

PHY 712 Electrodynamics

10-10:50 AM MWF Online

Notes for Lecture 23:

Continue reading Chap. 9 & 10

A. Electromagnetic waves due to specific sources

B. Dipole radiation examples

C. Scattered radiation

				Exam	
21	Mon: 03/22/2021	Chap. 8	EM waves in wave guides		
22	Wed: 03/24/2021	Chap. 9	Radiation from localized oscillating sources	#15	03/26/2021
23	Fri: 03/26/2021	Chap. 9	Radiation from oscillating sources	#16	03/29/2021
24	Mon: 03/29/2021	Chap. 9 & 10	Radiation and scattering		

PHY 712 – Problem Set #16

Continue reading Chapter 9 in **Jackson**

1. For HW # 15, you evaluated, using the dipole approximation, the radiation for a time harmonic charge density

$$\rho(\mathbf{r}) = \frac{2e}{\sqrt{6}a_0^4} r \exp\left(-\frac{3r}{2a_0}\right) Y_{00}(\hat{\mathbf{r}})Y_{10}(\hat{\mathbf{r}})e^{-i\omega_0 t}.$$

Note that in this case, ω_0 can be expressed in terms of the fine structure constant α , speed of light c , and Bohr radius a_0 as $\omega_0 = \frac{3}{8} \frac{\alpha c}{a_0}$. Compare the results for the scalar potential $\Phi(\mathbf{r}, t) = \tilde{\Phi}(\mathbf{r}, \omega_0)e^{-i\omega_0 t}$ in the far field limit, using

- (a) the dipole approximation (as in HW #15)
- (b) the exact result (in the far field limit).
- (c) Discuss how numerically close the two results are (or are not).

Your questions –

From Gao -- How to generate two situations: 1 Einc polarized perpendicular to scattering plane and 2 Einc polarized in scattering plane?
What physical mechanism leads to two different results?

Important results from last time – EM waves from time harmonic sources – open isotropic boundaries

For scalar potential (Lorentz gauge, $k \equiv \frac{\omega}{c}$)

$$\tilde{\Phi}(\mathbf{r}, \omega) = \tilde{\Phi}_0(\mathbf{r}, \omega) + \frac{1}{4\pi\epsilon_0} \int d^3r' \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \tilde{\rho}(\mathbf{r}', \omega),$$

where $\left(\nabla^2 + \frac{\omega^2}{c^2} \right) \tilde{\Phi}_0(\mathbf{r}, \omega) = 0$

For vector potential (Lorentz gauge, $k \equiv \frac{\omega}{c}$)

$$\tilde{\mathbf{A}}(\mathbf{r}, \omega) = \tilde{\mathbf{A}}_0(\mathbf{r}, \omega) + \frac{\mu_0}{4\pi} \int d^3r' \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \tilde{\mathbf{J}}(\mathbf{r}', \omega),$$

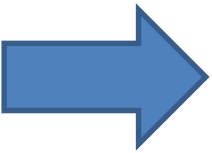
where $\left(\nabla^2 + \frac{\omega^2}{c^2} \right) \tilde{\mathbf{A}}_0(\mathbf{r}, \omega) = 0$

Useful expansion :

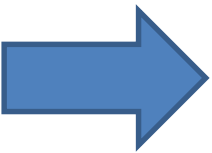
$$\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} = ik \sum_{lm} j_l(kr_<) h_l(kr_>) Y_{lm}(\hat{\mathbf{r}}) Y_{lm}^*(\hat{\mathbf{r}}')$$

Spherical Bessel function : $j_l(kr)$

Spherical Hankel function : $h_l(kr) = j_l(kr) + in_l(kr)$


$$\tilde{\Phi}(\mathbf{r}, \omega) = \tilde{\Phi}_0(\mathbf{r}, \omega) + \sum_{lm} \tilde{\phi}_{lm}(r, \omega) Y_{lm}(\hat{\mathbf{r}})$$

$$\tilde{\phi}_{lm}(r, \omega) = \frac{ik}{\epsilon_0} \int d^3r' \tilde{\rho}(\mathbf{r}', \omega) j_l(kr_<) h_l(kr_>) Y_{lm}^*(\hat{\mathbf{r}}')$$


$$\tilde{\mathbf{A}}(\mathbf{r}, \omega) = \tilde{\mathbf{A}}_0(\mathbf{r}, \omega) + \sum_{lm} \tilde{\mathbf{a}}_{lm}(r, \omega) Y_{lm}(\hat{\mathbf{r}})$$

$$\tilde{\mathbf{a}}_{lm}(r, \omega) = ik\mu_0 \int d^3r' \tilde{\mathbf{J}}(\mathbf{r}', \omega) j_l(kr_<) h_l(kr_>) Y_{lm}^*(\hat{\mathbf{r}}')$$

Example of dipole radiation source

$$\tilde{\mathbf{J}}(\mathbf{r}, \omega) = \hat{\mathbf{z}} J_0 e^{-r/R} \quad \tilde{\rho}(\mathbf{r}, \omega) = \frac{J_0}{-i\omega R} \cos \theta e^{-r/R}$$

Note that the continuity of charge and current must be satisfied. For the Fourier amplitudes, the relations are as bellow:

Recall continuity condition: $-i\omega \tilde{\rho}(\mathbf{r}, \omega) + \nabla \cdot \tilde{\mathbf{J}}(\mathbf{r}, \omega) = 0$

$$-i\omega \mathbf{r} \tilde{\rho}(\mathbf{r}, \omega) + \mathbf{r} \nabla \cdot \tilde{\mathbf{J}}(\mathbf{r}, \omega) \quad \text{Jackson's clever trick!}$$

$$\begin{aligned} \int d^3r \mathbf{r} \tilde{\rho}(\mathbf{r}, \omega) &= \frac{1}{i\omega} \int d^3r \mathbf{r} \nabla \cdot \tilde{\mathbf{J}}(\mathbf{r}, \omega) \\ &= -\frac{1}{i\omega} \int d^3r \tilde{\mathbf{J}}(\mathbf{r}, \omega) = \mathbf{p}(\omega) \end{aligned}$$

