}(\mathbf{r},t) = 0 \quad \Rightarrow -i\omega\tilde{\rho}(\mathbf{r},\omega) + \nabla\cdot\tilde{\mathbf{J}}(\mathbf{r},\omega) = 0$$

General source: $f(\mathbf{r},t) = \Re\left(\tilde{f}(\mathbf{r},\omega)e^{-i\omega t}\right)$

For $\tilde{f}(\mathbf{r},\omega) = \frac{1}{4\pi\varepsilon_0}\tilde{\rho}(\mathbf{r},\omega)$

or $\tilde{f}(\mathbf{r},\omega) = \frac{\mu_0}{4\pi}\tilde{J}_i(\mathbf{r},\omega)$

Electromagnetic waves from time harmonic sources – continued:

$$\Psi(\mathbf{r},t) = \Psi_{f=0}(\mathbf{r},t) +$$

$$\int d^3r' \int dt' \frac{1}{|\mathbf{r}-\mathbf{r}'|} \delta\left(t' - \left(t - \frac{1}{c}|\mathbf{r}-\mathbf{r}'|\right)\right) f(\mathbf{r}',t')$$

$$\widetilde{\Psi}(\mathbf{r},\omega)e^{-i\omega t} = \widetilde{\Psi}_{f=0}(\mathbf{r},\omega)e^{-i\omega t} +$$

$$\int d^3r' \int dt' \frac{1}{|\mathbf{r}-\mathbf{r}'|} \delta\left(t' - \left(t - \frac{1}{c}|\mathbf{r}-\mathbf{r}'|\right)\right) \widetilde{f}(\mathbf{r}',\omega)e^{-i\omega t'}$$

$$= \widetilde{\Psi}_{f=0}(\mathbf{r},\omega)e^{-i\omega t} + \int d^3r' \frac{e^{i\frac{\omega}{c}|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \widetilde{f}(\mathbf{r}',\omega)e^{-i\omega t}$$

Electromagnetic waves from time harmonic sources – continued:

For scalar potential (Lorentz gauge, $k \equiv \frac{\omega}{c}$)

$$\tilde{\Phi}(\mathbf{r},\omega) = \tilde{\Phi}_0(\mathbf{r},\omega) + \frac{1}{4\pi\varepsilon_0}\int d^3r' \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\tilde{\rho}(\mathbf{r}',\omega),$$

where $\left(\nabla^2 + \frac{\omega^2}{c^2}\right)\tilde{\Phi}_0(\mathbf{r},\omega) = 0$

For vector potential (Lorentz gauge, $k \equiv \frac{\omega}{c}$)

$$\tilde{\mathbf{A}}(\mathbf{r},\omega) = \tilde{\mathbf{A}}_0(\mathbf{r},\omega) + \frac{\mu_0}{4\pi}\int d^3r' \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\tilde{\mathbf{J}}(\mathbf{r}',\omega),$$

where $\left(\nabla^2 + \frac{\omega^2}{c^2}\right)\tilde{\mathbf{A}}_0(\mathbf{r},\omega) = 0$

Electromagnetic waves from time harmonic sources – continued:

Useful expansion :

$$\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} = ik\sum_{lm} j_l(kr_<)h_l(kr_>)Y_{lm}(\hat{\mathbf{r}})Y^*_{lm}(\hat{\mathbf{r}}')$$

Spherical Bessel function : $j_l(kr)$

Spherical Hankel function : $h_l(kr) = j_l(kr) + in_l(kr)$

$$\widetilde{\Phi}(\mathbf{r},\omega) = \widetilde{\Phi}_0(\mathbf{r},\omega) + \sum_{lm} \widetilde{\phi}_{lm}(r,\omega)Y_{lm}(\hat{\mathbf{r}})$$

$$\widetilde{\phi}_{lm}(r,\omega) = \frac{ik}{\varepsilon_0}\int d^3r' \,\widetilde{\rho}(\mathbf{r}',\omega)\, j_l(kr_<)h_l(kr_>)Y^*_{lm}(\hat{\mathbf{r}}')$$