Example of dipole radiation source

$$\tilde{\mathbf{J}}(\mathbf{r}, \omega) = \hat{\mathbf{z}} J_0 e^{-r/R} \quad \tilde{\rho}(\mathbf{r}, \omega) = \frac{J_0}{-i\omega R} \cos \theta e^{-r/R}$$

$$\mathbf{p}(\omega) = \hat{\mathbf{z}} \frac{J_0}{-i\omega R} \int d^3r \, r \cos(\theta) \left(\cos(\theta) e^{-r/R} \right)$$

$$= \hat{\mathbf{z}} \frac{J_0}{-i\omega R} \frac{4\pi}{3} \int_0^\infty dr \, r^3 e^{-r/R} = \hat{\mathbf{z}} J_0 \frac{8\pi R^3}{-i\omega}$$

$$= \frac{1}{-i\omega} \int d^3r \, \tilde{\mathbf{J}}(\mathbf{r}, \omega)$$

From the analysis valid for $kr \gg 1$ and $kR \ll 1$:

$$\tilde{\mathbf{A}}(\mathbf{r}, \omega) = -\frac{i\mu_0\omega}{4\pi} \mathbf{p}(\omega) \frac{e^{ikr}}{r} = \hat{\mathbf{z}} J_0 \mu_0 2R^3 \frac{e^{ikr}}{r}$$

$$\tilde{\Phi}(\mathbf{r}, \omega) = -\frac{ik}{4\pi\epsilon_0} \mathbf{p}(\omega) \cdot \hat{\mathbf{r}} \left(1 + \frac{i}{kr} \right) \frac{e^{ikr}}{r} = \frac{J_0 2R^3}{\epsilon_0 c} \left(1 + \frac{i}{kr} \right) \frac{e^{ikr}}{r} \cos \theta$$

Example of dipole radiation source -- exact results for $r \gg R$:

$$\tilde{\mathbf{J}}(\mathbf{r}, \omega) = \hat{\mathbf{z}} J_0 e^{-r/R} \quad \tilde{\rho}(\mathbf{r}, \omega) = \frac{J_0}{-i\omega R} \cos \theta e^{-r/R}$$

$$\tilde{\mathbf{A}}(\mathbf{r}, \omega) = \hat{\mathbf{z}} J_0 (ik\mu_0) \int_0^\infty r'^2 dr' e^{-r'/R} h_0(kr_>) j_0(kr_<)$$

$$\tilde{\Phi}(\mathbf{r}, \omega) = -\frac{J_0 k}{\epsilon_0 \omega R} \cos \theta \int_0^\infty r'^2 dr' e^{-r'/R} h_1(kr_>) j_1(kr_<)$$

Evaluation for $r \gg R$:

$$\tilde{\mathbf{A}}(\mathbf{r}, \omega) = \hat{\mathbf{z}} J_0 \mu_0 \frac{e^{ikr}}{r} \frac{2R^3}{(1 + k^2 R^2)^2}$$

Agrees with dipole approximation for $kR \ll 1$.

$$\tilde{\Phi}(\mathbf{r}, \omega) = \frac{J_0 k}{\epsilon_0 \omega} \cos \theta \frac{e^{ikr}}{r} \left(1 + \frac{i}{kr}\right) \frac{2R^3}{(1 + k^2 R^2)^2}$$

More details --

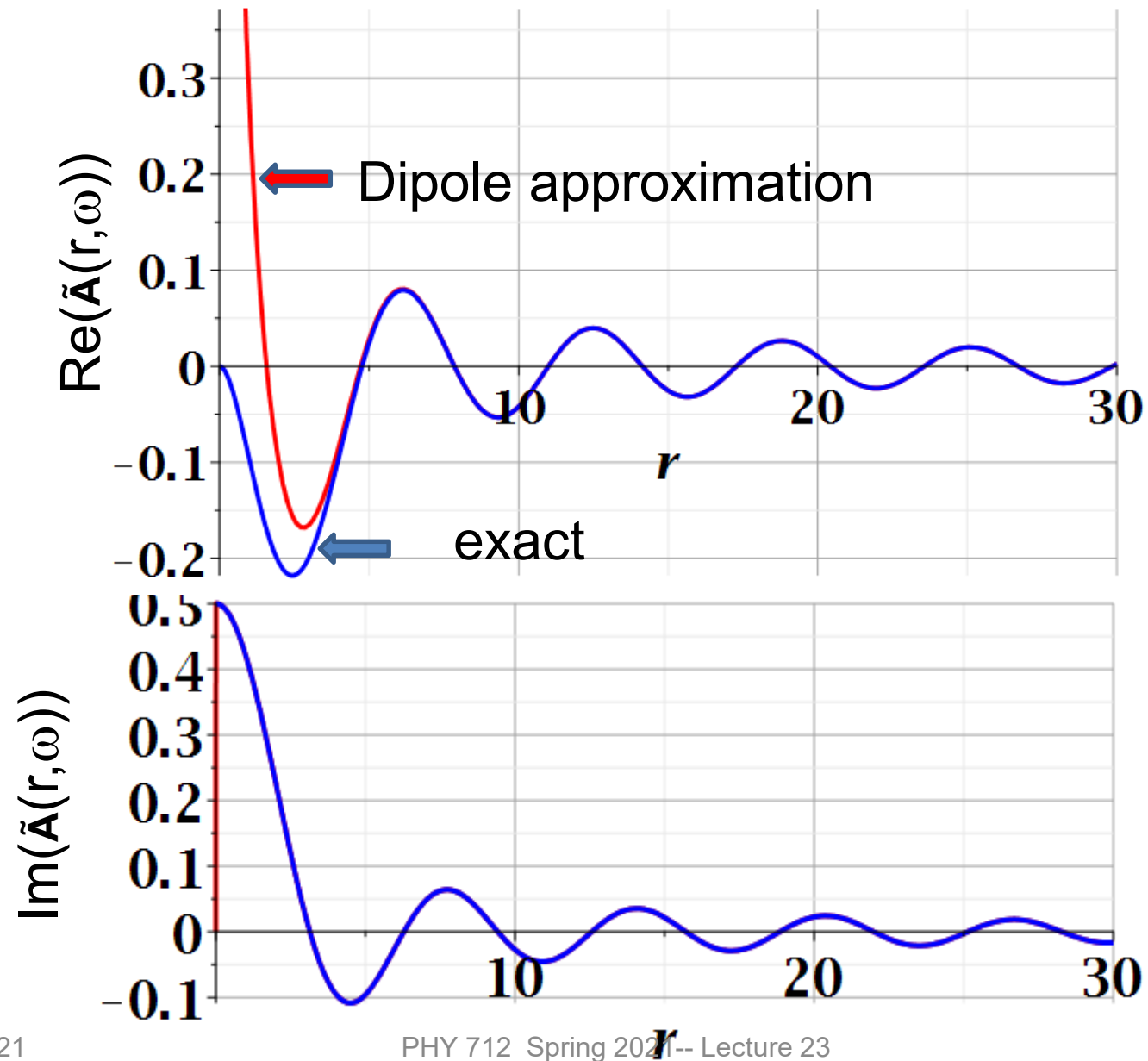
$$\tilde{\mathbf{J}}(\mathbf{r}, \omega) = \hat{\mathbf{z}} J_0 e^{-r/R} \quad \tilde{\rho}(\mathbf{r}, \omega) = \frac{J_0}{-i\omega R} \cos \theta e^{-r/R}$$

$$\tilde{\mathbf{A}}(\mathbf{r}, \omega) = \hat{\mathbf{z}} J_0 (ik\mu_0) \int_0^\infty r'^2 dr' e^{-r'/R} h_0(kr_>) j_0(kr_<)$$

$$\tilde{\Phi}(\mathbf{r}, \omega) = -\frac{J_0 k}{\varepsilon_0 \omega R} \cos \theta \int_0^\infty r'^2 dr' e^{-r'/R} h_1(kr_>) j_1(kr_<)$$

$$\begin{aligned} \tilde{\mathbf{A}}(r, \omega) &= \hat{\mathbf{z}} J_0 \mu_0 \left(\frac{e^{ikr}}{kr} \int_0^r r' dr' e^{-r'/R} \sin(kr') + \frac{\sin(kr)}{kr} \int_r^\infty r' dr' e^{-r'/R + ikr'} \right) \\ &\underset{r \gg R}{\approx} \hat{\mathbf{z}} J_0 \mu_0 \frac{e^{ikr}}{kr} \frac{2Rk^2 R^2}{(k^2 R^2 + 1)^2} \end{aligned}$$

Example continued



Continued review of dipole results --

Power in the dipole approximation; Section 9.2 of Jackson

Here we use our notation with $\mathbf{n} \rightarrow \hat{\mathbf{r}}$ and $Z_0 \equiv \sqrt{\frac{\mu_0}{\epsilon_0}}$

$$\frac{dP}{d\Omega} = \frac{r^2}{2} \Re \left| \left(\hat{\mathbf{r}} \cdot (\mathbf{E} \times \mathbf{H}^*) \right) \right|^2$$

Using the expressions for the dipole fields far from the source:

$$\mathbf{H} = \frac{ck^2}{4\pi} (\hat{\mathbf{r}} \times \mathbf{p}) \frac{e^{ikr}}{r} \quad \mathbf{E} = Z_0 \mathbf{H} \times \hat{\mathbf{r}}$$

The power can be written

$$\frac{dP}{d\Omega} = \frac{c^2 Z_0}{32\pi^2} k^4 \left| ((\hat{\mathbf{r}} \times \mathbf{p}) \times \hat{\mathbf{r}}) \right|^2$$

Defining the angle θ by $\mathbf{p} \cdot \hat{\mathbf{r}} = |\mathbf{p}| \cos \theta$,

$$\frac{dP}{d\Omega} = \frac{c^2 Z_0}{32\pi^2} k^4 |\mathbf{p}|^2 \sin^2 \theta \quad \text{integrating over solid angles} \quad P = \frac{c^2 Z_0}{12\pi} k^4 |\mathbf{p}|^2$$

Review:

Electromagnetic waves from time harmonic sources – continued:

Dipole radiation case:

Define dipole moment at frequency ω :

$$\mathbf{p}(\omega) \equiv \int d^3r \, \mathbf{r} \tilde{\rho}(\mathbf{r}, \omega) = -\frac{1}{i\omega} \int d^3r \, \tilde{\mathbf{J}}(\mathbf{r}, \omega)$$

For fields outside extent of source and $kr' \ll 1$ within the source:

$$\tilde{\mathbf{A}}(\mathbf{r}, \omega) = -\frac{i\mu_0\omega}{4\pi} \mathbf{p}(\omega) \frac{e^{ikr}}{r}$$

$$\tilde{\Phi}(\mathbf{r}, \omega) = -\frac{ik}{4\pi\epsilon_0} \mathbf{p}(\omega) \cdot \hat{\mathbf{r}} \left(1 + \frac{i}{kr} \right) \frac{e^{ikr}}{r}$$

Review:

Electromagnetic waves from time harmonic sources – continued:

$$\begin{aligned}\tilde{\mathbf{E}}(\mathbf{r}, \omega) &= -\nabla \tilde{\Phi}(\mathbf{r}, \omega) + i\omega \tilde{\mathbf{A}}(\mathbf{r}, \omega) \\ &= \frac{1}{4\pi\epsilon_0} \frac{e^{ikr}}{r} \left(k^2 \left((\hat{\mathbf{r}} \times \mathbf{p}(\omega)) \times \hat{\mathbf{r}} \right) + \left(\frac{3\hat{\mathbf{r}}(\hat{\mathbf{r}} \cdot \mathbf{p}(\omega)) - \mathbf{p}(\omega)}{r^2} \right) (1 - ikr) \right)\end{aligned}$$

$$\begin{aligned}\tilde{\mathbf{B}}(\mathbf{r}, \omega) &= \nabla \times \tilde{\mathbf{A}}(\mathbf{r}, \omega) \\ &= \frac{1}{4\pi\epsilon_0 c} \frac{e^{ikr}}{r} k^2 (\hat{\mathbf{r}} \times \mathbf{p}(\omega)) \left(1 - \frac{1}{ikr} \right)\end{aligned}$$

Power radiated for $kr \gg 1$:

$$\begin{aligned}\frac{dP}{d\Omega} &= r^2 \hat{\mathbf{r}} \cdot \langle \mathbf{S} \rangle_{avg} = \frac{r^2}{2\mu_0} \hat{\mathbf{r}} \cdot \Re \left(\tilde{\mathbf{E}}(\mathbf{r}, \omega) \times \tilde{\mathbf{B}}^*(\mathbf{r}, \omega) \right) \\ &= \frac{c^2 k^4}{32\pi^2} \sqrt{\frac{\mu_0}{\epsilon_0}} \left| (\hat{\mathbf{r}} \times \mathbf{p}(\omega)) \times \hat{\mathbf{r}} \right|^2\end{aligned}$$

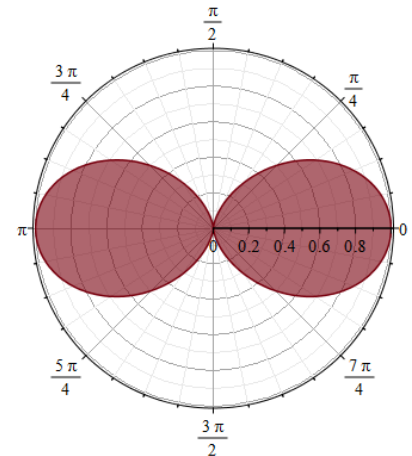
Properties of dipole radiation field for $kr \gg 1$:

$$\tilde{\mathbf{E}}(\mathbf{r}, \omega) = \frac{1}{4\pi\epsilon_0} \frac{e^{ikr}}{r} \left(k^2 ((\hat{\mathbf{r}} \times \mathbf{p}(\omega)) \times \hat{\mathbf{r}}) \right)$$

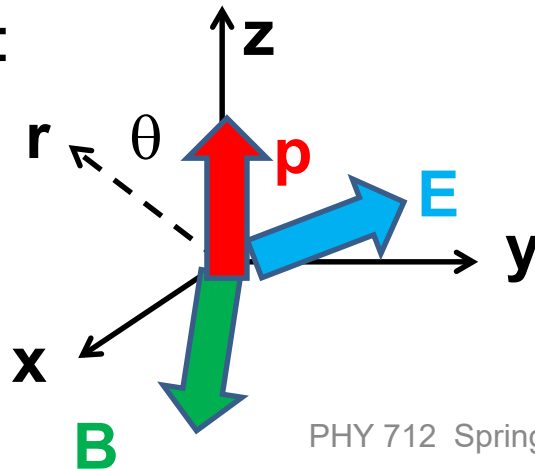
$$\tilde{\mathbf{B}}(\mathbf{r}, \omega) = \frac{1}{4\pi\epsilon_0 c^2} \frac{e^{ikr}}{r} k^2 (\hat{\mathbf{r}} \times \mathbf{p}(\omega))$$

Power radiated for $kr \gg 1$:

$$\frac{dP}{d\Omega} = r^2 \hat{\mathbf{r}} \cdot \langle \mathbf{S} \rangle_{avg} = \frac{c^2 k^4}{32\pi^2} \sqrt{\frac{\mu_0}{\epsilon_0}} |(\hat{\mathbf{r}} \times \mathbf{p}(\omega)) \times \hat{\mathbf{r}}|^2$$



Example:



Note that vectors $\mathbf{r}$, $\mathbf{E}$, $\mathbf{B}$ are mutually orthogonal

Alternative approach

Fields from time harmonic source:

$$\tilde{\Phi}(\mathbf{r}, \omega) = \frac{1}{4\pi\epsilon_0} \int d^3r' \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \tilde{\rho}(\mathbf{r}', \omega)$$

$$\tilde{\mathbf{A}}(\mathbf{r}, \omega) = \frac{\mu_0}{4\pi} \int d^3r' \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \tilde{\mathbf{J}}(\mathbf{r}', \omega)$$

For $r \gg r'$: $|\mathbf{r} - \mathbf{r}'| \approx r - \hat{\mathbf{r}} \cdot \mathbf{r}' + \dots$

$$\tilde{\Phi}(\mathbf{r}, \omega) \approx \frac{1}{4\pi\epsilon_0} \frac{e^{ikr}}{r} \int d^3r' e^{-ik\hat{\mathbf{r}} \cdot \mathbf{r}'} \tilde{\rho}(\mathbf{r}', \omega)$$

$$\tilde{\mathbf{A}}(\mathbf{r}, \omega) \approx \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \int d^3r' e^{-ik\hat{\mathbf{r}} \cdot \mathbf{r}'} \tilde{\mathbf{J}}(\mathbf{r}', \omega)$$

For our example:

$$\tilde{\mathbf{J}}(\mathbf{r}, \omega) = \hat{\mathbf{z}} J_0 e^{-r/R} \quad \tilde{\rho}(\mathbf{r}, \omega) = \frac{J_0}{-i\omega R} \cos \theta e^{-r/R}$$

For $r \gg r'$: $|\mathbf{r} - \mathbf{r}'| \approx r - \hat{\mathbf{r}} \cdot \mathbf{r}' + \dots$