Electromagnetic waves from time harmonic sources – continued:

Useful expansion :

$$\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} = ik\sum_{lm} j_l(kr_<)h_l(kr_>)Y_{lm}(\hat{\mathbf{r}})Y^*{}_{lm}(\hat{\mathbf{r}}')$$

Spherical Bessel function : $j_l(kr)$

Spherical Hankel function : $h_l(kr) = j_l(kr) + in_l(kr)$

$$\widetilde{\mathbf{A}}(\mathbf{r},\omega) = \widetilde{\mathbf{A}}_0(\mathbf{r},\omega) + \sum_{lm} \widetilde{\mathbf{a}}_{lm}(r,\omega)Y_{lm}(\hat{\mathbf{r}})$$

$$\widetilde{\mathbf{a}}_{lm}(r,\omega) = ik\mu_0\int d^3r'\,\widetilde{\mathbf{J}}(\mathbf{r}',\omega)j_l(kr_<)h_l(kr_>)Y^*{}_{lm}(\hat{\mathbf{r}}')$$

Forms of spherical Bessel and Hankel functions:

$$j_0(x) = \frac{\sin(x)}{x}$$

$$h_0(x) = \frac{e^{ix}}{ix}$$

$$j_1(x) = \frac{\sin(x)}{x^2} - \frac{\cos(x)}{x}$$

$$h_1(x) = -\left(1 + \frac{i}{x}\right)\frac{e^{ix}}{x}$$

$$j_2(x) = \left(\frac{3}{x^3} - \frac{1}{x}\right)\sin(x) - \frac{3\cos(x)}{x^2}$$

$$h_2(x) = i\left(1 + \frac{3i}{x} - \frac{3}{x^2}\right)\frac{e^{ix}}{x}$$

Asymptotic behavior:

$$x << 1 \qquad \Rightarrow j_l(x) \approx \frac{(x)^l}{(2l+1)!!}$$

$$x >> 1 \qquad \Rightarrow h_l(x) \approx (-i)^{l+1}\frac{e^{ix}}{x}$$

Digression on spherical Bessel functions --

Consider the homogeneous wave equation

$$\left(\nabla^2 + \frac{\omega^2}{c^2}\right)\tilde{\Phi}_0(\mathbf{r},\omega) = 0$$

Suppose $\tilde{\Phi}_0(\mathbf{r},\omega) = \psi_{lm}(r)Y_{lm}(\hat{\mathbf{r}})$

$\Rightarrow \psi_{lm}(r)$ must satisfy the following for $k = \omega / c$:

$$\left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{l(l+1)}{r^2} + k^2\right)\psi_{lm}(r) = 0$$

General Bessel function equation:

$$\left(\frac{d^2}{dx^2} + \frac{2}{x}\frac{d}{dx} - \frac{l(l+1)}{x^2} + 1\right)w_l(x) = 0 \qquad \Rightarrow \psi_{lm}(r) = w_l(kr)$$

Electromagnetic waves from time harmonic sources – continued:

$$\tilde{\Phi}(\mathbf{r},\omega)=\tilde{\Phi}_0(\mathbf{r},\omega)+\sum_{lm}\tilde{\phi}_{lm}(r,\omega)Y_{lm}(\hat{\mathbf{r}})$$

$$\tilde{\phi}_{lm}(r,\omega)=\frac{ik}{\varepsilon_0}\int d^3r'\,\tilde{\rho}(\mathbf{r}',\omega)\,j_l(kr_<)h_l(kr_>)Y^*{}_{lm}(\hat{\mathbf{r}}')$$

$$\tilde{\mathbf{A}}(\mathbf{r},\omega)=\tilde{\mathbf{A}}_0(\mathbf{r},\omega)+\sum_{lm}\tilde{\mathbf{a}}_{lm}(r,\omega)Y_{lm}(\hat{\mathbf{r}})$$

$$\tilde{\mathbf{a}}_{lm}(r,\omega)=ik\mu_0\int d^3r'\,\tilde{\mathbf{J}}(\mathbf{r}',\omega)\,j_l(kr_<)h_l(kr_>)Y^*{}_{lm}(\hat{\mathbf{r}}')$$