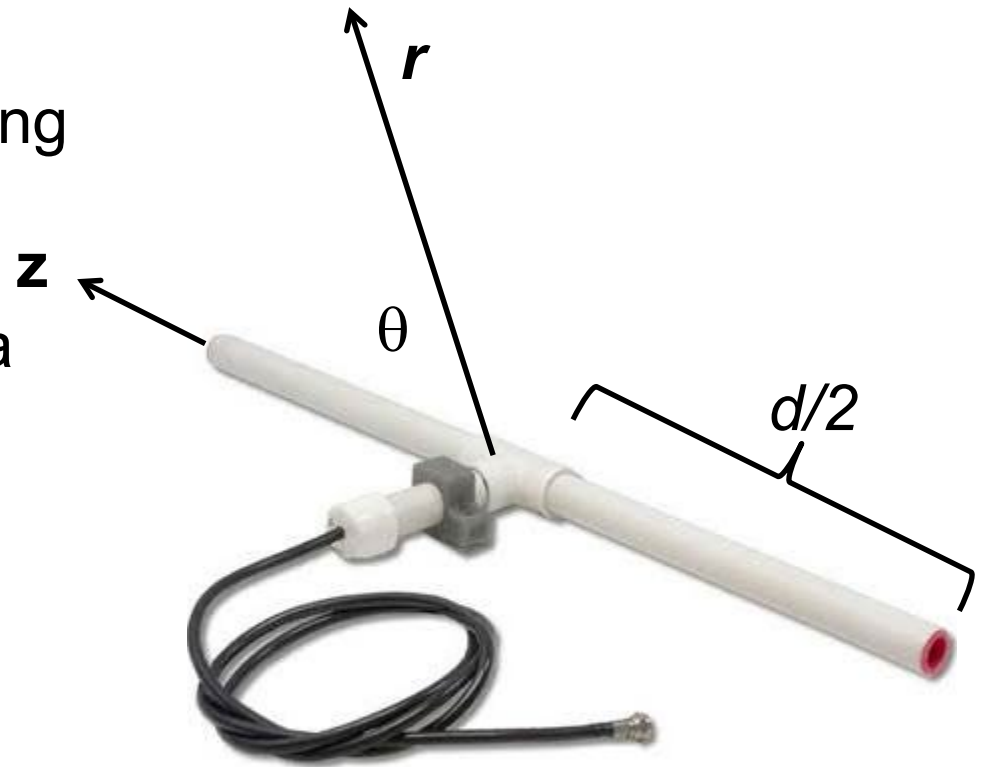
$$\tilde{\Phi}(\mathbf{r}, \omega) \approx \frac{1}{4\pi\epsilon_0} \frac{e^{ikr}}{r} \int d^3r' e^{-ik\hat{\mathbf{r}} \cdot \mathbf{r}'} \tilde{\rho}(\mathbf{r}', \omega)$$

$$\tilde{\mathbf{A}}(\mathbf{r}, \omega) \approx \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \int d^3r' e^{-ik\hat{\mathbf{r}} \cdot \mathbf{r}'} \tilde{\mathbf{J}}(\mathbf{r}', \omega)$$

→ Results equivalent to Bessel function expansion in the limit $kr \rightarrow \infty$.

Other radiation sources using
“alternative approach”

Linear center-fed antenna



$$\tilde{\mathbf{A}}(\mathbf{r}, \omega) \approx$$

$$\frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \int d^3r' e^{-ik\hat{\mathbf{r}} \cdot \mathbf{r}'} \tilde{\mathbf{J}}(\mathbf{r}', \omega)$$

$$\tilde{\mathbf{J}}(\mathbf{r}', \omega) = I_0 \sin\left(\frac{kd}{2} - k|z|\right) \delta(x) \delta(y) \hat{\mathbf{z}}$$

Alternative approach – linear center-fed antenna continued

$$\begin{aligned}\tilde{\mathbf{A}}(\mathbf{r}, \omega) &\approx \hat{\mathbf{z}} \frac{\mu_0 I_0}{4\pi} \frac{e^{ikr}}{r} \int_{-d/2}^{d/2} dz' e^{-ik \cos(\theta) z'} \sin\left(\frac{kd}{2} - k|z'|\right) \\ &= \hat{\mathbf{z}} \frac{\mu_0 I_0}{2\pi} \frac{e^{ikr}}{kr} \left(\frac{\cos\left(\frac{kd}{2} \cos \theta\right) - \cos\left(\frac{kd}{2}\right)}{\sin^2 \theta} \right)\end{aligned}$$

Time averaged power:

$$\frac{dP}{d\Omega} = I_0^2 \sqrt{\frac{\mu_0}{\epsilon_0}} \frac{1}{8\pi^2} \left| \frac{\cos\left(\frac{kd}{2} \cos \theta\right) - \cos\left(\frac{kd}{2}\right)}{\sin \theta} \right|^2$$

Alternative approach – linear center-fed antenna continued

Time averaged power:

$$\frac{dP}{d\Omega} = I_0^2 \sqrt{\frac{\mu_0}{\epsilon_0}} \frac{1}{8\pi^2} \left| \frac{\cos\left(\frac{kd}{2} \cos \theta\right) - \cos\left(\frac{kd}{2}\right)}{\sin \theta} \right|^2$$

for $kd = \pi$:

$$\frac{dP}{d\Omega} = I_0^2 \sqrt{\frac{\mu_0}{\epsilon_0}} \frac{1}{8\pi^2} \frac{\cos^2\left(\frac{\pi}{2} \cos \theta\right)}{\sin^2 \theta}$$

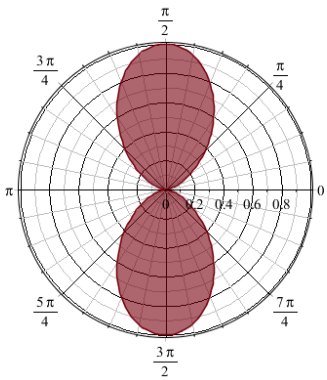
for $kd = 2\pi$:

$$\frac{dP}{d\Omega} = I_0^2 \sqrt{\frac{\mu_0}{\epsilon_0}} \frac{4}{8\pi^2} \frac{\cos^4\left(\frac{\pi}{2} \cos \theta\right)}{\sin^2 \theta}$$