For $r >>$ (extent of source)

$$\tilde{\phi}_{lm}(r,\omega)\approx\frac{ik}{\varepsilon_0}h_l(kr)\int d^3r'\,\tilde{\rho}(\mathbf{r}',\omega)\,j_l(kr')Y^*{}_{lm}(\hat{\mathbf{r}}')$$

$$\tilde{\mathbf{a}}_{lm}(r,\omega)\approx ik\mu_0 h_l(kr)\int d^3r'\,\tilde{\mathbf{J}}(\mathbf{r}',\omega)\,j_l(kr')Y^*{}_{lm}(\hat{\mathbf{r}}')$$

Some details:

$$\tilde{\Phi}(\mathbf{r},\omega)=\tilde{\Phi}_0(\mathbf{r},\omega)+\sum_{lm}\tilde{\varphi}_{lm}(r,\omega)Y_{lm}(\hat{\mathbf{r}})$$

$$\tilde{\varphi}_{lm}(r,\omega)=\frac{ik}{\varepsilon_0}\int d^3r'\,\tilde{\rho}(\mathbf{r}',\omega)\,j_l(kr_<)\,h_l(kr_>)\,Y^*_{lm}(\hat{\mathbf{r}}')$$

$$=\frac{ik}{\varepsilon_0}\int d\Omega' Y^*_{lm}(\hat{\mathbf{r}}')\left(h_l(kr)\int_0^r r'^2\,dr'\,j_l(kr')\,\tilde{\rho}(\mathbf{r}',\omega)+j_l(kr)\int_r^\infty r'^2\,dr'\,h_l(kr')\,\tilde{\rho}(\mathbf{r}',\omega)\right)$$

For r >> (extent of source)

$$\tilde{\phi}_{lm}(r,\omega)\approx\frac{ik}{\varepsilon_0}h_l(kr)\int d^3r'\,\tilde{\rho}(\mathbf{r}',\omega)j_l(kr')Y^*_{lm}(\hat{\mathbf{r}}')$$

$$\tilde{\mathbf{a}}_{lm}(r,\omega)\approx ik\mu_0 h_l(kr)\int d^3r'\,\tilde{\mathbf{J}}(\mathbf{r}',\omega)j_l(kr')Y^*_{lm}(\hat{\mathbf{r}}')$$

Electromagnetic waves from time harmonic sources – continued -- some details:

$$\tilde{\varphi}_{lm}(r,\omega) = \frac{ik}{\varepsilon_0}\int d^3r'\,\tilde{\rho}(\mathbf{r}',\omega)\, j_l(kr_<)\, h_l(kr_>)\, Y^*_{lm}(\hat{\mathbf{r}}')$$