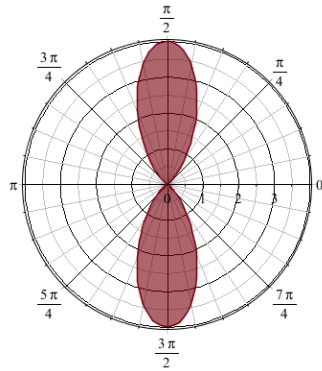
Alternative approach – linear center-fed antenna continued

Time averaged power:

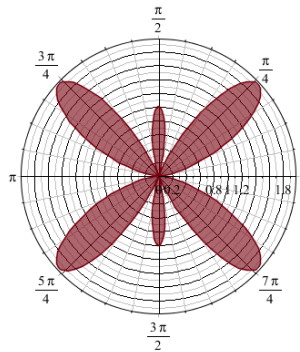
$$\frac{dP}{d\Omega} = I_0^2 \sqrt{\frac{\mu_0}{\epsilon_0}} \frac{1}{8\pi^2} \left| \frac{\cos\left(\frac{kd}{2} \cos \theta\right) - \cos\left(\frac{kd}{2}\right)}{\sin \theta} \right|^2$$



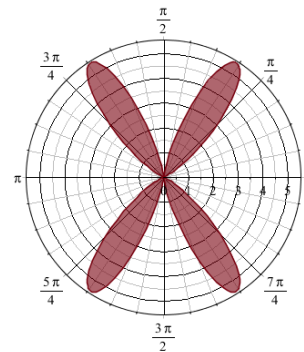
$kd = \pi$



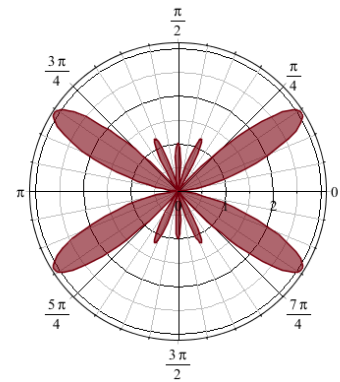
2π



3π

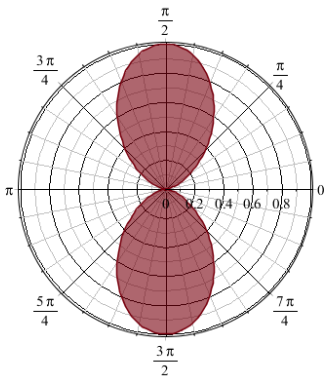


4π

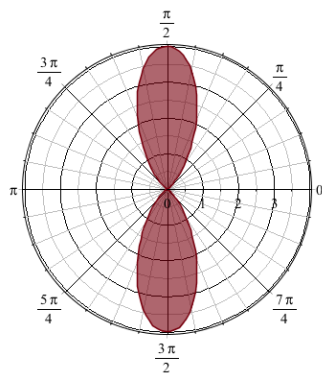


5π

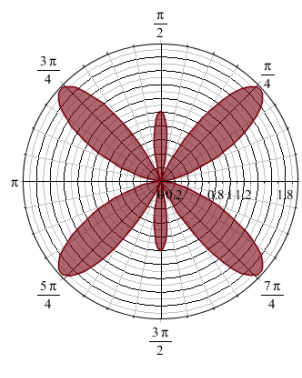
Some details --



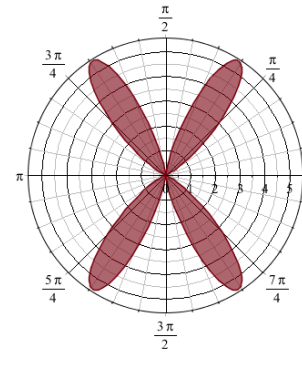
$kd=\pi$



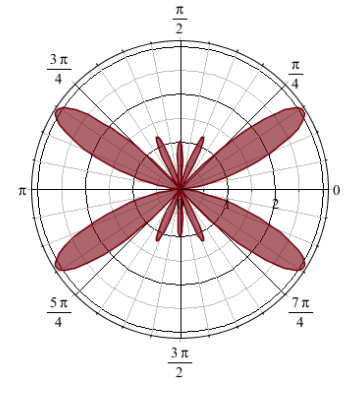
2π



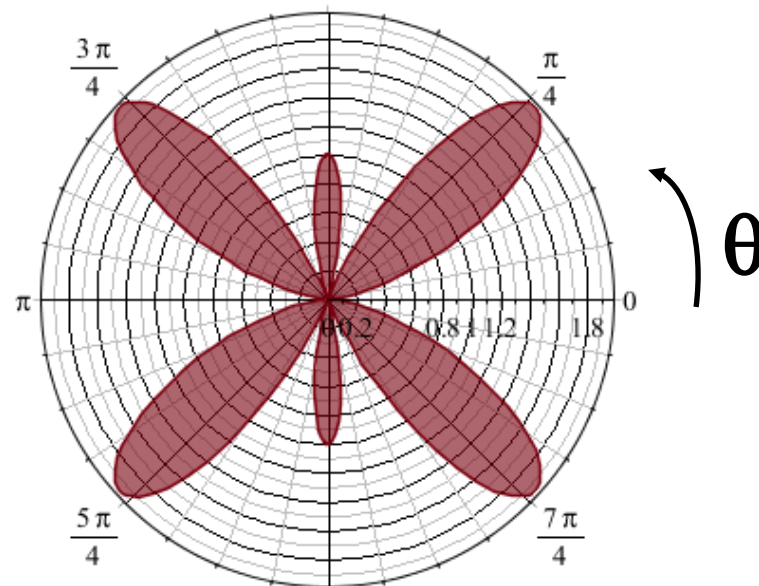
$\frac{3\pi}{2}$



4π



5π



Polar plot:
Angle indicates
values of theta

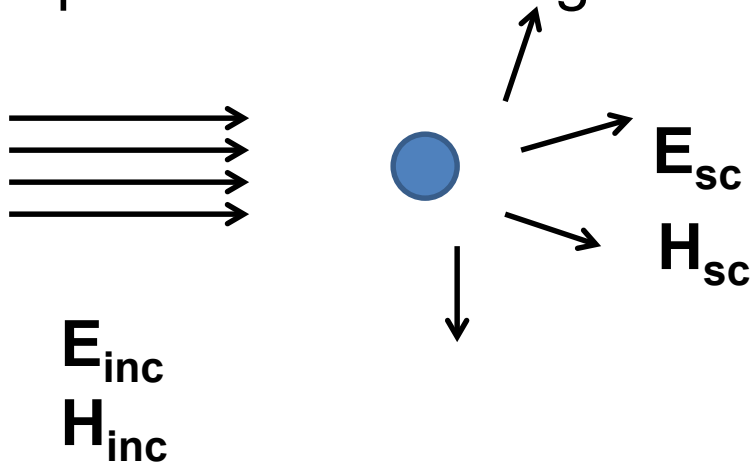
Radius indicates
value scaled to 1.

Another source of radiation –

Radiation due particles reacting to incident
electromagnetic waves – scattering processes

Chapter 10 in **Jackson**

Dipole radiation in light scattering by small (dielectric) particles



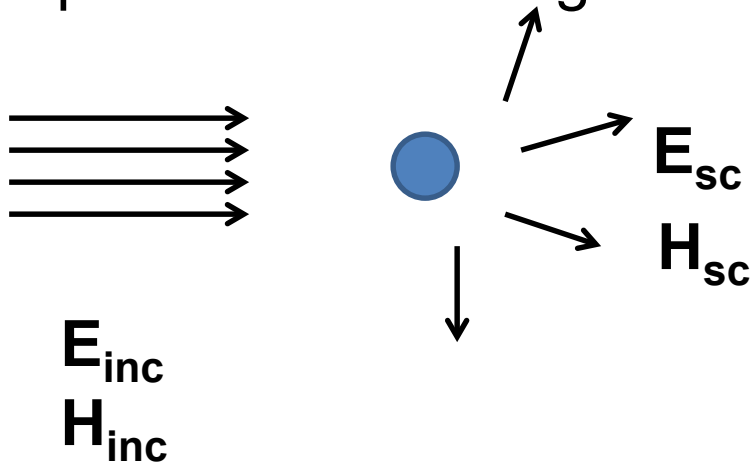
$$\mathbf{E}_{\text{inc}} = \hat{\mathbf{e}}_0 E_0 e^{ik\hat{\mathbf{k}}_0 \cdot \mathbf{r}}$$

$$\mathbf{H}_{\text{inc}} = \frac{1}{\mu_0 c} \hat{\mathbf{k}}_0 \times \mathbf{E}_{\text{inc}}$$