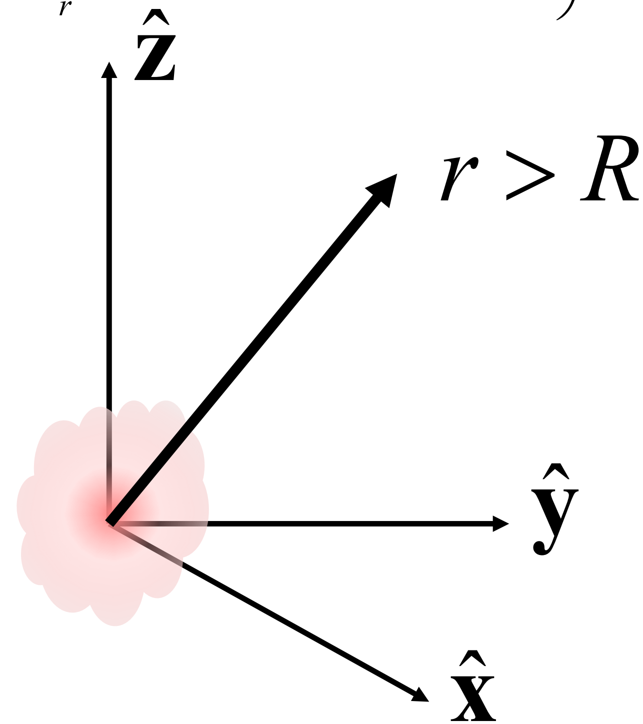
$$= \frac{ik}{\varepsilon_0}\left(h_l(kr)\int_0^r r'^2\, dr'\,\rho_{lm}(\mathbf{r}',\omega) j_l(kr') + j_l(kr)\int_r^\infty r'^2\, dr'\,\rho_{lm}(\mathbf{r}',\omega) h_l(kr') \right)$$

where $\rho_{lm}(\mathbf{r}',\omega) \equiv \int d\Omega'\,\tilde{\rho}(\mathbf{r}',\omega) Y^*_{lm}(\hat{\mathbf{r}}')$

note that for $r > R$, where $\tilde{\rho}(\mathbf{r},\omega) \approx 0$,

$$\tilde{\varphi}_{lm}(r,\omega) \approx \frac{ik}{\varepsilon_0} h_l(kr)\int_0^\infty r'^2\, dr'\,\rho_{lm}(\mathbf{r}',\omega) j_l(kr')$$

Similar relationships can be written for $\tilde{\mathbf{a}}_{lm}(r,\omega)$ and $\tilde{\mathbf{J}}(\mathbf{r}',\omega)$.



Electromagnetic waves from time harmonic sources – continued:

For $r \gg$ (extent of source)

$$\tilde{\phi}_{lm}(r,\omega) \approx \frac{ik}{\varepsilon_0} h_l(kr) \int d^3r' \, \tilde{\rho}(\mathbf{r}',\omega) j_l(kr') Y^*_{lm}(\hat{\mathbf{r}}')$$

$$\tilde{\mathbf{a}}_{lm}(r,\omega) \approx ik\mu_0 h_l(kr) \int d^3r' \, \tilde{\mathbf{J}}(\mathbf{r}',\omega) j_l(kr') Y^*_{lm}(\hat{\mathbf{r}}')$$

Note that $\tilde{\rho}(\mathbf{r}',\omega)$ and $\tilde{\mathbf{J}}(\mathbf{r}',\omega)$ are connected via the continuity condition: $\quad -i\omega\,\tilde{\rho}(\mathbf{r},\omega) + \nabla \cdot \tilde{\mathbf{J}}(\mathbf{r},\omega) = 0$

$$\tilde{\phi}_{lm}(r,\omega) \approx \frac{ik}{\varepsilon_0} h_l(kr) \int d^3r' \, \tilde{\rho}(\mathbf{r}',\omega) j_l(kr') Y^*_{lm}(\hat{\mathbf{r}}')$$

$$= -\frac{k}{\omega\varepsilon_0} h_l(kr) \int d^3r' \, \tilde{\mathbf{J}}(\mathbf{r}',\omega) \cdot \nabla' \left(j_l(kr') Y^*_{lm}(\hat{\mathbf{r}}') \right)$$

Electromagnetic waves from time harmonic sources – continued:

Various approximations:

$$kr >> 1 \qquad \Rightarrow h_l\left(kr\right) \approx \left(-i\right)^{l+1} \frac{e^{ikr}}{kr}$$

$$kr' << 1 \qquad \Rightarrow j_l\left(kr'\right) \approx \frac{\left(kr'\right)^l}{\left(2l+1\right)!!}$$

Lowest (non-trivial) contributions in l expansions:

$$\tilde{\varphi}_{1m}\left(r,\omega\right) \approx \frac{ik}{\varepsilon_0} h_1(kr) \int d^3r' \tilde{\rho}\left(\mathbf{r}',\omega\right) \frac{kr'}{3} Y^*_{1m}\left(\hat{\mathbf{r}}'\right)$$

$$\tilde{\mathbf{a}}_{00}\left(r,\omega\right) \approx ik\mu_0 h_0(kr) \int d^3r' \tilde{\mathbf{J}}\left(\mathbf{r}',\omega\right) Y^*_{00}\left(\hat{\mathbf{r}}'\right)$$

Some details -- continued: (assuming confined source)

Recall continuity condition: $-i\omega\,\tilde{\rho}(\mathbf{r},\omega)+\nabla\cdot\tilde{\mathbf{J}}(\mathbf{r},\omega)=0$

$$-i\omega\mathbf{r}\;\tilde{\rho}(\mathbf{r},\omega)+\mathbf{r}\nabla\cdot\tilde{\mathbf{J}}(\mathbf{r},\omega)$$

$$\int d^3r\;\mathbf{r}\;\tilde{\rho}(\mathbf{r},\omega)=\frac{1}{i\omega}\int d^3r\;\mathbf{r}\nabla\cdot\tilde{\mathbf{J}}(\mathbf{r},\omega)$$

$$=-\frac{1}{i\omega}\int d^3r\;\tilde{\mathbf{J}}(\mathbf{r},\omega)=\mathbf{p}(\omega)$$

Here we have used the identity:

$$\nabla\cdot(\psi\mathbf{V})=\nabla\psi\cdot\mathbf{V}+\psi(\nabla\cdot\mathbf{V})$$

We have also assumed that

$$\lim_{r\to\infty}\left(x\tilde{\mathbf{J}}(\mathbf{r},\omega)\right)=0$$

Electromagnetic waves from time harmonic sources – continued:

Lowest order contribution; dipole radiation:

Define dipole moment at frequency ω:

$$\mathbf{p}(\omega) \equiv \int d^3r\ \mathbf{r}\tilde{\rho}(\mathbf{r},\omega) = -\frac{1}{i\omega}\int d^3r\ \tilde{\mathbf{J}}(\mathbf{r},\omega)$$

$$\tilde{\mathbf{A}}(\mathbf{r},\omega) = -\frac{i\mu_0\omega}{4\pi}\mathbf{p}(\omega)\frac{e^{ikr}}{r}$$

$$\tilde{\Phi}(\mathbf{r},\omega) = -\frac{ik}{4\pi\varepsilon_0}\mathbf{p}(\omega)\cdot\hat{\mathbf{r}}\left(1+\frac{i}{kr}\right)\frac{e^{ikr}}{r}$$

Note: in this case we have assumed a restricted extent of the source such that *kr'<<1* for all *r'* with significant charge/current density.

Electromagnetic waves from time harmonic sources – continued:

$$\tilde{\mathbf{E}}(\mathbf{r},\omega) = -\nabla\tilde{\Phi}(\mathbf{r},\omega) + i\omega\tilde{\mathbf{A}}(\mathbf{r},\omega)$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{e^{ikr}}{r}\left(k^2\left(\left(\hat{\mathbf{r}}\times\mathbf{p}(\omega)\right)\times\hat{\mathbf{r}}\right) + \left(\frac{3\hat{\mathbf{r}}\left(\hat{\mathbf{r}}\cdot\mathbf{p}(\omega)\right) - \mathbf{p}(\omega)}{r^2}\right)(1-ikr)\right)$$

$$\tilde{\mathbf{B}}(\mathbf{r},\omega) = \nabla\times\tilde{\mathbf{A}}(\mathbf{r},\omega)$$

$$= \frac{1}{4\pi\varepsilon_0 c^2}\frac{e^{ikr}}{r}k^2\left(\hat{\mathbf{r}}\times\mathbf{p}(\omega)\right)\left(1-\frac{1}{ikr}\right)$$