In electric dipole approximation :

$$\mathbf{E}_{\text{sc}} = \frac{1}{4\pi\epsilon_0} k^2 \frac{e^{ikr}}{r} ((\hat{\mathbf{r}} \times \mathbf{p}) \times \hat{\mathbf{r}}) \quad \mathbf{H}_{\text{sc}} = \frac{1}{\mu_0 c} \hat{\mathbf{r}} \times \mathbf{E}_{\text{sc}}$$

Dipole radiation in light scattering by small (dielectric) particles



Scattering cross section :

$$\begin{aligned}\frac{d\sigma}{d\Omega}(\hat{\mathbf{r}}, \hat{\boldsymbol{\epsilon}}; \hat{\mathbf{k}}_0, \hat{\boldsymbol{\epsilon}}_0) &= \frac{r^2 \hat{\mathbf{r}} \cdot \langle \mathbf{S}_{sc} \rangle_{avg}}{\hat{\mathbf{k}}_0 \cdot \langle \mathbf{S}_{inc} \rangle_{avg}} \\ &= \frac{r^2 |\hat{\boldsymbol{\epsilon}} \cdot \mathbf{E}_{sc}|^2}{|\hat{\boldsymbol{\epsilon}}_0 \cdot \mathbf{E}_{inc}|^2} = \frac{k^4}{(4\pi\epsilon_0 E_0)^2} |\hat{\boldsymbol{\epsilon}} \cdot \mathbf{p}|^2\end{aligned}$$

Estimation of scattering dipole moment:

Suppose the scattering particle is a dielectric sphere with permittivity ε and radius a :

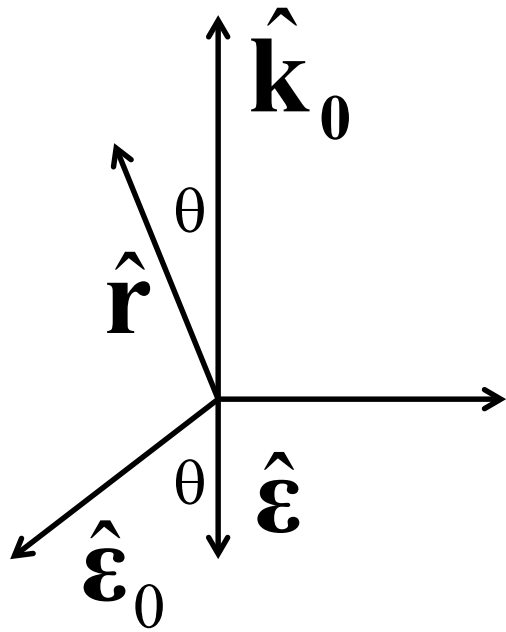
$$\mathbf{p} = 4\pi a^3 \varepsilon_0 \left(\frac{\varepsilon / \varepsilon_0 - 1}{\varepsilon / \varepsilon_0 + 2} \right) \mathbf{E}_{inc} \quad \mathbf{E}_{inc} = \hat{\mathbf{\varepsilon}}_0 E_0 e^{ik\hat{\mathbf{k}}_0 \cdot \mathbf{r}}$$