Power radiated for $kr >> 1$:

$$\frac{dP}{d\Omega} = r^2\hat{\mathbf{r}}\cdot\langle\mathbf{S}\rangle_{avg} = \frac{r^2}{2\mu_0}\hat{\mathbf{r}}\cdot\Re\left(\tilde{\mathbf{E}}(\mathbf{r},\omega)\times\tilde{\mathbf{B}}^*(\mathbf{r},\omega)\right)$$

$$= \frac{c^2k^4}{32\pi^2}\sqrt{\frac{\mu_0}{\varepsilon_0}}\left|\left(\hat{\mathbf{r}}\times\mathbf{p}(\omega)\right)\times\hat{\mathbf{r}}\right|^2$$

Example of dipole radiation source

$$\tilde{\mathbf{J}}(\mathbf{r},\omega)=\hat{\mathbf{z}}J_0 e^{-r/R} \qquad \tilde{\rho}(\mathbf{r},\omega)=\frac{J_0}{-i\omega R}\cos\theta e^{-r/R}$$

$$\tilde{\mathbf{A}}(\mathbf{r},\omega)=\hat{\mathbf{z}}J_0\,(ik\mu_0)\int_0^\infty r'^2\,dr'\,e^{-r'/R}h_0(kr_>)j_0(kr_<)$$

$$\tilde{\Phi}(\mathbf{r},\omega)=-\frac{J_0 k}{\varepsilon_0\omega R}\cos\theta\int_0^\infty r'^2\,dr'\,e^{-r'/R}h_1(kr_>)j_1(kr_<)$$

Evaluation for $r >> R$:

$$\tilde{\mathbf{A}}(\mathbf{r},\omega)=\hat{\mathbf{z}}J_0\mu_0\frac{e^{ikr}}{r}\;\frac{2R^3}{\left(1+k^2R^2\right)^2}$$

$$\tilde{\Phi}(\mathbf{r},\omega)=\frac{J_0 k}{\varepsilon_0\omega}\cos\theta\frac{e^{ikr}}{r}\left(1+\frac{i}{kr}\right)\;\frac{2R^3}{\left(1+k^2R^2\right)^2}$$

Example of dipole radiation source -- continued

Evaluation for $r >> R$:

$$\tilde{\mathbf{A}}(\mathbf{r},\omega) = \hat{\mathbf{z}} J_0 \mu_0 \frac{e^{ikr}}{r} \; \frac{2R^3}{\left(1+k^2R^2\right)^2}$$

$$\tilde{\Phi}(\mathbf{r},\omega) = \frac{J_0 k}{\varepsilon_0 \omega} \cos\theta \, \frac{e^{ikr}}{r} \left(1+\frac{i}{kr}\right) \; \frac{2R^3}{\left(1+k^2R^2\right)^2}$$

Relationship to pure dipole approximation (exact when *kR*➔0)

$$\mathbf{p}(\omega) \equiv \int d^3r \; \mathbf{r}\tilde{\rho}(\mathbf{r},\omega) = -\frac{1}{i\omega}\int d^3r \; \tilde{\mathbf{J}}(\mathbf{r},\omega) = -\frac{8\pi R^3 J_0}{i\omega}\hat{\mathbf{z}}$$

Corresponding dipole fields: $\tilde{\mathbf{A}}(\mathbf{r},\omega) = -\frac{i\mu_0\omega}{4\pi}\mathbf{p}(\omega)\frac{e^{ikr}}{r}$

$$\tilde{\Phi}(\mathbf{r},\omega) = -\frac{ik}{4\pi\varepsilon_0}\mathbf{p}(\omega)\cdot\hat{\mathbf{r}}\left(1+\frac{i}{kr}\right)\frac{e^{ikr}}{r}$$