Scattering cross section :

$$\begin{aligned} \frac{d\sigma}{d\Omega}(\hat{\mathbf{r}}, \hat{\mathbf{\varepsilon}}; \hat{\mathbf{k}}_0, \hat{\mathbf{\varepsilon}}_0) &= \frac{r^2 |\hat{\mathbf{\varepsilon}} \cdot \mathbf{E}_{sc}|^2}{|\hat{\mathbf{\varepsilon}}_0 \cdot \mathbf{E}_{inc}|^2} = \frac{k^4}{(4\pi\varepsilon_0 E_0)^2} |\hat{\mathbf{\varepsilon}} \cdot \mathbf{p}|^2 \\ &= k^4 a^6 \left| \frac{\varepsilon / \varepsilon_0 - 1}{\varepsilon / \varepsilon_0 + 2} \right|^2 |\hat{\mathbf{\varepsilon}} \cdot \hat{\mathbf{\varepsilon}}_0|^2 \end{aligned}$$

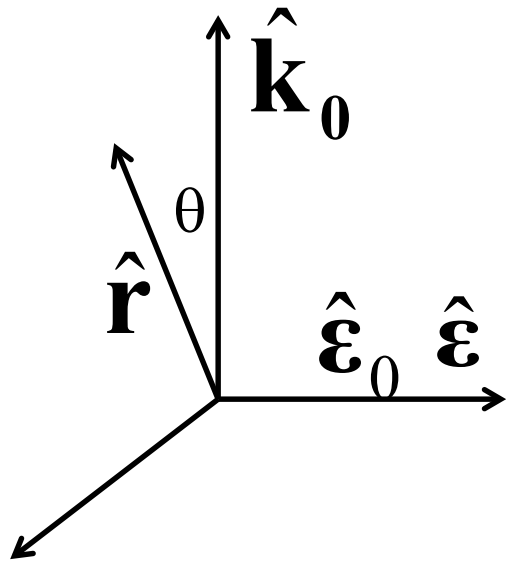
Scattering by dielectric sphere with permittivity ϵ and radius a :

For $\mathbf{E}_{\text{inc}}$ polarized in scattering plane:


$$\frac{d\sigma}{d\Omega}(\hat{\mathbf{r}}, \hat{\boldsymbol{\epsilon}}; \hat{\mathbf{k}}_0, \hat{\boldsymbol{\epsilon}}_0) = k^4 a^6 \left| \frac{\epsilon / \epsilon_0 - 1}{\epsilon / \epsilon_0 + 2} \right|^2 |\hat{\boldsymbol{\epsilon}} \cdot \hat{\boldsymbol{\epsilon}}_0|^2$$
$$= k^4 a^6 \left| \frac{\epsilon / \epsilon_0 - 1}{\epsilon / \epsilon_0 + 2} \right|^2 \cos^2 \theta$$

Scattering by dielectric sphere with permittivity ϵ and radius a :

For $\mathbf{E}_{\text{inc}}$ polarized perpendicular to scattering plane:



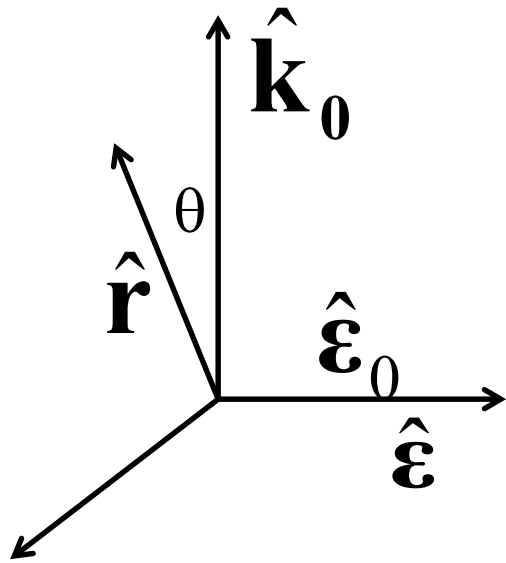
$$\frac{d\sigma}{d\Omega}(\hat{\mathbf{r}}, \hat{\boldsymbol{\epsilon}}; \hat{\mathbf{k}}_0, \hat{\boldsymbol{\epsilon}}_0) = k^4 a^6 \left| \frac{\epsilon / \epsilon_0 - 1}{\epsilon / \epsilon_0 + 2} \right|^2 |\hat{\boldsymbol{\epsilon}} \cdot \hat{\boldsymbol{\epsilon}}_0|^2$$

$$= k^4 a^6 \left| \frac{\epsilon / \epsilon_0 - 1}{\epsilon / \epsilon_0 + 2} \right|^2$$

Assuming both polarizations are equally likely, average cross section is given by :

$$\frac{d\sigma}{d\Omega}(\hat{\mathbf{r}}, \hat{\boldsymbol{\epsilon}}; \hat{\mathbf{k}}_0, \hat{\boldsymbol{\epsilon}}_0) = \frac{k^4 a^6}{2} \left| \frac{\epsilon / \epsilon_0 - 1}{\epsilon / \epsilon_0 + 2} \right|^2 (\cos^2 \theta + 1)$$

Scattering by dielectric sphere with permittivity ε and radius a :



$$\frac{d\sigma}{d\Omega}(\hat{\mathbf{r}}, \hat{\boldsymbol{\varepsilon}}; \hat{\mathbf{k}}_0, \hat{\boldsymbol{\varepsilon}}_0) = \frac{k^4 a^6}{2} \left| \frac{\varepsilon / \varepsilon_0 - 1}{\varepsilon / \varepsilon_0 + 2} \right|^2 (\cos^2 \theta + 1)$$

